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A NOTE ON THE CANCELLATION PROPERTIES OF SEMISTAR OPERATIONS

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Abstract. If D is an integral domain with quotient field K, then let $\bar{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K, $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D and $\mathrm{f}(D)$ be the set of non-zero finitely generated D-submodules of K. A semistar operation $\star$ on D is called arithmetisch brauchbar (or a.b.) if, for every $H \in \mathrm{f}(D)$ and every $H_1, H_2 \in \bar{\mathrm{F}}(D)$, $(HH_1)^\star = (HH_2)^\star$ implies $H_1^\star = H_2^\star$, and $\star$ is called endlich arithmetisch brauchbar (or e.a.b.) if the same holds for every $F, F_1, F_2 \in \mathrm{f}(D)$. In this note, we introduce the notion of strongly arithmetisch brauchbar (or s.a.b.) and consider relationships among semistar operations suggested by other related cancellation properties.

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Let D be an integral domain with quotient field K. Let $\bar{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K and let $\mathrm{F}(D)$ be the set of non-zero fractional ideals of D (i.e., $E \in \mathrm{F}(D)$ if $E \in \bar{\mathrm{F}}(D)$ and there is a non-zero element $d \in D$ with $dE \subset D$). We also let $\mathrm{f}(D)$ be the set of non-zero finitely generated D-submodules of K. A star operation on D is a mapping

$$\star : \quad \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D)$$
$$G \longmapsto G^\star$$

such that, for every $x \in K - \{0\}$ and every $G, G_1, G_2 \in \bar{\mathrm{F}}(D)$, the following properties hold:

(1) $(x)^\star = (x)$,
(2) $(xG)^\star = xG^\star$,
(3) $G_1 \subset G_2$ implies $G_1^\star \subset G_2^\star$,
(4) $G \subset G^\star$, and
(5) $(G^\star)^\star = G^\star$.

A star operation $\star$ on D is called arithmetisch brauchbar (or a.b.) if, for every $F \in \mathrm{f}(D)$ and every $G_1, G_2 \in \mathrm{F}(D)$, $(FG_1)^\star = (FG_2)^\star$ implies $G_1^\star = G_2^\star$, and $\star$ is called endlich arithmetisch brauchbar (or e.a.b.) if the same holds for every

$F, F_1, F_2 \in \mathrm{f}(D)$. A semistar operation on D is a mapping

$$\star : \quad \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D)$$
$$H \longmapsto H^\star,$$

such that, for every $x \in K - \{0\}$ and every $H_1, H_2 \in \bar{\mathrm{F}}(D)$, properties (2) through (5) above hold. Similar to the situation above, a semistar operation $\star$ on D is called a.b. if, for every $H \in \mathrm{f}(D)$ and every $H_1, H_2 \in \bar{\mathrm{F}}(D)$, $(HH_1)^\star = (HH_2)^\star$ implies $H_1^\star = H_2^\star$, and $\star$ is called e.a.b. if the same holds for every $F, F_1, F_2 \in \mathrm{f}(D)$. The mapping

$$e : \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D)$$
$$H \longmapsto H^e = K$$

is a semistar operation called the e-semistar operation on D. A good general reference on star and semistar operations is the monograph [3].

Definition 1. A semistar operation $\star$ on D is called cancellative if, for every $E, F, G \in \bar{\mathrm{F}}(D)$, $(EF)^\star = (EG)^\star$ implies $F^\star = G^\star$ (see [5]).

Proposition 2. *Let $\star$ be a semistar operation on D. Then $\star$ is cancellative if and only if $\star = e$.*

Proof. Clearly ($\Leftarrow$) holds. For ($\Rightarrow$), let $H \in \bar{\mathrm{F}}(D)$. Since $KH = K$, we have $(KH)^\star = (KK)^\star$, and hence $H^\star = K^\star$. Since $K^\star = K$, we have $H^\star = K$, and hence $\star = e$. □

Definition 3. Let $\star$ be a semistar operation on D, and let T be an overring of D. Then the mapping

$$\alpha_{T/D}(\star) : \quad \bar{\mathrm{F}}(T) \longrightarrow \bar{\mathrm{F}}(T)$$
$$H \longmapsto H^\star$$

(or, simply, $\alpha(\star)$) is a semistar operation on T, and is called the *ascent* of $\star$ to T. Let $\star'$ be a semistar operation on T. Then the mapping

$$\delta_{T/D}(\star') : \quad \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D)$$
$$h \longmapsto (hT)^{\star'}$$

(or, simply, $\delta(\star')$)is a semistar operation on D, and is called the *descent* of $\star'$ to D.

The following three Theorems were proved by G.Picozza in [5] and are a starting point for our work.

Theorem 4. *Let D be an integral domain, let T be an overring of D, let $\star$ be a semistar operation on D, and let $\alpha(\star)$ be the ascent of $\star$ to T.*

(1) *If $\star$ is cancellative, then $\alpha(\star)$ is cancellative.*
(2) *If $\star$ is a.b., then $\alpha(\star)$ is a.b.*
(3) *Assume that $T = D^\star$ or $T \in f(D)$. If $\star$ is e.a.b., then $\alpha(\star)$ is e.a.b.*

Theorem 5. *Let D be an integral domain, let T be an overring of D, let $\star$ be a semistar operation on T, and let $\delta(\star)$ be the descent of $\star$ to D.*

(1) *If $\star$ is cancellative, then $\delta(\star)$ is cancellative.*
(2) *If $\star$ is a.b., then $\delta(\star)$ is a.b.*
(3) *If $\star$ is e.a.b., then $\delta(\star)$ is e.a.b.*

Theorem 6. *Let D be an integral domain, and let $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings of D.*

(1) *There is a canonical bijection between the set of cancellative semistar operations on D and the set $\cup_\lambda \{\star \mid \star$ is a cancellative semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*
(2) *There is a canonical bijection between the set of a.b. semistar operations on D and the set $\cup_\lambda \{\star \mid \star$ is an a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*
(3) *There is a canonical bijection between the set of e.a.b. semistar operations on D and the set $\cup_\lambda \{\star \mid \star$ is an e.a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*

Let D be an integral domain, and let $\star$ be a semistar operation on D. Set $(\mathrm{f}(D))^\star = \{E^\star \mid E \in \mathrm{f}(D)\}$. If T is an overring of D and if $T = D^\star$ or $T \in \mathrm{f}(D)$, then we have $T^\star \in (\mathrm{f}(D))^\star$. In the next Proposition, we generalize Theorem 4(3) and consider more closely the sets explored in Theorem 6.

Proposition 7. *Let D be an integral domain, T be an overring of D, $\star$ be a semistar operation on D, and $\alpha(\star)$ be the ascent of $\star$ to T.*

(1) *Assume that $T^\star \in (f(D))^\star$. If $\star$ is e.a.b., then $\alpha(\star)$ is e.a.b.*
(2) *The set $\{\star \mid \star$ is a cancellative semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$ is an empty set unless $T_\lambda = K$.*

Proof. (1) Let $(FF_1)^{\alpha(\star)} = (FF_2)^{\alpha(\star)}$, where $F, F_1, F_2 \in \mathrm{f}(T)$. There are elements $f, f_1, f_2, f_0 \in \mathrm{f}(D)$ such that $F = fT, F_1 = f_1T, F_2 = f_2T$, and $T^\star = f_0^\star$. It follows that $(ff_1f_0)^\star = (ff_2f_0)^\star$, and hence $f_1^\star = f_2^\star$. Hence we have $F_1^{\alpha(\star)} = F_2^{\alpha(\star)}$.

(2) Let $\star$ be a cancellative semistar operation on T_λ with $T_\lambda^\star = T_\lambda$. By Proposition 2, $\star$ is the e-semistar operation on T_λ. It follows that $T_\lambda = K$. $\square$

The notion of a cancellative semistar operation suggests a stronger property. Hence, we make the following definition.

Definition 8. We say that a semistar operation $\star$ on D is s.a.b. (or, strongly arithmetisch brauchbar) if, for every $G \in \mathrm{F}(D)$, and $H_1, H_2 \in \bar{\mathrm{F}}(D), (GH_1)^\star = (GH_2)^\star$ implies $H_1^\star = H_2^\star$.

Clearly, the e-semistar operation is an s.a.b. semistar operation, and an s.a.b. semistar operation is an a.b. semistar operation.

Remark 9. An s.a.b. semistar operation need not be the e-semistar operation. To see this, let D be a principal ideal domain which is not a field, and let $\star$ be a semistar operation on D with $\star \neq e$. Then $\star$ is s.a.b.

The identity mapping d on $\bar{\mathrm{F}}(D)$ is a semistar operation called the d-semistar operation on D.

Remark 10. An a.b. semistar operation need not be an s.a.b. semistar operation. To see this, let $D = V$ be a valuation domain which is not a field, let M be the maximal ideal with $M = M^2$, and let $\star = d$. Then $\star$ is a.b., and $\star$ is not s.a.b., in fact, $(MM)^\star = (MD)^\star$ and $M^\star \neq D^\star$.

Proposition 11. (1) *Let D be an integral domain, T be an overring of D with $T \in F(D)$, $\star$ be a semistar operation on D, and $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is s.a.b., then $\alpha(\star)$ is s.a.b.*

(2) *Let D be an integral domain, T be an overring of D, $\star$ be a semistar operation on T, and $\delta(\star)$ be the descent of $\star$ to D. If $\star$ is s.a.b., then $\delta(\star)$ is s.a.b.*

(3) *Let D be an integral domain, and $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \in F(D)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is an s.a.b. semistar operations $\star$ on D with $D^\star \in F(D)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is an s.a.b. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*

Proof. (1) Let $(GH_1)^{\alpha(\star)} = (GH_2)^{\alpha(\star)}$, where $G \in \mathrm{F}(T)$ and $H_1, H_2 \in \bar{\mathrm{F}}(T)$. Then we have $G \in \mathrm{F}(D), H_1, H_2 \in \bar{\mathrm{F}}(D)$, and $(GH_1)^\star = (GH_2)^\star$. It follows that $H_1^\star = H_2^\star$, and hence $H_1^{\alpha(\star)} = H_2^{\alpha(\star)}$.

(2) Let $(gh_1)^{\delta(\star)} = (gh_2)^{\delta(\star)}$, where $g \in \mathrm{F}(D)$ and $h_1, h_2 \in \bar{\mathrm{F}}(D)$. Then we have $gT \in \mathrm{F}(T), h_1T, h_2T \in \bar{\mathrm{F}}(T)$, and $(gTh_1T)^\star = (gTh_2T)^\star$. Since $\star$ is s.a.b., we have $(h_1T)^\star = (h_2T)^\star$, and hence $h_1^{\delta(\star)} = h_2^{\delta(\star)}$. Hence $\delta(\star)$ is s.a.b.

(3) Let $\star \in A$, and $\alpha(\star)$ be the ascent of $\star$ to $D^\star$. Then $\alpha(\star) \in B$ by (1). For every $h \in \bar{\mathrm{F}}(D)$, we have $h^\star = (hD^\star)^{\alpha(\star)}$. Assume that $\alpha(\star_1) = \alpha(\star_2)$, where $\star_1, \star_2 \in A$. Since $\alpha(\star_1)$ (resp., $\alpha(\star_2)$) is a semistar operation on $D^{\star_1}$ (resp., $D^{\star_2}$), we have $D^{\star_1} = D^{\star_2}$. Then we have $h^{\star_1} = (hD^{\star_1})^{\alpha(\star_1)}$ and $h^{\star_2} = (hD^{\star_2})^{\alpha(\star_2)}$.

Hence we have $\star_1 = \star_2$. Assume that $\star \in B$, and let $\delta(\star)$ be the descent of $\star$ to D. Then we have $\delta(\star) \in A$ by (2), and $\alpha(\delta(\star)) = \star$. □

To expand our investigation, we introduce five additional cancellation properties of semistar operations.

Definition 12. Let D be a domain, and let $\star$ be a semistar operation on D. Set $f(D) = X_1$, $F(D) = X_2$, and set $\bar{F}(D) = X_3$. If, for every $A \in X_i$ and $B, C \in X_j$, $(AB)^\star = (AC)^\star$ implies $B^\star = C^\star$, then $\star$ is called $f_i.f_j$. Since an element of $f(D)$ is finitely generated, we set also $f_1 = f$ and, considering the alphabetical order, we set also $f_2 = g$ and $f_3 = h$. Thus, we define as follows

(1) $\star$ is called h.g. if, for every $H \in \bar{\mathrm{F}}(D)$ and $G_1, G_2 \in \mathrm{F}(D)$, $(HG_1)^\star = (HG_2)^\star$ implies $G_1^\star = G_2^\star$.
(2) $\star$ is called g.g. if, for every $G, G_1, G_2 \in \mathrm{F}(D)$, $(GG_1)^\star = (GG_2)^\star$ implies $G_1^\star = G_2^\star$.
(3) $\star$ is called f.g. if, for every $F \in \mathrm{f}(D)$ and $G_1, G_2 \in \mathrm{F}(D)$, $(FG_1)^\star = (FG_2)^\star$ implies $G_1^\star = G_2^\star$.
(4) $\star$ is called h.f. if, for every $H \in \bar{\mathrm{F}}(D)$ and $F_1, F_2 \in \mathrm{f}(D), (HF_1)^\star = (HF_2)^\star$ implies $F_1^\star = F_2^\star$.
(5) $\star$ is called g.f. if, for every $G \in \mathrm{F}(D)$ and $F_1, F_2 \in \mathrm{f}(D), (GF_1)^\star = (GF_2)^\star$ implies $F_1^\star = F_2^\star$.

If $\star$ is cancellative (resp., s.a.b., a.b., e.a.b.), then we call it h.h. (resp., g.h., f.h., f.f.).

The d-semistar operation on a quasi-local domain D is f.g. if and only if D is a Bezout domain (cf., [2, p.67]).

Proposition 13. *Let $\star$ be a semistar operation on a domain D. The following conditions are equivalent.*

(1) *$\star$ is h.g.*
(2) *$\star$ is h.f.*
(3) *$\star = e$.*

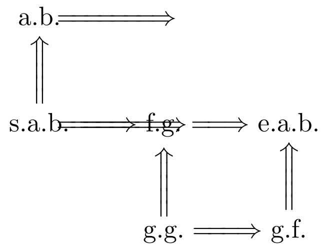
Proof. Assume that $\star$ is h.f. and let $F_1, F_2 \in \mathrm{f}(D)$. Since $(KF_1)^\star = (KF_2)^\star$, we have $F_1^\star = F_2^\star$. Hence there is $H \in \bar{\mathrm{F}}(D)$ such that $H = F^\star$ for every $F \in \mathrm{f}(D)$. Let $a \in K - \{0\}$. Since $a \in (Da)^\star = H$, we have $H = K$. Hence $\star = e$. □

Remark 14. The following implications can be routinely verified and the arguments are left to the reader.

(1) s.a.b. implies g.g.

(2) g.g. implies g.f.
(3) a.b. implies f.g.
(4) f.g. implies e.a.b.
(5) g.g. implies f.g.
(6) g.f. implies e.a.b.

We offer the following diagram which illustrates the relationships described above.



The proofs of the following three Propositions are similar to that of Proposition 11 and are also left to the reader.

Proposition 15. (1) *Let D be an integral domain, T be an overring of D with $T \in F(D)$, $\star$ be a semistar operation on D, and $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is g.g., then $\alpha(\star)$ is g.g.*

(2) *Let D be an integral domain, T be an overring of D, $\star$ be a semistar operation on T, and $\delta(\star)$ be the descent of $\star$ to D. If $\star$ is g.g., then $\delta(\star)$ is g.g.*

(3) *Let D be an integral domain, and $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \in F(D)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is a g.g. semistar operation on D with $D^\star \in F(D)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a g.g. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*

Proposition 16. (1) *Let D be an integral domain, T be an overring of D with $T \in F(D)$, $\star$ be a semistar operation on D, and $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is f.g., then $\alpha(\star)$ is f.g.*

(2) *Let D be an integral domain, T be an overring of D, $\star$ be a semistar operation on T, and $\delta(\star)$ be the descent of $\star$ to D. If $\star$ is f.g., then $\delta(\star)$ is f.g.*

(3) *Let D be an integral domain, and $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \in F(D)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is a f.g. semistar operation on D with $D^\star \in F(D)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a f.g. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*

Proposition 17. (1) *Let D be an integral domain, T be an overring of D with $T \in F(D)$, $\star$ be a semistar operation on D, and $\alpha(\star)$ be the ascent of $\star$ to T. If $\star$ is g.f., then $\alpha(\star)$ is g.f.*

(2) *Let D be an integral domain, T be an overring of D, $\star$ be a semistar operation on T, and $\delta(\star)$ be the descent of $\star$ to D. If $\star$ is g.f., then $\delta(\star)$ is g.f.*

(3) *Let D be an integral domain, and $\mathcal{T} = \{T_\lambda \mid \lambda \in \Lambda\}$ be the set of overrings T of D with $T \in F(D)$. Then there is a canonical bijection between the set $A = \{\star \mid \star$ is a g.f. semistar operation on D with $D^\star \in F(D)\}$ and the set $B = \cup_\lambda \{\star \mid \star$ is a g.f. semistar operation on T_λ with $T_\lambda^\star = T_\lambda\}$.*

Let D be an integral domain, and let $\star$ be a semistar operation on D. Set $(\mathrm{F}(D))^\star = \{G^\star \mid G \in \mathrm{F}(D)\}$.

Remark 18. Proposition 11 (1) (resp., Proposition 15 (1)) may be generalized as follows. Let D be an integral domain, T be an overring of D and $\star$ be a semistar operation on D with $(\mathrm{F}(T))^\star \subset (\mathrm{F}(D))^\star$. If $\star$ is s.a.b. (resp., g.g.), then $\alpha_{T/D}(\star)$ is s.a.b. (resp., g.g.).

We proceed to consider more relationships between the cancellation properties of semistar operations introduced in Definition 12. We first require a lemma, the details of which are left to the reader.

Lemma 19. (*cf.*, [1, Lemma 2.7 (iii)]) *Let $\star$ be a semistar operation on D. Then $\star$ is g.h. if and only if, for every $G \in F(D)$ and $H \in \bar{\mathrm{F}}(D)$, $G \subset (GH)^\star$ implies $1 \in H^\star$. A Similar characterization holds for every g.g., g.f., f.h., f.g., and f.f. semistar operation $\star$. For instance, $\star$ is f.g. if and only if, for every $F \in f(D)$ and $G \in \mathrm{F}(D)$, $F \subset (FG)^\star$ implies $1 \in G^\star$.*

Proposition 20. (1) *a.b. need not imply g.f.*

(2) *g.f. need not imply g.g.*

Proof. (1) Let $D = V$ be a 2-dimensional valuation domain, $M \supsetneqq P \supsetneqq (0)$ be the prime ideals of V, and $\star = d$. Let $x \in M - P$, and set $F_1 = (x)$ and $F_2 = D$. Then we have $(PF_1)^\star = (PF_2)^\star$ and $F_1^\star \neq F_2^\star$. It follows that $\star$ is a.b., and that $\star$ is not g.f.

(2) Let $D = V$ be an $\boldsymbol{R}$-valued valuation domain, v be the valuation belonging to V with value group $\boldsymbol{R}$, and $\star = d$. We have $(MM)^\star = (MD)^\star$ and $M^\star \neq D^\star$, and hence $\star$ is not g.g.

Let $(GF_1)^\star = (GF_2)^\star$, where $G \in \mathrm{F}(D)$ and $F_1, F_2 \in \mathrm{f}(D)$. Let $F_1 = Va$ and $F_2 = Vb$ with $a, b \in K$, and set $\inf v(G) = v(x)$ with $x \in K$. Then $\inf v(GF_1) =$

$v(x) + v(a)$ and $\inf v(GF_2) = v(x) + v(b)$. It follows that $v(a) = v(b)$, hence $Va = Vb$, and hence $F_1 = F_2$. Therefore $\star$ is g.f. □

Remark 21. (cf., [4, Proposition 6, (1)]) Let D be a 1-dimensional Prüfer domain with exactly two maximal ideals M and N. Assume that M is principal, and that N is not principal. Let $\star$ be a semistar operation $H \longmapsto HD_N$ on D. Then $\star \neq d$, $\star$ is f.h., $\star$ is g.f., and $\star$ is not g.g.

Note that if every finitely generated ideal of D is principal ([4, Lemma 4]), then $(NN)^\star = (ND)^\star$, and $N^\star \neq D^\star$.

Proposition 22. (1) *(cf.,* [2, (32.8) Corollary]*) Let $\star$ be a semistar operation on D. If $D^\star$ is not integrally closed, then $\star$ is not e.a.b.*

(2) *(cf.,* [2, (32.5) Theorem]*) Let D be an integrally closed domain. Then there is an f.h. semistar operation $\star$ on D such that $D^\star = D$.*

Proof. (1) Suppose that $\star$ is e.a.b. Let $\alpha(\star)$ be the ascent of $\star$ to $D^\star$. Then $\alpha(\star)$ is an e.a.b. semistar operation on $D^\star$. Then the restriction of $\alpha(\star)$ to $\mathrm{F}(D^\star)$ is an e.a.b. star operation on $D^\star$, and hence $D^\star$ is integrally closed; a contradiction.

(2) Let $\{V_\lambda \mid \lambda \in \Lambda\}$ be the set of valuation overrings of D. Let $\star$ be the semistar operation $H \longmapsto \cap_\lambda HV_\lambda$. For every λ_0, we have $HV_{\lambda_0} = H^\star V_{\lambda_0}$. Moreover, $H^\star V_\lambda = (\cap_\lambda HV_\lambda)V_{\lambda_0} \subset (HV_{\lambda_0})V_{\lambda_0} = HV_{\lambda_0}$.

Assume that $(FH_1)^\star = (FH_2)^\star$ for $F \in \mathrm{f}(D)$ and $H_1, H_2 \in \bar{\mathrm{F}}(D)$. Then, for every λ, $FH_1V_\lambda = (FH_1)^\star V_\lambda = (FH_2)^\star V_\lambda = FH_2V_\lambda$. Since FV_λ is a principal ideal of V_λ, we have $H_1V_\lambda = H_2V_\lambda$. Therefore, $H_1^\star = \cap_\lambda H_1V_\lambda = \cap_\lambda H_2V_\lambda = H_2^\star$. □

Finally, we will call a star operation $\star$ on D $g_0 \cdot g_0$. if, for every $G, G_1, G_2 \in \mathrm{F}(D)$, $(GG_1)^\star = (GG_2)^\star$ implies $G_1^\star = G_2^\star$. We call a star operation $\star$ on D $g_0 \cdot f_0$. if, for every $G \in \mathrm{F}(D)$ and $F_1, F_2 \in \mathrm{f}(D)$, $(GF_1)^\star = (GF_2)^\star$ implies $F_1^\star = F_2^\star$.

If $\star$ is an a.b. star operation (resp., e.a.b. star operation), then we call $\star$ an $f_0 \cdot g_0$ star operation (resp., $f_0 \cdot f_0$. star operation).

Proposition 23. *Let D be a quasi-local domain which is not a field. The following statements are equivalent.*

(1) *the d-semistar operation is $g \cdot g$.*
(2) *D is a discrete valuation ring with rank 1.*

Proof. Every $G \in \mathrm{F}(D)$ is a cancellation ideal. Hence G is principal by a well known result of A. Kaplansky. □

Proposition 24. *Let $\star$ be a star operation on D.*

(1) (a) *$g_0 \cdot g_0$ implies a.b.*

(b) *$g_0 \cdot g_0$ implies $g_0 \cdot f_0$.*
(c) *$g_0 \cdot f_0$ implies e.a.b.*
(2) (a) *a.b. need not imply $g_0 \cdot f_0$.*
(b) *$g_0 \cdot f_0$ need not imply $g_0 \cdot g_0$.*

Proof. (2) (a) In the example for Proposition 20 (1), let $\star_0$ be the restriction of $\star$ to $\mathrm{F}(D)$. Then $\star_0$ is an a.b. star operation, which is not $g_0 \cdot f_0$.

(b) In the example for Proposition 20 (2), let $\star_0$ be the restriction of $\star$ to $\mathrm{F}(D)$. Then $\star_0$ is $g_0 \cdot f_0$., which is not $g_0 \cdot g_0$. □

Proposition 25. *The following statetments are equivalent.*

(1) *Every e.a.b. semistar operation is an f.g. semistar operation.*
(2) *Every e.a.b. star operation is an a.b. star operation.*

Proof. (1) $\Rightarrow$ (2): There is an e.a.b. star operation $\star$ which is not a.b. Let $\star'$ be the canonical extension of $\star$ to a semistar operation on D. Then $\star'$ is an e.a.b. semistar operation on D which is not f.g.

(2) $\Rightarrow$ (1): There is an e.a.b. semistar operation $\star$ on a domain D which is not f.g. For every $G \in \mathrm{F}(D^\star)$, set $G^{\star'} = G^\star$. Then $\star'$ is an e.a.b. star operation on $D^\star$ which is not a.b. □

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