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INC-EXTENSIONS AMID ZERO-DIVISORS

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Abstract. All rings considered are commutative with identity. Let R be a complemented ring with integral closure R' (in its total quotient ring K). Then $R \subseteq S$ satisfies INC for each overring S of R (inside K) if and only if R' is a Prüfer ring. If R' is a Prüfer ring and T is a complemented ring that contains R as a subring such that each regular element of T is a root of a polynomial in $R[X]$ with a regular coefficient and T is torsion-free over R, then $R \subseteq T$ satisfies INC. As a consequence, a new generalization for rings with nontrivial zero-divisors is found of Prüfer's result on the integral closure of a Prüfer domain in a field extension of the quotient field.

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1. Introduction

All rings considered below are commutative with $1 \neq 0$; all inclusions of rings, ring extensions, subrings, algebras and ring/algebra homomorphisms are unital. If A is a ring, then $\text{Spec}(A)$ denotes the set of prime ideals of A; $\text{Max}(A)$ the set of maximal ideals of A; $\text{Reg}(A)$ the set of regular (that is, non-zero-divisor) elements of A; $\text{tq}(A) := A_{\text{Reg(A)}}$ the total quotient ring of A; and A' the integral closure of A (in $\text{tq}(A)$). By an *overring* of a ring A, we mean an A-subalgebra of $\text{tq}(A)$, that is, a ring B such that $A \subseteq B \subseteq \text{tq}(A)$. As in [8, page 28], INC, GU and LO denote the incomparable, going-up and lying-over properties, respectively, of ring extensions.

Our starting point is the classical fact that a (commutative integral) domain R which is integrally closed is a Prüfer domain if and only if $R \subseteq T$ satisfies INC for each overring T of R [5, Theorem 26.2]. More generally, Papick [9] defined a domain R to be an INC-*domain* if $R \subseteq T$ satisfies INC for each overring T of R;

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and showed [9, Proposition 2.26] that a domain R is an INC-domain if and only if R' is a Prüfer domain. We are thus led to say that a ring R is an INC-*ring* if $R \subseteq T$ satisfies INC for each overring T of R; and we ask the natural question whether a ring R is an INC-ring if and only if R' is a Prüfer ring (in the sense of [6]). One of our main results, Theorem 2.4, answers this question in the affirmative in case R is a complemented ring. Recall that a ring R is said to be *complemented* if, for each element $a \in R$, there exists $b \in R$ such that $ab = 0$ and $a + b \in \mathrm{Reg}(R)$; when this holds, b is called a *complement to* a (*in* R). It is known (cf. [1, Theorem 2.3]) that a ring R is a complemented ring if and only if $\mathrm{tq}(R)$ is a von Neumann regular ring. As it is clear that each domain is a complemented ring, Theorem 2.4 forms part of a program of generalizing results about domains to the context of complemented rings. As explained below, this entire note can be seen as part of that program.

Perhaps the most striking stability result about the class of Prüfer domains is Prüfer's result [11] that if T is the integral closure of a Prüfer domain R in a field extension L of the quotient field K of R, then T is a Prüfer domain. In [6, Proposition 14], Griffin generalized this result to the case of a Prüfer ring R (with K replaced by $\mathrm{tq}(R)$), at the cost of assuming that $K \subseteq L$ is an integral ring extension. In Corollary 2.6, we provide another generalization of Prüfer's result, by assuming that the Prüfer ring R is also a complemented ring and that L is a von Neumann regular ring (but without an explicit hypothesis of integrality on the extension $K \subseteq L$).

The proof of Corollary 2.6 depends on Theorem 2.4 (which was discussed above) and our other main result, Theorem 2.2. The latter result generalizes the statement [3, Proposition 3] that if R is an INC-domain and T is a domain that contains R as a subring and is algebraic over R, then $R \subseteq T$ satisfies INC. Notice that one purpose of this result from [3] was to widen the context of [9] by considering domain extensions that are possibly not overrings of a given base domain. Continuing in that spirit, Theorem 2.2 assumes that both rings involved in the given ring extension are complemented, that the algebraicity polynomials in question each have a regular coefficient, and that the given ring extension is (module-theoretically) torsion-free (the latter condition being, of course, automatic for domains). Theorem 2.2 not only generalizes [3, Proposition 3] but also gives a ring-theoretic companion for a result [3, Proposition 6] about ring extensions of a domain, the latter being accomplished at the reasonable cost of replacing a hypothesis about the going-down property with a hypothesis of torsion-freeness.

In addition to the notation mentioned above, we let X denote an indeterminate over the ambient coefficient ring(s); and $\subset$ denotes proper inclusion. Any unexplained material is as in [5], [8].

2. Results

We begin with a lemma that will be used several times in this work.

Lemma 2.1. *Let R be a complemented ring and a Prüfer ring. If $P \in \mathrm{Spec(R)}$, then R_P is a valuation domain.*

Proof. $K :=\mathrm{tq}(R)$ is a von Neumann regular ring; that is, R is "quasi-regular" in the sense of [4]. Hence, by [4, Proposition 2], $\mathrm{tq}(R_P)$ can be identified with $K_{R\setminus P}$, which is a von Neumann regular ring. Furthermore, since R is integrally closed, it follows from [4, Propositions 5 and 6] that R_P is an integrally closed domain. Hence, it suffices to prove that R_P is a Prüfer domain/ring. By [12, Theorem 4] (cf. also [6, Theorem 13]), it suffices to show that each overring T of R_P is R_P-flat. By the above description of $\mathrm{tq}(R_P)$, T can be identified with $S_{R\setminus P}$ for some overring S of R. As R is a Prüfer ring, S is R-flat by [6, Theorem 13], whence $T = S_{R\setminus P}$ is R_P-flat, as desired. □

We next present our first main result. Theorem 2.2 extends the context of [3] from ring extensions of domains to ring extensions of complemented rings. Recall that an R-module E is said to be *torsion-free* (*over* R) if $0 \neq r \in \mathrm{Reg}(R)$, $e \in E, re = 0$ implies $e = 0$.

Theorem 2.2. *Let R be a complemented ring such that R' is a Prüfer ring. Let T be a complemented ring that contains R as a subring such that T is torsion-free (as a module) over R. Assume also that each regular element of T is a root of a polynomial in $R[X]$ with a regular coefficient. Then $R \subseteq T$ satisfies INC.*

Proof. Let K and L be the total quotient rings of R and T, respectively. Since T is torsion-free over R, we can view K as a subring of L (up to R-algebra isomorphism). Consider the ring composite $S := R'T$ inside L. Since $T \subseteq S$ inherits integrality from $R \subseteq R'$, it follows that $T \subseteq S$ satisfies LO and GU (cf. [8, Theorem 44]). As integrality also ensures that $R \subseteq R'$ satisfies INC, it clearly suffices to prove that $R' \subseteq S$ satisfies INC.

We claim that each regular element s of S is a root of some polynomial in $R'[X]$ with a regular coefficient. We can write s as a finite sum $\sum(r_i/z)t_i$, where $r_i \in R$, $z \in \mathrm{Reg}(R)$ and $t_i \in T$ for each i. Thus, $s = t/z$, with $t := \sum r_i t_i \in T$. Since

$s \in \mathrm{Reg}(S)$ and S is an overring of T, we have $t \in \mathrm{Reg}(T)$, and so the hypothesis gives a polynomial $f \in R[X]$ such that $f(t) = 0$ and some coefficient of f is regular. Dividing through by $z^{\deg(f)}$ leads to a polynomial $g \in R_z[X]$ such that $g(s) = g(t/z) = 0$ and some coefficient of g is a unit of K. (As usual, R_z denotes the ring of fractions obtained by localizing R at the multiplicatively closed subset generated by z.) Clearing denominators, we see that by multiplying g by a suitable integral power of z, we obtain $h \in R[X]$ such that $h(s) = 0$ and some coefficient of h is regular. This proves the above claim. Thus, by *abus de langage*, we can replace (R,T) with (R',S), and so without loss of generality, $R = R'$ is a Prüfer ring.

Suppose that the assertion fails. Then there exist distinct prime ideals $Q_1 \subset Q_2$ of T such that $Q_1 \cap R = Q_2 \cap R =: P \in \mathrm{Spec}(R)$. Since Q_2 is not a minimal prime ideal of T and L has (Krull) dimension 0, Q_2 contains some regular element u of T. Moreover, T is a Marot ring by [7, Theorem 7.4]; that is, each regular ideal of T is generated by the set of its regular elements. It follows easily that we can assume $u \notin Q_1$. As $u \in (Q_2 \cap R[u]) \setminus (Q_1 \cap R[u])$, we have that $R \subseteq R[u]$ does not satisfy INC. Hence, $R_P \subseteq (R[u])_{R\setminus P} = (R_P)[u/1]$ does not satisfy INC. Therefore, by [2, Theorem] (cf. also [13, Corollary 3.3]), $u/1$ is not the root of any polynomial in $R_P[X]$ with a unit coefficient.

By hypothesis, u is a root of some polynomial $f \in R[X]$ with a regular coefficient. Then f induces a polynomial $g \in R_P[X]$ with a regular coefficient such that $u/1$ is a root of g. Note that R_P is a valuation domain by Lemma 2.1. Therefore $g = c\cdot h$, for some nonzero $c \in R_P$ and polynomial $h \in R_P[X]$ such that at least one coefficient of h is 1. Observe that $c \cdot h(u/1) = 0 \in T_{R\setminus P}$. However, in view of the hypothesis that T is torsion-free over R, it follows from the proof of [4, Proposition 7] that $T_{R\setminus P}$ is a torsion-free R_P-module. Thus $h(u/1) = 0$, the desired contradiction. □

The above theorem has the following immediate consequence. As explained in the Introduction, Corollary 2.3 generalizes [3, Proposition 3] and can be viewed as a companion for [3, Proposition 6].

Corollary 2.3. *Let R be a domain such that R' is a Prüfer domain, and let T be a complemented ring that contains R as a subring such that T is torsion-free over R. Assume also that each regular element of T is algebraic over R (for instance, assume that T is algebraic over R.) Then $R \subseteq T$ satisfies INC.*

As mentioned in the Introduction, the next result generalizes Papick's INC-theoretic characterization of domains with Prüferian integral closure [9, Proposition 2.26] to the context of complemented rings.

Theorem 2.4. *Let R be a complemented ring. Then R is an INC-ring if and only if R' is a Prüfer ring.*

Proof. Assume first that R' is a Prüfer ring. Let T be an overring of R. Our task is to show that $R \subseteq T$ satisfies INC; equivalently, that $R_P \subseteq T_{R\setminus P}$ satisfies INC for each $P \in \mathrm{Spec}(R)$. By considering the composite $R'T$ as in the first paragraph of the proof of Theorem 2.2, we can assume, without loss of generality, that $R = R'$ is a Prüfer ring. Now, by Lemma 2.1, R_P is a valuation domain. As in the proof of Lemma 2.1, [4, Proposition 2] shows that $T_{R\setminus P}$ can be identified with an overring S of R_P. Since R_P is a valuation domain, the classical case of the motivating result [5, Theorem 26.2] ensures that $R_P \subseteq S$ satisfies INC.

Conversely, suppose that R is an INC-ring. We must show that R' is a Prüfer ring. Let $K := \mathrm{tq}(R)$. We first reduce to the case $R = R'$. Notice that R' is a complemented ring, since $\mathrm{tq}(R') = K$ is von Neumann regular. Also, if T is an overring of R', then $R' \subseteq T$ satisfies INC since $R \subseteq T$ satisfies INC. As $(R')' = R'$, we can replace R with R'. In other words, without loss of generality, R is integrally closed. We proceed to show that R is a Prüfer ring.

By [6, Theorem 13], it is enough to show that each overring T of R is R-flat; equivalently, that $T_{R\setminus P}$ is R_P-flat for each $p \in \mathrm{Spec}(R)$. By Lemma 2.1 (and its proof), R_P is a valuation domain and $T_{R\setminus P}$ can be viewed as an overring of R_P. Accordingly, the assertion follows because torsion-free modules over Prüfer domains are flat. □

We pause to give a useful sufficient condition for a ring to be complemented.

Lemma 2.5. *Let T be a subring of a von Neumann regular ring L such that T is integrally closed in L. Then T is a complemented ring.*

Proof. Since each finitely generated ideal of a von Neumann regular ring is generated by an idempotent, it is easy to see that each element of L has a complement that is idempotent. However, each idempotent element of L belongs to T because T is integrally closed in L. Therefore, the assertion follows from the definition of "complemented ring" and the simple observation that $\mathrm{Reg}(L) \cap T \subseteq \mathrm{Reg}(T)$. □

We next generalize Prüfer's classic ascent result. As explained in the Introduction, Corollary 2.6 can be viewed as a companion for [6, Proposition 14].

Corollary 2.6. *Let R be a complemented ring and a Prüfer ring, with $K := tq(R)$. Let L be a von Neumann regular ring which contains K as a subring. Let T be the integral closure of R in L. Then T is a Prüfer ring.*

Proof. By Lemma 2.5, T is a complemented ring. Also, we claim that $\text{Reg}(T) \subseteq \text{Reg}(L)$. To see this, consider any $c \in \text{Reg}(T)$. As noted in the proof of Lemma 2.5, we can choose an idempotent element $e \in T$ that is a complement to c in L. Since $c \in \text{Reg}(T)$ and $ce = 0$, we conclude that $e = 0$. Thus $c = c + e \in \text{Reg}(L)$, proving the claim. Since K is von Neumann regular, L is K-flat and, hence, torsion-free over K. (For an alternate way to get the last conclusion, apply [4, Proposition 7].) Therefore, L is torsion-free over R and, *a fortiori*, T is torsion-free over R. As a result, K can be viewed, up to R-algebra isomorphism, as a subring of $\text{tq}(T)$, and so we can replace L with $T_{\text{Reg(R)}} = TK$ (which is contained in the former L). At this point, one could obtain the assertion by applying [6, Proposition 14] (which applies since $L = TK$ is integral over K and T is the integral closure of R in TK). However, one can avoid using [6] by first showing that $\text{tq}(T)= TK$ and then essentially repeating the proof of [3, Corollary 4]. For the first of these, let us begin by observing that TK is reduced (since it is a subring of the reduced ring L) and zero-dimensional (since TK is integral over the zero-dimensional ring K). Thus, TK is a von Neumann regular ring; in particular, $\text{tq}(TK)= TK$. On the other hand, we have seen that $K \subseteq \text{tq}(T)$, and so $TK \subseteq \text{tq}(T)$. As TK is therefore an overring of T, we have that $\text{tq}(T) = \text{tq}(TK) = TK$, as asserted. As for the second point, the proof of [3, Corollary 4] now carries over with the following two changes. Replace the appeal to the criterion of Papick [9, Proposition 2.26] with Theorem 2.4; and replace [3, Proposition 3] with Theorem 2.2. Note that the last step is permissible since the integrality of TK over $K = \text{tq}(R)$ ensures that each element of $\text{Reg}(TK)$ is a root of some polynomial in $R[X]$ with a regular coefficient. □

Remark 2.7. (a) Recall from [9] that a domain R is said to be an i-domain if the canonical map $\text{Spec}(T) \to \text{Spec}(R)$ is an injection for each overring T of R. It is a classical fact that an integrally closed i-domain is the same as a Prüfer domain [5, Theorem 26.2]. More generally, Papick showed [9, Propositions 2.14 and 2.26] that a domain R is an i-domain if and only if R is an INC-domain such that the canonical map $\text{Spec}(R') \to \text{Spec}(R)$ is an injection. While any i-domain is an INC-domain, the converse is false [9, Example 2.17]. This material leads us to define a ring R to be an i-ring if the canonical map $\text{Spec}(T) \to \text{Spec}(R)$ is an injection for each overring T of R. It is clear that any i-ring is an INC-ring (and the converse is false). More significantly, our work implicitly shows that if a complemented ring R is a Prüfer ring, then R is an i-ring. Indeed, one need only verify that if $P \in \text{Spec}(R)$ and T is an overring of R, then the canonical map $\text{Spec}(T_{R\setminus P}) \to \text{Spec}(R_P)$ is an injection. As we have seen in the proof of Lemma 2.1, R_P is a valuation domain and (up to isomorphism) $T_{R\setminus P}$ is an overring of R_P, and so the assertion therefore follows from the above-mentioned classical fact.

We would like to thank the referee for raising the following question. Given a complemented ring R, must it be the case that R is an i-ring if and only if the canonical map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an injection and R' is a Prüfer ring? The answer is in the affirmative and thus generalizes the above-mentioned application of [9, Propositions 2.14 and 2.26] from domains to complemented rings. To see this, note that the "only if" assertion follows from the definition of an i-ring and Theorem 2.4; and, since we showed in the preceding paragraph that any complemented Prüfer ring must be an i-ring, the "if" assertion follows as in the proof of [9, Proposition 2.14].

(b) Much of the above reasoning depended on the fact that complemented rings are the rings having von Neumann regular total quotient rings. This class of rings has been studied for several reasons and under several different names. For instance, as noted above, Endo [4] called them "quasi-regular" rings; and Picavet and Picavet-L'Hermitte [10] called them "decent" rings. We note that some of the above reasoning can be replaced by alternate proofs, owing to the variety of known characterizations of the rings having von Neumann regular total quotient rings. In addition, alternate proofs of some of our assertions involving Prüfer rings are also available. For instance, in view of [4, Proposition 7], the use of flat overrings in the proofs of Theorem 2.4 and Corollary 2.6 can be replaced by the characterization of Prüfer rings in terms of integrally closed overrings [6, Theorem 13]. Finally, the reader is encouraged to find alternate proofs in which the above roles of Lemma 2.1 are played by the characterization [6, Theorem 13] of Prüfer rings as the rings A whose large quotient rings $A_{[M]}$ are (Manis) valuation rings for all $M \in \mathrm{Max}(A)$.

References

[1] D. F. Anderson and A. Badawi, *Divisibility conditions in commutative rings with zerodivisors*, Comm. Algebra, 30 (2002), 4031–4047.

[2] D. E. Dobbs, *On INC-extensions and polynomials with unit content*, Canad. Math. Bull., 23 (1980), 37–42.

[3] D. E. Dobbs, *Prüfer's ascent result via INC*, Comm. Algebra, 23 (1995), 109–119.

[4] S. Endo, *On semi-hereditary rings*, J. Math. Soc. Japan, 13 (1961), 5413–5417.

[5] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.

[6] M. Griffin, *Prüfer rings with zero divisors*, J. Reine Angew. Math., 239/240 (1969), 55–67.

[7] J. A. Huckaba, Commutative Rings with Zero Divisors, Marcel Dekker, New York, 1988.

[8] I. Kaplansky, Commutative Rings, rev. ed., Univ. Chicago Press, Chicago, 1974.

[9] I. J. Papick, *Topologically defined classes of going-down domains*, Trans. Amer. Math. Soc., 219 (1976), 1–37.

[10] G. Picavet and M. Picavet-L'Hermitte, *Seminormal or t-closed schemes and Rees rings*, Algebr. Represent. Theory, 1 (1998), 255–309.

[11] H. Prüfer, *Untersuchungen über Teilbarkeitseigenschaften in Körpern*, J. Reine Angew. Math., 168 (1932), 1–36.

[12] F. Richman, *Generalized quotient rings*, Proc. Amer. Math. Soc.,16 (1965), 794–799.

[13] H. Uda, *Incomparability in ring extensions*, Hiroshima Math. J., 9 (1979), 451–463.

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