International Electronic Journal of Algebra
Volume 7 (2010) 161-166

A NOTE ON TENSOR PRODUCT OF VALUED DIVISION RINGS

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Received: 20 October 2009; Revised: 31 December 2009
Communicated by A. Çiğdem Özcan

Abstract. The aim of this note is to apply the notion of a *value function* on simple Artinian rings to give a short proof of a result on the tensor product of valued division rings.

Mathematics Subject Classification (2000): 16K20, 16K40, 16W60, 12E15
Keywords: division ring, valuation, value function, Artinian ring

1. Introduction

It has been proved in [2] that if (D_1, v_1) and (D_2, v_2) are valued division rings with $F \subseteq Z(D_i),\ i = 1, 2,\ v_1|_F = v_2|_F$ and $[D_1 : F] < \infty$, furthermore if D_1 is defectless over F , $\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$ and $\overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ is a division ring, then $D = D_1 \otimes_F D_2$ is a division ring with a valuation extending v_1, v_2. In [2] Morandi has defined a function v and shown to be a valuation. The proof was quite long due to the difficulty in proving the multiplicative property $v(ab) = v(a) + v(b)$. In [1] Morandi has used the notion of value functions on central simple algebras and has given a simpler proof of this theorem when D is finite dimensional over F.

We use value functions on simple Artinian rings and prove that if the residue ring B/J of the valuation ring B of a value function on a simple Artinian ring is a division ring, then the value function is a valuation. Then we use the results to give a simpler proof of this theorem in general case. For the rest of this section we give some definitions and elementary properties.

Let D be a division ring and put $D^* = D \setminus \{0\}$. A (Krull) valuation on D is a function $v : D^* \longrightarrow \Gamma$, where Γ is a totally ordered abelian group, such that for all $a, b \in D^*$

i) $v(ab) = v(a) + v(b)$,

ii) $v(a + b) \geq \min\{v(a), v(b)\}$ if $a \neq -b$.

For convenience we extend v to D by setting $v(0) = \infty$, where $\infty > \alpha$ and $\infty + \alpha = \alpha + \infty = \infty$ for all $\alpha \in \Gamma$.

Given a valuation v on D, one obtains a value group $\Gamma_D = v(D^*)$, valuation ring $V_D = \{a \in D \mid v(a) \geq 0\}$ with unique maximal left and right ideal $M_D = \{a \in D \mid v(a) > 0\}$, the group of units $U_D = V_D \setminus M_D = \{a \in D \mid v(a) = 0\}$, and the residue division ring $\overline{D} = V_D/M_D$. If K is a division subring of D, then $v|_K$ is a valuation on K and v is called an extension of $v|_K$ to D. In this case, Γ_K is a subgroup of Γ_D and $\overline{K}$ is a division subring of $\overline{D}$. If $[D : K] < \infty$ is the dimension of D as a left K-vector space, then by [3] one has the fundamental inequality

$$[\overline{D} : \overline{K}] \cdot |\Gamma_D : \Gamma_K| \leq [D : K] \qquad (1)$$

D is called *defectless* over K if the equality holds in (1).

2. Value Functions on Simple Artinian Rings

We begin our study with

Definition 2.1. Let S be a simple Artinian ring and Γ a totally ordered abelian group. A function $\omega : S \longrightarrow \Gamma \cup \{\infty\}$ is called a *value function on* S provided that,

1. $\omega(-1) = 0$,
2. $\omega(a) = \infty \iff a = 0$,
3. $\omega(ab) \geq \omega(a) + \omega(b)$, for all $a, b \in S$,
4. $\omega(a + b) \geq \min\{\omega(a), \omega(b)\}$, for all $a, b \in S$,
5. For all $0 \neq s \in S$ there is an element $a \in st(\omega)$ such that $\omega(s) = \omega(a)$, where $st(\omega) = \{s \in S^* \mid \omega(s^{-1}) = -\omega(s)\}$.

The following lemma is proved in [1].

Lemma 2.2. *Let S be a simple Artinian ring and ω a value function on S. Then the following statements hold:*

1. *$\omega(-a) = \omega(a)$, for all $a \in S$.*
2. *If $a \in st(\omega)$ and $b \in S$, then $\omega(ab) = \omega(ba) = \omega(a) + \omega(b)$.*
3. *$st(\omega)$ is a subgroup of S^*,the multiplicative group of units in S.*
4. *$B_\omega = \{a \in S \mid \omega(a) \geq 0\}$ is a subring of S and $J_\omega = \{a \in S \mid \omega(a) > 0\}$ is a proper two-sided ideal of B_ω.*

B_ω (resp. J_ω) is called the ring (resp. ideal) associated to ω. The key result in the proof of the theorem is the following lemma.

Lemma 2.3. *Let S be a simple Artinian ring and ω a value function on S with associated ring B and ideal J. If the residue ring B/J has no left and no right zero divisors, then S is a division ring and ω is a Krull valuation on S.*

Proof. We show that $\omega(ab) = \omega(a) + \omega(b)$, for all $a, b \in S$. We first suppose that $a, b \in S$ and $\omega(a) = \omega(b) = 0$. If $\omega(ab) > 0$ then $ab \in J$. Hence $\overline{a}\overline{b} = \overline{ab} = \overline{0}$ in B/J. Now since B/J has no zero divisors, then $\overline{a} = \overline{0}$ or $\overline{b} = \overline{0}$, so $a \in J$ or $b \in J$, which would contradict the choice of a, b. Thus

$$\omega(ab) = 0 = \omega(a) + \omega(b).$$

Now suppose $a, b \in S$ are arbitrary. By the Definition and Lemma 2.2, there exist elements $r, s \in st(\omega)$ such that $\omega(bs) = \omega(ra) = 0$, therefore

$$\omega(rabs) = \omega((ra)(bs)) = \omega(ra) + \omega(bs) = \omega(r) + \omega(a) + \omega(b) + \omega(s).$$

On the other hand, $\omega(rabs) = \omega(r) + \omega(ab) + \omega(s)$. Putting the last two equalities together, we obtain

$$\omega(ab) = \omega(a) + \omega(b).$$

This equality implies that S has no left and no right zero divisors and so S is a division ring. □

3. The Main Theorem

We now use the Lemma 2.3 to give another proof of Theorem 1 of [2] which be shorter than the original one.

Theorem 3.1. *Let (D_1, v_1) and (D_2, v_2) be valued division rings with $F \subseteq Z(D_i)$, $i = 1, 2$, $v_1|_F = v_2|_F$. and $[D_1 : F] < \infty$. Suppose*

1. *D_1 is defectless over F;*
2. *$\Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$;*
3. *$\overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ is a division ring.*

Then $D = D_1 \otimes_F D_2$ is a division ring with a valuation extending v_1, v_2 and with $\Gamma_{D_1 \otimes_F D_2} = \Gamma_{D_1} + \Gamma_{D_2}$, and $\overline{D_1 \otimes_F D_2} = \overline{D_1} \otimes_{\overline{F}} \overline{D_2}$.

Proof. Since $[D_1 : F] < \infty$, the injection $\Gamma_F \longrightarrow \Delta_F = \Gamma_F \otimes_Z Q$, where Δ_F is the divisible hull of Γ_F, extends uniquely to an order preserving injection $\Gamma_{D_1} \longrightarrow \Delta_F \subset \Delta = \Gamma_{D_2} \otimes_Z Q$. Hence we could consider Γ_{D_1} and Γ_{D_2} as subgroups of Δ . As usual the intersection $\Gamma_{D_1} \cap \Gamma_{D_2}$ is computed in Δ.

As in the proof given in [2], suppose that $a_1, \ldots, a_r, u_1, \ldots, u_s \in D_1^*$ with $a_1 = u_1 = 1$ such that $\Gamma_{D_1}/\Gamma_F = \{v_1(a_i) + \Gamma_F \mid 1 \leq i \leq r\}$ and $\{\overline{u_1}, \ldots, \overline{u_s}\}$ is a basis of $\overline{D_1}$ over $\overline{F}$.

It is easy to see that the rs elements $\{a_i u_j \mid 1 \leq i \leq r, , 1 \leq j \leq s\}$ are linearly independent over F, so form a basis for D_1 since D_1 is defectless over F. Therefore every element of D has a unique representation in the form $\sum_{i,j} a_i u_j \otimes b_{ij}$. Now we define a function $\omega : D \longrightarrow \Delta$ by

$$\omega(\sum_{i,j} a_i u_j \otimes b_{ij}) = \min_{i,j}\{v_1(a_i) + v_2(b_{ij}) \ \mid \ b_{ij} \neq 0\}.$$

We will show that ω is a value function on the simple Artinian ring D.
(1) and (2) of Definition 2.1 are clear. The proof of (4) is similar to step (1) in the proof given in [2]. Furthermore, ω extends v_1 and v_2, as it is shown in step (2) of the proof of Theorem 1 in [2]. We prove (3) in several cases.

1. If $0 \neq a = \sum_{i,j} \alpha_{ij} a_i u_j \in D_1$ with $\alpha_{ij} \in F$ and $0 \neq b \in D_2$, then $a \otimes b = \sum_{i,j} a_i u_j \otimes \alpha_{ij} b$, and so

$$\begin{aligned}\omega(a \otimes b) &= \min_{i,j}\{v_1(a_i) + v_2(\alpha_{ij} b)\} = \min_{i,j}\{v_1(a_i) + v_2(\alpha_{ij}) + v_2(b)\} \\ &= \min_{i,j}\{v_1(a_i) + v_1(\alpha_{ij})\} + v_2(b) = v_1(a) + v_2(b).\end{aligned}$$

2. For every $a, c \in D_1$ and $b, d \in D_2$, we have

$$\begin{aligned}\omega((a \otimes b)((c \otimes d)) &= \omega(ac \otimes bd) = v_1(ac) + v_2(bd) \\ &= v_1(a) + v_1(c) + v_2(b) + v_2(d) \\ &= (v_1(a) + v_2(b)) + (v_1(c) + v_2(d)) \\ &= \omega(a \otimes b) + \omega(c \otimes d).\end{aligned}$$

3. Suppose $a = \sum_{i,j} a_i u_j \otimes a_{ij}$, $b = \sum_{i,j} a_i u_j \otimes b_{ij} \in D$, using (4) and the case (2) we obtain

$$\begin{aligned}\omega(ab) &= \omega\left((\sum_{i,j} a_i u_j \otimes a_{ij})(\sum_{k,l} a_k u_l \otimes b_{kl})\right) \\ &= \omega\left(\sum_{i,j,k,l} (a_i u_j \otimes a_{ij})(a_k u_l \otimes b_{kl})\right) \\ &\geq \min_{i,j,k,l}\{\omega\left((a_i u_j \otimes a_{ij})(a_k u_l \otimes b_{kl})\right)\} \\ &= \min_{i,j,k,l}\{\omega(a_i u_j \otimes a_{ij}) + \omega(a_k u_l \otimes b_{kl})\} \\ &= \min_{i,j}\{\omega(a_i u_j \otimes a_{ij})\} + \min_{k,l}\{\omega(a_k u_l \otimes b_{kl})\} \\ &= \omega(a) + \omega(b).\end{aligned}$$

This settles (3).

To prove (5) suppose $0 \neq a = \sum_{i,j} a_i u_j \otimes b_{ij} \in D$ and

$$v_1(a_{i_0}) + v_2(b_{i_0 j_0}) = \min_{i,j}\{v_1(a_i) + v_2(b_{ij})\} = \omega(a).$$

By case (1), we have $\omega(a_{i_0} \otimes b_{i_0 j_0}) = v_1(a_{i_0}) + v_2(b_{i_0 j_0}) = \omega(a)$. Similarly,

$$\omega(a_{i_0}^{-1} \otimes b_{i_0 j_0}^{-1}) = v_1(a_{i_0}^{-1}) + v_2(b_{i_0 j_0}^{-1}) = -v_1(a_{i_0}) - v_2(b_{i_0 j_0}) = -\omega(a_{i_0} \otimes b_{i_0 j_0}).$$

Thus $a_{i_0} \otimes b_{i_0 j_0} \in st(\omega)$.

Therefore ω is a value function on D. It is clear, by definition of ω, that $\Gamma_D = \Gamma_{D_1} + \Gamma_{D_2}$. Now let B be the ring associated to ω with ideal J. We show that the residue ring B/J has no zero divisors.

Suppose $a = \sum_{i,j} a_i u_j \otimes b_{ij}$ is an arbitrary element of D such that $\overline{a} \in B/J$ is not zero. So $\omega(a) = 0$. On the other hand , if

$$v_1(a_{i_0}) + v_2(b_{i_0 j_0}) = \min_{i,j}\{v_1(a_i) + v_2(b_{ij})\} = \omega(a) = 0$$

then $v_1(a_{i_0}) = -v_2(b_{i_0 j_0}) \in \Gamma_{D_1} \cap \Gamma_{D_2} = \Gamma_F$.
Hence $a_{i_0} = 1$ and $i_0 = 1$, the minimum term occurs for $i = 1$ and for every $i, j, i \neq 1$, $\omega(a_i u_j \otimes b_{ij}) > 0$, since the $v_1(a_i)$ are distinct mod Γ_F. Similarly $v_2(b_{1j}) \geq 0$ for all j. Therefore, $\sum_{i,j,i\neq 1} a_i u_j \otimes b_{ij} \in J$ and so

$$\overline{a} = \overline{\sum_j u_j \otimes b_{1j}} = \sum_j \overline{u_j \otimes b_{1j}} = \sum_j \overline{u_j} \otimes \overline{b_{1j}} \in \overline{D_1} \otimes_{\overline{F}} \overline{D_2}.$$

Thus $B/J \subseteq \overline{D_1} \otimes_{\overline{F}} \overline{D_2}$. Since $\overline{D_1} \otimes_{\overline{F}} \overline{D_2}$ is a division ring, then B/J has no zero divisors. Our Lemma now implies that D is a division ring and ω a Krull valuation on D. The rest of the proof is easy and it may be seen in [2]. □

Acknowledgment. I would like to thank the referee for the valuable suggestions and comments.

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