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PLANARITY OF INTERSECTION GRAPHS OF IDEALS OF RINGS

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Abstract. In this paper we characterize planar intersection graphs of ideals of a commutative ring with 1.

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1. Introduction

For graph theory terminology in general we follow [6]. Specifically, let $G = (V, E)$ be a graph with vertex set V of order n and edge set E. We denote the degree of a vertex v in G by $d_G(v)$, which is the number of edges incident to v. A graph G is *complete* if there is an edge between every pair of the vertices. A subset X of the vertices of a graph G is called *independent* if there is no edge with two endpoints in X. A graph G is called *bipartite* if its vertex set can be partitioned into two subsets X and Y such that every edge of G has one endpoint in X and other endpoint in Y. A *complete bipartite graph* is a bipartite graph in which any vertex of a partite set is adjacent to all vertices in another partite set. A graph G is said to be *star* if G contains one vertex in which all other vertices are joined to this vertex and G has no other edges. The *complement* $\overline{G}$ of G is the graph with vertex set $V(\overline{G}) = V(G)$, and $E(\overline{G}) = \{uv : uv \notin E(G)\}$. The complement of a complete graph is the *null graph.*

Let $F = \{S_i : i \in I\}$ be an arbitrary family of sets. The *intersection graph* $G(F)$ is the one-dimensional skeleton of the nerve of F , i.e., $G(F)$ is the graph whose vertices are S_i, $i \in I$ and in which the vertices S_i and S_j $(i, j \in I)$ are adjacent if and only if $S_i \neq S_j$ and $S_i \cap S_j \neq \emptyset$. It is shown that every simple graph is an intersection graph, ([5]).

It is interesting to study the intersection graphs $G(F)$ when the members of F have an algebraic structure. Bosak [2] in 1964 studied graphs of semigroups. Then Cskny and Pollk [4] in 1969 studied the intersection graphs of subgroups of a finite group. Zelinka [7] in 1975 continued the work on intersection graphs of nontrivial subgroups of finite abelian groups.

Chakrabarty et al. [3] studied *intersection graphs of ideals of rings*. The intersection graph of ideals of a ring R, denoted $\Gamma(R)$, is the undirected simple graph (without loops and multiple edges) whose vertices are in a one-to-one correspondence with all nontrivial left ideals of R and two distinct vertices are joined by an edge if and only if the corresponding left ideals of R have a nontrivial (nonzero) intersection. Clearly the set of vertices is empty for left simple rings. In this case we refer $\Gamma(R)$ as the empty graph.

Chakrabarty et al. [3] studied planarity of intersection graphs of the ring $\mathbb{Z}_n$. In this paper we will characterize all commutative rings with 1 which $\Gamma(R)$ is planar. Throughout this paper for an ideal I in a ring R, the vertex in $\Gamma(R)$ corresponded to I is also denoted by I. All rings we handle are commutative with 1.

We denote by K_n the complete graph on n vertices, and by $K_{m,n}$ the complete bipartite graph which one partite set is of cardinality m and another partite set is of cardinality n.

2. Results

We will repeatedly use Kuratowski's theorem, which states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ (see [6, Theorem 6.2.2]).

Let R be a commutative ring with 1. We begin with the following lemma.

Lemma 2.1. *If $\Gamma(R)$ is planar, then any chain of ideals of R has length at most five.*

Proof. Let $I_1 \subset I_2 \subset ... \subset I_5$ be a chain of nontrivial proper ideals of R. Then $I_1, I_2, ..., I_5$ induce a K_5 as an induced subgraph in $\Gamma(R)$. This completes the proof. □

Corollary 2.2. *If $\Gamma(R)$ is planar, then R is both Noetherian and Artinian.*

Lemma 2.3. *If $\Gamma(R)$ is null and R contains at least two proper nontrivial distinct ideals, then $R \cong R_1 \times R_2$, where R_1, R_2 are fields.*

Proposition 2.4. *$\Gamma(R_1 \times R_2)$ is planar if and only if one of $\Gamma(R_1), \Gamma(R_2)$ is empty, and another is empty or null with at most two vertices.*

Proof. ($\Longrightarrow$) Let I_1, I_2 be two nontrivial ideals of R_2 with $I_1 \subseteq I_2$. Then $0 \times I_1, 0 \times I_2, 0 \times R_2, R_1 \times I_1, R_1 \times I_2$ form a K_5, a contradiction. So $\Gamma(R_1), \Gamma(R_2)$ are null or empty. We show that $\Gamma(R_1)$ or $\Gamma(R_2)$ is empty. Suppose that both $\Gamma(R_1)$ and $\Gamma(R_2)$

are null. Let $I \lhd R_1, J \lhd R_2$, (nontrivial). Then $0 \times R_2, 0 \times J, I \times R_2, I \times J, R_1 \times J$ form a K_5, a contradiction. Assume that $\Gamma(R_1)$ is empty. Suppose that $\Gamma(R_2)$ is null. By Lemma 2.3, $\Gamma(R_2)$ has at most two vertices.

($\Longleftarrow$) Is straightforward. $\square$

Corollary 2.5. *$\Gamma(R_1 \times R_2 \times R_3)$ is planar if and only if R_i is a field for $i = 1, 2, 3$.*

Proof. Notice that if R_3 is not a field and $I \leq R_3$, then $R_2 \times 0, R_2 \times I$ is an edge in $\Gamma(R_2 \times R_3)$, and by Proposition 2.4, $\Gamma(R_1 \times R_2 \times R_3)$ is not planar. $\square$

Let $Max(R)$ be the set of all maximal ideals of R.

Lemma 2.6. *If $\Gamma(R)$ is planar, then $|Max(R)| \leq 3$.*

Proof. Let $\Gamma(R)$ is planar. Suppose that $|Max(R)| \geq 4$. Let M_1, M_2, M_3 be three distinct maximal ideals of R. Let $I = M_1 \cap M_2 \cap M_3$. Since $|Max(R)| \geq 4$, we have $I \neq 0$. Then $M_1, M_2, M_3, M_1 \cap M_2, M_1 \cap M_3$, and $M_2 \cap M_3$ form a K_6, as an induced subgraph, a contradiction. $\square$

We divide the rest of the paper into two subsection according to $|Max(R)|$.

2.1. $|Max(R)| \neq 1$**.** Let $J(R)$ be the Jacobson radical of R. We first consider the case $|Max(R)| = 3$.

Corollary 2.7. *If $|Max(R)| = 3$ and $\Gamma(R)$ is planar, then $J(R) = 0$.*

Corollary 2.8. *If $|Max(R)| = 3$ and $\Gamma(R)$ is planar, then $R = R_1 \times R_2$.*

Proof. Let $Max(R) = \{M_1, M_2, M_3\}$. By Corollary 2.7, $M_1 \cap (M_2 \cap M_3) = 0$. On the other hand $M_1 + (M_2 \cap M_3) = R$. So the result follows. $\square$

Theorem 2.9. *If $|Max(R)| = 3$, then $\Gamma(R)$ is planar if and only if $R = R_1 \times R_2 \times R_3$, where R_i is a field for $i = 1, 2, 3$.*

Proof. Follows from Corollary 2.8, Lemma 2.3 and Proposition 2.4. $\square$

We next assume that $|Max(R)| = 2$.

Lemma 2.10. (Nakayama, [1]) *Let M be a finitely generated R-module. If $J(R)M = M$, then $M = 0$.*

Lemma 2.11. *If $Max(R) = \{M_1, M_2\}$ and $\Gamma(R)$ is planar, then $R \cong M_1^3 \times M_2^3$.*

Proof. We first show that $M_1^3 \cap M_2^3 = 0$. Suppose that $M_1^3 \cap M_2^3 \neq 0$. By Corollary 2.2, M_1, M_2 are finitely generated R-modules. By Nakayama's lemma $M_1, M_2, (M_1 \cap M_2), (M_1 \cap M_2)^2$, and $(M_1 \cap M_2)^3$ are all mutually distinct. Then $M_1, M_2, (M_1 \cap M_2), (M_1 \cap M_2)^2, (M_1 \cap M_2)^3$ form a K_5 as an induced subgraph, a contradiction. So $M_1^3 \cap M_2^3 = 0$. On the other hand $M_1^3 + M_2^3 = R$. This completes the result. □

Theorem 2.12. *If* $|Max(R)| = 2$, *then* $\Gamma(R)$ *is planar if and only if one of* $\Gamma(R_1), \Gamma(R_2)$ *is empty, and another is empty or null with one vertex.*

Proof. Notice that by Lemma 2.11, $R \cong R_1 \times R_2$. Now the result follows by Proposition 2.4. □

2.2. $|Max(R)| = 1$. In this subsection R is a local ring. Let M be the unique maximal ideal of R. The following lemmas are easily verified.

Lemma 2.13. *If* $\Gamma(R)$ *is planar, then* $M^5 = 0$.

Lemma 2.14. *Let* $I \trianglelefteq R$. *Then* $\frac{I}{IM}$ *is a vector space over* $\frac{R}{M}$. *Further, any subspace of* $\frac{I}{IM}$ *is in the form* $\frac{J}{IM}$, *where* $J \trianglelefteq R$ *and* $IM \subseteq J \subseteq I$.

Lemma 2.15. *Let* $I \trianglelefteq R$. *If* $\dim(\frac{I}{IM}) \geq 3$, *then* $\Gamma(R)$ *is not planar.*

Proof. Let u_1, u_2, u_3 be three linear independent vectors in $\frac{I}{IM}$. Let $W = \langle u_1, u_2, u_3 \rangle$. Since $\dim(\frac{W}{\langle u_1 \rangle}) = 2$, $\frac{W}{\langle u_1 \rangle}$ contains exactly $|\frac{R}{M}| + 1$ subspaces of dimension 1. This implies that W contains at least 3 subspaces W_1, W_2, W_3 of dimension 2 containing u_1. On the other hand $W_4 = \langle u_2, u_3 \rangle$ is another subspace of W of dimension 2. We obtain that W_1, W_2, W_3, W_4 are for subspaces of dimension 2 such that $W_i \cap W_j \neq 0$ for $i, j \in \{1, 2, 3, 4\}$. Suppose that $W_i = \frac{J_i}{IM}$ for $i = 1, 2, 3, 4$. Now J_1, J_2, J_3, J_4, M form a K_5. □

Corollary 2.16. *Let* $M^2 = 0$. *Then* $\Gamma(R)$ *is planar if and only if* $\dim(M) = 1$ *or* 2 *as a vector space over* $\frac{R}{M}$.

Proof. Follows by Lemma 2.15 with putting $I = M$. □

Corollary 2.17. *Let* $M^2 = 0$. *Then* $\Gamma(R)$ *is planar if and only if* $\Gamma(R)$ *is either an star or* K_1.

Lemma 2.18. *Let* $M^2 \neq 0$. *If* $\Gamma(R)$ *is planar, then* $\dim(\frac{M}{M^2}) = 1$ *and* $\frac{M}{M^2} \cong \frac{M^2}{M^3}$ *as an isomorphism of* R*-modules.*

Proof. By Lemma 2.15, $\dim(\frac{M}{M^2}) \leq 2$. Suppose that $\dim(\frac{M}{M^2}) = 2$. It follows that $\frac{M}{M^2}$ contains at least three subspaces W_1, W_2, W_3 of dimension 1. Let $W_i = \frac{J_i}{M^2}$ for $i = 1, 2, 3$. Then J_1, J_2, J_3, M, M^2 form a K_5, a contradiction. Thus $\dim(\frac{M}{M^2}) = 1$. As a consequent $M = \langle a \rangle$ for some $a \in R$. We define the map $\phi : \frac{M}{M^2} \longrightarrow \frac{M^2}{M^3}$ by $\phi(ra + M^2) = ra^2 + M^3$. Since $\frac{M}{M^2}$ is a simple R-module, it is straightforward to see that ϕ is an R-isomorphism. □

Corollary 2.19. *Let $M^2 \neq 0$ and $M^3 = 0$. Then $\Gamma(R)$ is planar if and only if $\Gamma(R) = K_2$.*

Proof. Let $\Gamma(R)$ be planar. By Lemma 2.18, $M = Ra$ where $a \in R$. Let I be a minimal ideal of R. We show that $I = M^2$. Since I is a simple R-module, we obtain $I \cong \frac{R}{M}$. Then $I = \langle x \rangle$, where $x \in R$. If $x \in M \setminus M^2$, then $x = ra$, where $r \in R \setminus M$. So r is invertiable and $\langle x \rangle = \langle a \rangle = M$, a contradiction. We deduce that $x \in M^2$, and so $I \subseteq M^2$. Since M^2 is simple, we obtain $I = M^2$. Thus M^2 is the unique minimal ideal of R, and $\Gamma(R) = K_2$. The converse is obvious. □

Lemma 2.20. *Let $M^3 \neq 0$ and $M^4 = 0$. If $\Gamma(R)$ is planar, then $\dim(\frac{M}{M^2}) = 1$ and $\frac{M}{M^2} \cong \frac{M^2}{M^3} \cong \frac{M^3}{M^4}$.*

Corollary 2.21. *Let $M^3 \neq 0$ and $M^4 = 0$. Then $\Gamma(R)$ is planar if and only if $\Gamma(R)$ is K_3 or K_4.*

Proof. By Lemma 2.20, $M = Ra$ where $a \in R$. Let I be a minimal ideal of R. We show that $I = M^3$. Since I is a simple R-module, we obtain $I \cong \frac{R}{M}$. Then $I = \langle x \rangle$, where $x \in R$. If $x \in M \setminus M^2$, then $x = ra$, where $r \in R \setminus M$. So r is invertiable and $\langle x \rangle = \langle a \rangle = M$, a contradiction. If $x \in M^2 \setminus M^3$, then $x = ra^2$, where $r \in R \setminus M$. As before we can see that $\langle x \rangle = \langle a^2 \rangle = M^2$, a contradiction. We deduce that $x \in M^3$, and so $I \subseteq M^3$. Since M^3 is simple, we obtain $I = M^3$. Thus M^3 is the unique minimal ideal of R, and $\Gamma(R)$ is complete. Now the result follows. □

Lemma 2.22. *Let $M^4 \neq 0$ and $M^5 = 0$. If $\Gamma(R)$ is planar, then $\dim(\frac{M}{M^2}) = 1$ and $\frac{M}{M^2} \cong \frac{M^2}{M^3} \cong \frac{M^3}{M^4} \cong \frac{M^4}{M^5}$.*

Corollary 2.23. *Let $M^4 \neq 0$ and $M^5 = 0$. Then $\Gamma(R)$ is planar if and only if $\Gamma(R) = K_4$.*

Proof. By Lemma 2.22, $M = Ra$ where $a \in R$. Let I be a minimal ideal of R. Similar to the proof of Corollary 2.21, we obtain $I = M^4$. Thus M^4 is the unique minimal ideal of R, and $\Gamma(R)$ is complete. Now the result follows. □

As a consequent of Corollaries 2.17, 2.19, 2.21 and 2.23 we obtain the following.

Theorem 2.24. *If* $|Max(R)| = 1$, *then* $\Gamma(R)$ *is planar if and only if* $\Gamma(R)$ *is an star,* K_1, K_3, *or* K_4.

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