International Electronic Journal of Algebra
Volume 7 (2010) 47-58

$\mathcal{L}_2$-PRIME AND DIMENSIONAL MODULES

M. R. Vedadi

Received: 15 December 2008; Revised: 4 September 2009
Communicated by Abdullah Harmancı

Abstract. We introduce a map κ that generalizes Krull and Noetherian dimensions. If M_R finitely generates all fully invariant submodules and has acc on them, there are only a finite number of minimal $\mathcal{L}_2$-prime submodules $P_i (1 \leq i \leq n)$ and when defined, $\kappa(M) = \kappa(M/P_j)$ for some j. Here, each M/P_i is a prime R-module, and in particular, M has finite length if every irreducible prime submodule of M is maximal. Quasi-projective $\mathcal{L}_2$-prime R-module with non-zero socle are investigated and some applications are then given when $\kappa(M)$ means the Krull dimension or the injective dimension.

Mathematics Subject Classification (2000): Primary 16D10, 16P10; Secondary 16P40
Keywords: dimension map, hereditary ring, Krull dimension, $\mathcal{L}_2$-Noetherian, $\mathcal{L}_2$-prime submodule

1. Introduction

Throughout rings will have unit elements and modules will be right unitary. Let R be a ring. A non-empty class of R-modules is called *weak hereditary* if it is closed under homogeneous finite direct sums, homomorphic images and taking submodules. A map κ from a weak hereditary class of R-modules to the set of all cardinal numbers, is called a *dimension* map, if κ takes the same value on finite direct sum of isomorphic modules, with the conditions $\kappa(M/N) \leq \kappa(M) \in \{\kappa(N), \kappa(M/N)\}$ for every $N \leq M$. An R-module M is called *dimensional* if it belongs to the domain of some dimension map κ. In this case, we say that M is κ-dimensional and $\kappa(M)$ means the κ-*dimension* of M. Any module with Krull dimension (resp. Noetherian dimension) is a κ-dimensional module where $\kappa(M)$ means the Krull dimension (resp. Noetherian dimension) of M, see [5, Chapter 15], [10], [1], [8]. By [10, Corollaire 6], an R-module has Krull dimension if and only if it has Noetherian dimension. However, there are dimensional R-modules which have not necessarily Krull dimensions. Defining $\kappa(M)$ = the injective dimension

Dedicated to Professor O.A.S. Karamzadeh on the occasion of his sixty fifth birthday.

of M_R for all M in Mod-R, the class of all R-modules, we obtain a dimension map κ : Mod-$R \to \{0,1\}$ if R is a right hereditary ring. Hence, modules over right hereditary rings are dimensional. More generally, for an arbitrary ring R, let $\kappa'(M)$ = -1 if M_R has a finite length, otherwise $\kappa'(M) = 1$. Then it is easy to verify that κ' : Mod-$R \to \{-1,1\}$ is a dimension map and so all R-modules are κ'-dimensional. In this paper, it is shown that the computation of κ-dimension of certain κ-dimensional modules reduces to that of some prime factor modules. Some applications are then obtained when $\kappa(M)$ means either Krull dimension or Noetherian dimension or injective dimension of a module. In [2], a module M_R has been called " prime " if $N_1 \star N_2 = 0$ implies $N_1 = 0$ or N_2 for all submodules N_1, N_2 of M where $N_1 \star N_2 = \mathrm{Hom}_R(M, N_1)N_2$. These prime modules have been called $\star$-*prime modules* in [11] where some applications of them are obtained. If P is a proper submodule of a module M_R, we say that P is an $\mathcal{L}_2$-*prime submodule* of M if $W_1 \star W_2 \subseteq P$ implies $W_1 \subseteq P$ or $W_2 \subseteq P$ where W_1, W_2 belong to $\mathcal{L}_2(M)$, the set of all *fully invariant submodules* of M_R. Also, *minimal* $\mathcal{L}_2$-prime submodule means minimal among all $\mathcal{L}_2$-prime submodules of M. If (0) is an $\mathcal{L}_2$-prime submodule of M_R then M is called an $\mathcal{L}_2$-*prime R-module.* Clearly, every $\star$-prime module M is $\mathcal{L}_2$-prime and the converse is true if M is a *DUO module* in the sense of [13] (i.e., all of submodules of M are fully invariant). By a *prime module* M_R we mean the "classical" notion of a prime module, that is, $\mathrm{ann}_R(N) = \mathrm{ann}_R(M)$ for any $0 \neq N \leq M$.

We first study some properties of $\mathcal{L}_2$-prime submodules of a module and show, among other things, that a quasi-projective $\mathcal{L}_2$-prime module with non-zero socle is semisimple if it has acc on direct summands (Theorem 2.4). Next, we introduce $\mathcal{L}_2$-*Noetherian* modules (page 5) and prove that if M is an $\mathcal{L}_2$- Noetherian κ-dimensional R-module, then M has only a finite number of minimal $\mathcal{L}_2$-prime fully invariant submodules $P_1, \cdots, P_n$ such that $\kappa(M) = \mathrm{Max}\{\kappa(M/P_1), \cdots, \kappa(M/P_n)\}$ (Theorem 3.1). In particular, a generalization of Lambek-Michler Theorem [9, Theorem 3.6] is obtained (Proposition 3.2(i)) and a module theoretic version of Ginn-Moss Theorem [5, Theorem 8.16] is offered (Corollary 3.11).

Any unexplained terminology, and all the basic results on rings and modules that are used in the sequel can be found in [12] and [16].

2. $\mathcal{L}_2$-prime submodules

Let K and P be proper submodules of a module M_R. We say that P is a *minimal* $\mathcal{L}_2$*-prime* submodule over K if P is minimal among all $\mathcal{L}_2$-prime submodules which contain K, in this case, we write $K \overset{min}{\leq} P$. We begin with the following result.

Proposition 2.1. *Let M be a non-zero R-module.*
(i) If N is any $\mathcal{L}_2$-prime submodule of M then for every $K \leq N$ there exists an $\mathcal{L}_2$-prime submodule P of M such that $K \overset{min}{\leq} P \leq N$.
(ii) If P is a proper fully invariant $\mathcal{L}_2$-prime submodule of M_R, then M/P is a prime R-module.

Proof. (i) By Zorn's Lemma, pick a maximal chain $\{N_i\}_{i\in I}$ of $\mathcal{L}_2$-prime submodules of M such that $K \leq N_i \subseteq N$ for all $i \in I$. Then N contains the $\mathcal{L}_2$-prime submodule $P = \bigcap_i N_i$ of M and also we have $K \overset{min}{\leq} P$ by the maximality of $\{N_i\}_{i\in I}$ (to see P is an $\mathcal{L}_2$-prime submodule, let $W_1 \star W_2 \subseteq P$ for some $W_1, W_2 \in \mathcal{L}_2(M)$. If $W_1, W_2 \not\subseteq P$, then there exists $i \in I$ such that both $W_1, W_2 \not\subseteq N_i$. But $W_1 \star W_2 \subseteq N_i$ and N_i is $\mathcal{L}_2$ prime, a contradiction).
(ii) By [11, Lemma 5.3]. □

If M_R has acc on fully invariant submodules, we don't know whether M has an $\mathcal{L}_2$-prime submodule. However, the following result shows that the set of all minimal $\mathcal{L}_2$-prime submodules of M is a finite set. Proposition 2.5 provides examples of modules with $\mathcal{L}_2$-prime submodules.

Proposition 2.2. *Let M_R be a module with acc on fully invariant submodules.*
(i) If N is an $\mathcal{L}_2$-prime submodule of M, then N contains an $\mathcal{L}_2$-prime fully invariant submodule of M.
(ii) The set of all minimal $\mathcal{L}_2$-prime submodules of M is a finite set and each minimal $\mathcal{L}_2$-prime submodule is a fully invariant submodule.

Proof. (i) Let K be maximal among all fully invariant submodules of M which contained in N. Then K is an $\mathcal{L}_2$-prime submodule of M. In fact, if there exist $W_1, W_2 \in \mathcal{L}_2(M)$ such that $W_1 \star W_2 \subseteq K$. Then $W_1 \star W_2 \subseteq N$ and hence $W_1 \subseteq N$ or $W_2 \subseteq N$. Thus $W_1 + K = K$ or $W_2 + K = K$ by maximality of K. It follows that $W_1 \subseteq K$ or $W_2 \subseteq K$, proving that K is an $\mathcal{L}_2$-prime submodule of M.
(ii) Suppose that M has infinite number of minimal $\mathcal{L}_2$-prime submodules and set $\mathcal{A} = \{L \leq M \mid L$ is proper fully invariant and there exist infinite number of $\mathcal{L}_2$-prime submodule of M which are minimal over $L\}$. Then $(0) \in \mathcal{A}$ and by hypothesis $\mathcal{A}$ has a maximal member N. Thus N is not an $\mathcal{L}_2$-prime submodule of M. It follows

that there exist $W_1, W_2 \in \mathcal{L}_2(M)$ such that $W_1 \star W_2 \subseteq N$ and $W_1, W_2 \nsubseteq N$. Let $C_1 = W_1 + N$ and $C_2 = W_2 + N$. Because $W_1 \star W_2 \subseteq N$, we have $\{P \mid N \overset{min}{\leq} P\} \subseteq \{P \mid C_1 \overset{min}{\leq} P\} \cup \{P \mid C_2 \overset{min}{\leq} P\}$, but the union is a finite (even the empty) set by maximality of N, a contradiction. Hence M cannot have infinite number of minimal $\mathcal{L}_2$-prime submodules. The last statement is clear by (i). □

Proposition 2.3. *Let M be quasi-projective R-module.*
(i) Every minimal $\mathcal{L}_2$-prime submodule of M is fully invariant.
(ii) A fully invariant submodule P of M is an $\mathcal{L}_2$-prime submodule of M if and only if $(M/P)_R$ is an $\mathcal{L}_2$-prime module.

Proof. (i) Let N be a minimal $\mathcal{L}_2$-prime submodule of M and let $P = \mathrm{Rej}(M, M/N)$, then P is a fully invariant submodule of M and $P \subseteq N$. We shall show that P is an $\mathcal{L}_2$-prime submodule of M. Let $W_1 \star W_2 \subseteq P$ for some $W_1, W_2 \in \mathcal{L}_2(M)$. Then $W_1 \star W_2 \subseteq N$ and hence $W_1 \subseteq N$ or $W_2 \subseteq N$. On the other hand, because M_R is quasi-projective, $\mathrm{Rej}(M, M/N) = \{m \in M \mid Tm \subseteq N\}$ where $T = \{f \in \mathrm{End}_R(M) \mid f(N) \subseteq N\}$. Also for each $i = 1, 2$, $W_i \subseteq N$ implies $TW_i \subseteq TN \subseteq N$. Therefore, we have $W_1 \subseteq P$ or $W_2 \subseteq P$. This shows that P is $\mathcal{L}_2$ prime, as desired.
(ii) This is true because for every $P \leq N \leq M_R$, the hypothesis implies that M is N-projective, hence we have $\mathrm{Hom}_R(M/P, N/P) = \{F \mid \exists f : M_R \to N_R$ such that $F(m + P) = f(m) + P \; \forall m \in M\}$. □

Theorem 2.4. *Let M be a quasi-projective $\mathcal{L}_2$-prime R-module with non-zero socle. If either $Soc(M)$ is finitely generated or M has acc on direct summands then M_R is homogenous semisimple.*

Proof. Because M is an $\mathcal{L}_2$-prime module, $0 \neq K \star L \subseteq K \cap L$, for all non-zero fully invariant submodules K and L of M. It follows that $W := \mathrm{Soc}(M_R)$ is a homogeneous semisimple R-module. Since $W \star W \neq 0$, there exists $f : M \to W$ such that $f(W) \neq 0$. Let $W = \oplus S_i$ where $\{S_i\}_i$ are isomrphic simple submodules of M. Then $f = \sum f_i$ where $f_i : M \to S_i$. Since $W \nsubseteq \ker f = \bigcap \ker f_i$, there exists f_i such that $W \nsubseteq \ker f_i$. Thus $\ker f_i$ is a maximal submodule of M which not essential. It follows that $M = S_{i_1} \oplus L_1$ for some $L_1 \leq M_R$ and simple submodule S_{i_1}. Suppose that L_1 is non-zero and let $W_1 = \mathrm{Soc}(L_1)$. We claim $\mathrm{Hom}_R(L_1, W_1) \neq 0$. If $\mathrm{Hom}_R(L_1, W_1) = 0$, then $\mathrm{Hom}_R(L_1, S_{i_1}) = 0$. This in turn follows that L_1 and hence W_1 are fully invariant submodules of M. Thus by the above, $W_1 = W \cap L_1$ is a non-zero fully invariant submodule of M. It follows by hypothesis that $\mathrm{Hom}_R(M, W_1)L_1 \neq 0$, a contradiction. Therefore, $\mathrm{Hom}_R(L_1, W_1) \neq 0$, as claimed. Consequently, there exists a simple submodule U of W_1 which is homomorphic image of L_1. Because $U \simeq S_{i_1}$ and M is quasi-projective, $L_1 = S_{i_2} \oplus L_2$ for some $L_2 \leq L_1$ and simple submodule S_{i_2}. By hypothesis, this process is stopped (i.e. $L_n = 0$ for some $n \geq 1$) and so $M = \mathrm{Soc}(M)$. □

Proposition 2.5. *Let M finitely generate fully invariant submodules. If N is maximal among all proper fully invariant submodules, then N is an $\mathcal{L}_2$-prime submodule.*

Proof. Note that if $W_1 \star W_2 \subseteq N$ and $W_2 + N = M$ for some fully invariants submodules W_1, W_2, then $W_1 = \mathrm{Hom}(M, W_1)(W_2 + N) \subseteq N$. □

An R-module M is called $\mathcal{L}_2$-*Noetherian* if it finitely generates all of its fully invariant submodules and has acc on them. Self generator Noetherian modules and modules without non-trivial fully invariant submodules are clearly $\mathcal{L}_2$-Noetherian. It is easy to verify that the R-module R is $\mathcal{L}_2$-Noetherian if and only if every two sided ideal of R is finitely generated as a right ideal.

Example 2.6. (i) For a non-trivial example of $\mathcal{L}_2$-Noetherian modules which are not Noetherian, consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}$. Then $M_{\mathbb{Z}}$ is not Noetherian and each fully invariant submodule of $M_{\mathbb{Z}}$ has the form $\mathbb{Q} \oplus (n\mathbb{Z})$ for some $n \geq 0$, hence $M_{\mathbb{Z}}$ is $\mathcal{L}_2$-Noetherian. This can easily be seen from the fact that $\mathrm{End}_{\mathbb{Z}}(M) = \left[\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$.

(ii) Let R be a non-Artinian right primitive ring with simple faithful R-module M. Then there is no ring T such that ${}_TM_R$ is a bimodule and ${}_TM$ is Noetherian (or even ${}_TM$ is finitely generated). Because if ${}_TM$ is finitely generated, then R embedding in $M_R^{(n)}$ for some $n \geq 1$, is Artinian. Consequently, M_R is an $\mathcal{L}_2$-Noetherian such that ${}_SM$ is not Noetherian where $S = \mathrm{End}_R(M)$.

It is known that a prime ideal is either a minimal prime ideal or essential as a right ideal. In [6], a ring R is said to satisfy *r.min π-condition* if no minimal prime ideal of R is essential as a right ideal. Generalizing this to module M_R, we say M satisfies *min $\mathcal{L}_2$-π condition* if no minimal $\mathcal{L}_2$-prime submodule of M is an essential submodule. Clearly, $\mathcal{L}_2$-prime modules and semisimple modules satisfy the min $\mathcal{L}_2$-π condition. Also, it is well known that every semiprime right Goldie ring satisfies the r.min π-condition [5, Propositions 6.13, 7.3], however there are non semiprime rings which satisfy the r.min π-condition [6, Example 3.10]. If M is a quasi-projective module which satisfies the min $\mathcal{L}_2$-π condition and $\mathrm{Soc}(M)$ is an essential submodule of M, then M/P is an $\mathcal{L}_2$-prime module with non-zero socle, for every minimal $\mathcal{L}_2$-prime submodule P of M. In view of Theorem 2.4, it is then not out of place to consider the min $\mathcal{L}_2$-π condition.

Theorem 2.7. *Let M be a non-zero quasi-projective $\mathcal{L}_2$-Noetherian R-module with Krull dimension. Then the following statements are equivalent.*
(i) M is a semisimple R-module.
(ii) Soc(M) is an essential submodule of M and $\bigcap P = 0$ where the intersection runs through the set of minimal $\mathcal{L}_2$-prime submodules of M.
(iii) Soc(M) is an essential submodule of M and M satisfies the min $\mathcal{L}_2$-π condition.

Proof. (i)⇒(ii). This is true by Proposition 2.3(i) and the fact that fully invariant submodules of M are its components.
(ii)⇒(iii). By Proposition 2.2, M has only a finite number of minimal $\mathcal{L}_2$-prime submodules $P_i(1 \leq i \leq n)$. Thus, the condition $\bigcap_i P_i = 0$ implies that M satisfies the min $\mathcal{L}_2$-π condition.
(iii)⇒(i). Because M satisfies the min $\mathcal{L}_2$-π condition and Soc(M) is essential, no minimal $\mathcal{L}_2$-prime submodule of M contains Soc(M). It follows that Soc(M/P) is non-zero for every minimal $\mathcal{L}_2$-prime submodule P. Now let Soc$(M) \neq M$. Because M is $\mathcal{L}_2$-Noetherian, Soc(M) lies in a submodule N of M such that N is maximal among all proper fully invariant submodules of M. By Proposition 2.5, N is an $\mathcal{L}_2$-prime submodule of M. Hence, by Propositions 2.1, N contains a minimal $\mathcal{L}_2$-prime submodule P and so M/P is a homogeneous semisimple module by Theorem 2.4. It follows that $P = N$. But then P containing Soc(M) must be an essential submodule of M, a contradiction. Therefore Soc$(M) = M$. □

Remark 2.8. Let R be a right Noetherian ring such that Soc(R_R) is an essential right ideal of R. Then by [3], R is an Artinian ring if Soc$(R_R) \subseteq$ Soc$({}_RR)$. But in fact, Theorem 2.7 shows that R is a semisimple Artinian ring if R satisfies the right min-π condition.

3. Modules with κ-dimension

In [15], it is proved that if ${}_SM_R$ is a bimodule such that ${}_SM$ is finitely generated and M/N has the same Krull dimension as a left S- and as a right R-module for every sub-bimodule N of M, then the Krull dimension of M_R is equal to that of a suitable prime factor module. Generally, if M_R is an $\mathcal{L}_2$-Noetherian module with Krull dimension, there does not exist a ring S such that M becomes a left S-, right R-bimodule with the mentioned properties, see example 2.6(ii). Motivated by [15], we show that the computation of κ-dimension of an $\mathcal{L}_2$-Noetherian module M (if it exists) reduces to that of some prime factor module. This generalizes a well known result that states if R is a right Noetherian ring then k.dim$(R_R) =$ Max{k.dim$(R/I_1), \cdots,$ k.dim(R/I_n)} where I_1, ..., I_n are all minimal prime ideals of R, see for example [5, Proposition, 15.5].

Theorem 3.1. *Let M be a non-zero $\mathcal{L}_2$-Noetherian κ-dimensional R-module. Then M has only a finite number minimal $\mathcal{L}_2$-prime fully invariant submodules $P_1, \cdots, P_n$ such that $\kappa(M) = Max\{\kappa(M/P_1), \cdots, \kappa(M/P_n)\} = \kappa(M/MI)$ for some minimal prime ideal I of R.*

Proof. Let $\mathcal{A} = \{L \leq M \mid L$ is proper fully invariant and $\kappa(M) = \kappa(M/L)\}$. Clearly, $\mathcal{A}$ is a non-empty set. Let $P = \text{Max } \mathcal{A}$. We claim that P is an $\mathcal{L}_2$-prime submodule of M. Suppose that $W_1 \star W_2 \subseteq P$ for some $W_1, W_2 \in \mathcal{L}_2(M)$. We shall show that $W_1 \subseteq P$ or $W_2 \subseteq P$. Let $C_1 = W_1 + P$ and $C_2 = W_2 + P$. If $C_2 = P$ then clearly $W_2 \subseteq P$ and if $C_2 = M$ then $W_1 = \text{Hom}_R(M, W_1)M = \text{Hom}_R(M, W_1)(W_2 + P) \subseteq P$. Thus, suppose that $P < C_2 < M$. Hence $\kappa(M) > \kappa(M/C_2)$ by maximality of P. By hypothesis, there exist $n \geq 1$ and a surjective homomorphism $\varphi : M^{(n)} \to W_1$. Because $\text{Hom}_R(M, W_1)C_2 \subseteq P$, the map φ induces a surjective homomorphism $\alpha : M^{(n)}/C_2^{(n)} \to C_1/P$ such that $\alpha(\{m_i\}_{i=1}^n + C_2^{(n)}) = \sum_{i=1}^n \varphi\iota_i(m_i)$ where $\iota_i : M \to M^{(n)}$ $(1 \leq i \leq n)$ are the canonical injections. It follows that $\kappa(M/C_2) = \kappa([M/C_2]^{(n)}) \geq \kappa(C_1/P)$. Consequently, $\kappa(M/P) = \kappa(M) > \kappa(C_1/P)$ Thus $M \neq C_1$ and since $\kappa(M/P) \in \{\kappa(C_1/P), \kappa[(M/P)/(C_1/P)]$, we must have $\kappa(M/P) = \kappa(M/C_1)$. This implies that $C_1 = P$ by maximality of P. Thus $W_1 \subseteq P$, as claimed.

Now by Proposition 2.1(i), P contains a minimal $\mathcal{L}_2$-prime submodule P_1 and by Proposition 2.2(ii), P_1 is a fully invariant submodule of M. Hence M/P_1 is a prime R-module by Proposition 2.1(ii). It follows that the ideal $A := \text{ann}_R(M/P_1)$ is a prime ideal of R. Let I be a minimal prime ideal of R contained in A. Thus $\kappa(M) \geq \kappa(M/MI) \geq \kappa(M/P_1) \geq \kappa(M/P) = \kappa(M)$. The proof is now completed by Proposition 2.2(ii). □

We now give a number of applications of Theorem 3.1. A submodule N of a module M_R is called *irreducible* (resp. *prime*) if (M/N) is a uniform (resp. prime) R-module. Also M_R is called *finitely annihilated* if there exist elements $m_i \in M$ $(1 \leq i \leq n)$ such that $A := \text{ann}_R(M) = \cap_{i=1}^n \text{ann}_R(m_i)$, equivalently R/A embeds in $M_R^{(n)}$.

Proposition 3.2. *Let M be an $\mathcal{L}_2$-Noetherian R-module with Krull dimension.*
(i) If every irreducible prime submodule of M_R is maximal, then M_R has finite length. The converse is true if M_R is finitely annihilated.

(ii) If M_R is quasi-projective then M_R has finite length if and only if Soc(M/P) is non-zero for every minimal $\mathcal{L}_2$-prime submodule P of M.

Proof. (i) Let P be a minimal $\mathcal{L}_2$-prime submodule of M. Because M has Krull dimension, $V := M/P$ has finite uniform dimension [12, Lemma 6.2.6]. Also by Proposition 2.1, V is a prime module. Thus, by [14, Corollary 2.4], $0 = N_1 \cap \cdots \cap N_n$ for some positive integer n and irreducible prime submodules $N_i (1 \leq i \leq n)$ of V. Clearly V embeds in the module $(V/N_1) \oplus \cdots \ (V/N_n)$. It follows that k.dim$(V) =$ k.dim(V/N_j) for some N_j. Thus k.dim$(V) = 0$ by our assumption. Hence M_R is Artinian by Theorem 3.1. On the other hand, M has also Noetherian dimension by [10, Corollaire 6]. Thus a similar argument shows that M_R is Noetherian. Hence M_R has finite length.

Conversely, suppose that M_R is finitely annihilated with finite length and N is an irreducible prime submodule of M_R. Let $I = \text{ann}_R(M/N)$ and $A = \text{ann}_R(M)$. Then I is a prime ideal of R and R/A is an Artinian ring. Since $A \subseteq I$, R/I is a semisimple Artinian ring. It follows that M/N is a simple R-module. Thus N is maximal.

(ii) One direction is clear. Conversely, let P be any minimal $\mathcal{L}_2$-prime submodule of M. By hypothesis, Soc(M/P) is non-zero and finitely generated. Hence, M/P is a semisimple Artinian R-module with finite length by Theorem 2.4. Thus result is now obtained by Theorem 3.1. □

A well known result of Lambek and Michler Theorem [9, Theorem 3.6], can be obtained as a corollary of Proposition 3.2.

Corollary 3.3. *A right Noetherian ring R is right Artinian if and only if every irreducible prime right ideal of R is maximal.*

Proof. Apply Proposition 3.2(i) for $M = R$. □

A ring R is said to be *right semi-Artinian* if every non-zero factor ring of R has a non-zero right socle.

Corollary 3.4. *Over a right semi-Artinian ring, a quasi-projective $\mathcal{L}_2$-Noetherian module has Krull dimension if and only if it has finite length.*

Proof. Apply Proposition 3.2(ii). □

We now state further conclusions when the base ring is either commutative or hereditary.

Corollary 3.5. *Let R be a commutative ring and M_R finitely generate fully invariant submodules. Then M_R has finite length if and only if M_R is finitely generated with Krull dimension and every irreducible prime submodule of M is maximal.*

Proof. If M_R is finitely generated and fully invariant submodules are finitely generated by M, we can conclude that M is $\mathcal{L}_2$-Noetherian. Also, M_R is finitely annihilated because R is commutative. Thus, the result is proved by Proposition 3.2(i). □

Corollary 3.6. *Let M_R be projective and R be a commutative ring. Then M_R has finite length if M_R has Krull dimension, every fully invariant submodule is finitely generated and every irreducible prime submodule of M is maximal.*

Proof. Since M_R is finitely generated, it is a generator in Mod-$R/\mathrm{ann}_R(M)$ [16, 18.11] and so every fully invariant submodule is (finitely) generated by M. The result is now obtained by Corollary 3.5. □

The following result may be compared with Theorem 2.7.

Proposition 3.7. *Let M_R be a quasi-projective $\mathcal{L}_2$-Noetherian R-module with Krull dimension. If R is a hereditary Noetherian ring, then M_R has finite length if and only if Soc(M) is an essential submodule of M.*

Proof. Let $\mathrm{Soc}(M)$ be an essential submodule of M and let P be a minimal $\mathcal{L}_2$-prime submodule of M. In view of Proposition 3.2(ii), we shall show that $\mathrm{Soc}(M/P)$ is non-zero. If $\mathrm{Soc}(M) \nsubseteq P$ then clearly $\mathrm{Soc}(M/P) \neq 0$ and if $\mathrm{Soc}(M) \subseteq P$ then $(M/P)_R$ being singular has a non-zero socle by [12, Proposition 5.4.5, page 150]. □

Proposition 3.8. *Let R be a right hereditary ring.*
(i) An $\mathcal{L}_2$-Noetherian R-module M is injective if M/P is injective for any minimal $\mathcal{L}_2$-prime submodule P of M.
(ii) If every two sided ideal of R is finitely generated as a right ideal and $(R/P)_R$ is injective for every minimal prime ideal P, then R is a semisimple Artinian ring.

Proof. (i) Because R is right hereditary, every R-module has injective dimension 1 or 0. It follows that κ :Mod-$R \to \{0,1\}$ is a dimension map where $\kappa(M)$ means the injective dimension of M_R. Thus, our assumption with Theorem 3.1 imply that

M_R is injective.
(ii) By hypothesis and part (i), R is a right hereditary right self injective ring. Thus R is a semisimple ring by [4, Corollary 7.15]. □

Let M be an R-module. If M is $M^{(\Lambda)}$-projective for every index set Λ, then we say that M is $\sum$-*projective.* The following Lemma is needed.

Lemma 3.9. *Let M_R be* $\sum$-projective and P be maximal among all proper fully invariant submodule of M. Then P lies in a maximal submodule of M

Proof. Let $L = M/P$ and L has no maximal submodule (i.e. $\mathrm{J}(L) = L$). Let $x \in M \setminus P$, $W = xR + P$ and $K = W/P$. By hypothesis, $\mathrm{Tr}(K, L) = L$, hence there exist a set Λ and surjective homomorphism $\varphi : K^{(\Lambda)} \to L$. Because M is $M^{(\Lambda)}$-projective and $K^{(\Lambda)} \simeq W^{(\Lambda)}/P^{(\Lambda)}$, M is also $K^{(\Lambda)}$-projective [16, 18.2]. Hence, there exists $h : M \to K^{(\Lambda)}$ such that $\varphi h = p$ where $p : M \to L$ is the canonical projection. Again, by hypothesis, there exists $\bar{h} : M \to W^{(\Lambda)}$ such that $\pi\bar{h} = h$ where $\pi : W^{(\Lambda)} \to K^{(\Lambda)}$ is the natural projection. Because P is fully invariant, $\forall\lambda \in \Lambda$, $\pi_\lambda\bar{h}(P) \subseteq P$ where $\pi_\lambda : W^{(\Lambda)} \to W$ are canonical projections. Thus $\bar{h}(P) \subseteq P^{(\Lambda)} = \ker(\pi)$. It follows that $h(P) = 0$. Now $h(xR) \subseteq K^{(A)}$ for some finite subset $A \subseteq \Lambda$. Define $\theta : L \to L$ by $\theta(m + P) = \varphi[\pi_A(h(m))]$ where $\pi_A : K^{(\Lambda)} \to K^{(A)}$ is the canonical projection. Thus θ is an element of $\mathrm{End}_R(L)$ such that $\theta(L)$ is contained in the finitely generated (proper) submodule $\varphi(K^{(A)}) \subseteq L$. It follows that the image of θ is a small submodule of L and hence θ belongs to the Jacobson radical of $\mathrm{End}_R(L)$ [16, 22.2]. Consequently (1 - θ) is a one to one map. On the other hand, $\forall\ w \in W$, $h(w) \in K^{(A)}$ and so $\theta(w + P) = \varphi(h(w)) = p(w) = w + P$. Hence $(1 - \theta)(K) = 0$, a contradiction. □

A well known result due to Ginn and Moss states that a Noetherian ring whose right socle is essential as a right ideal is (right) Artinian [5, Theorem 8.16]. Using this result, we offer a module theoretic version of it in Corollary 3.11. An R-module M is said to be *retractable* if $\mathrm{Hom}_R(M, N) \neq 0$ for all $0 \neq N \leq M_R$.

Theorem 3.10. *Let M_R be a $\sum$-projective retractable module such that it finitely generates fully invariant submodules. Then the following statements are equivalent.*
(i) M_R has Krull dimension, $End_R(M)$ is a Noetherian ring and $Soc(M)$ is an essential submodule of M.
(ii) $End_R(M)$ is a Artinian ring and M_R has finite length.

Proof. We only prove, (i)⇒(ii). Let $S = \mathrm{End}_R(M)$. By [7, Propositions 2.1(b), 2.4], $\mathrm{Soc}(S_S)$ is an essential right ideal of S and so by [5, Theorem 8.16], S is an

Artinian ring. On the other hand, by [16, 18.4(3)], $I = \mathrm{Hom}_R(M, IM)$ for every (right) ideal of S and $N = \mathrm{Hom}_R(M, N)M$ for every $N \in \mathcal{L}_2(M)$ by our assumption. This yields a one-to-one order-preserving correspondence between $\mathcal{L}_2(M)$ and $\mathcal{L}_2(S)$. Consequently, M_R is $\mathcal{L}_2$-Noetherian and $\mathcal{L}_2$-prime submodules of M correspond to prime ideals of S. Because S is Artinian, we can conclude that M/P has no non-trivial fully invariant submodules where P is an $\mathcal{L}_2$-prime submodule. Hence, the Jacobson radical of $(M/P)_R$ is zero by Lemma 3.9. It follows that the quasi-projective R-module M/P is retractable [4, 3.4]. Therefore, $\mathrm{Soc}(M/P) \neq 0$ by [7, Proposition 2.4(b)]. The proof is now completed by Proposition 3.2(ii). □

Corollary 3.11. *Let M_R be quasi-projective retractable with $S = End_R(M)$. If $Soc(M)$ is essential in M_R and the bimodule ${}_SM_R$ is Noetherian on each side, then M_R is Artinian.*

Proof. By hypothesis, $\mathrm{Hom}_R(M, IM) = I$ for every right ideal of $\mathrm{End}_R(M)$. It follows that $\mathrm{End}_R(M)$ is a right Noetherian ring. Also, since M_R is finitely generated and M is a faithful left S-module, S embedded in ${}_SM$ and so it is a left Noetherian ring. The result is now obtained by Theorem 3.10. □

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] T. Albu and P. F. Smith, *Dual Krull dimension and duality*, Rocky Mountain J. Math., 29(4) (1999), 1153–1165.

[2] L. Bican, P. Jambor, T. Kepka and P. Němec, *Prime and coprime modules*, Fund. Math., 57 (1980), 33–45.

[3] J. Chen, N. Ding and M. F. Yousif, *On Noetherian rings with essential socle*, J. Aust. Math. Soc., 76(1)(2004), 39–49.

[4] N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Extending Modules, Longman, Harlow, 1994.

[5] K. R. Goodearl and R. B. Warfield, Jr., An Introduction to Noncommutative Noetherian Rings, Second Edition, London Mathematical Society Student Texts, Vol. 16, Cambridge University Press, Cambridge, 2004.

[6] A. Haghany and M. R. Vedadi, *Modules whose injective endomorphisms are essential*, J. Algebra, 243 (2001), 765–779.

[7] A. Haghany and M. R. Vedadi, *Study of semi-projective retractable modules*, Algebra Colloq., 14 (2007), 489–496.

[8] O. A. S. Karamzadeh and N. Shirali, *On the countability of Noetherian dimension of modules*, Comm. Algebra, 32(10) (2004), 4073–4083.

[9] J. Lambek and G. Michler, *The torsion theory at a prime ideal of a right Noetherian ring*, J. Algebra, 25 (1973) 364–389.

[10] B. Lemonnier, *Deviation des ensembless et groupes Abeliens Totalement Ordonnes*, Bull. Sci. Math., 96 (1972) 289–303.

[11] C. Lomp, *Prime elements in partially ordered groupoids applied to modules and Hopf algebra actions*, J. Algebra Appl., 4(1) (2005), 77–97.

[12] J. C. McConnell and J. C. Robson, Noncommutative Noetherian Rings, Wiley-Interscience, New York, 1987.

[13] A. Ç. Özcan, A. Harmanci and P. F. Smith, *Duo modules*, Glasg. Math. J., 48(3) (2006), 533–545.

[14] P. F. Smith, *Radical submodules and uniform dimension of modules*, Turkish J. Math., 28(3) (2004), 255–270.

[15] P. F. Smith and A. R. Woodward, *Krull dimension of bimodules*, J. Algebra, 310(1) (2007), 405–412.

[16] R. Wisbauer, Foundations of Module and Ring Theory, Gordon and Breach, Reading, Philadelphia, 1991.

M. R. Vedadi
Department of Mathematical Sciences
Isfahan University of Technology
Isfahan, 84156-83111, Iran
e-mail: mrvedadi@cc.iut.ac.ir