International Electronic Journal of Algebra
Volume 8 (2009) 80-113

IDEALS AND OVERRINGS OF DIVIDED DOMAINS

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Received: 27 November 2009; Revised: 3 June 2010
Communicated by Abdullah Harmancı

Abstract. New properties of divided domains R are established by looking at multiplicatively closed subsets associated to ring morphisms. Let I be an ideal of R. We exhibit primary ideals, like $I\sqrt{I}$ and I^n if I is primary. We show that $\mathrm{Ass}(I) = \mathcal{V}(I) \cap \mathrm{Spec}(R_{\mathrm{Max}(\mathrm{Ass}(I))})$. Moreover, the image of the maximal spectrum of $(I : I)$ is contained in $\mathrm{Ass}(I)$. We show that certain intersections of ideals are primary ideals. Goldman prime ideals are prime g-ideals. The characterization of maximal flat epimorphic subextensions gives as a result that R is a valuation subring of Prüfer hulls. We characterize Fontana-Houston divided Ω-domains, divided APVDs and divided PPC-domains.

Mathematics Subject Classification (2000): Primary 13G05; Secondary 13A15, 13B24, 13F05

Keywords: affine open subset, almost pseudo-valuation domain, antesharp prime ideal, complete integral closure, conductor overring, divided domain (ring), (flat) epimorphism, fragmented domain, g-ideal ring, G-ideal ring, going-down domain, i-domain, Kaplansky transform, Kasch ring, Manis pair, maximal flat epimorphic (sub)extension, Ω-domain, open domain, PPC-domain, power-Ahmes domain, propen domain, primal ideal, primary ideal, treed domain, unbranched prime ideal, valuation pair.

1. Introduction and notation

This paper deals with commutative unital rings and their (homo)morphisms. Dobbs introduced the divided property [7]. Let R be an integral domain with quotient field K, with $R \neq K$ (*i.e.* R is not a field). A prime ideal P of R is called *divided* if $PR_P = P$ (equivalently, P is comparable under inclusion to any (principal) ideal of R). Then R is termed *divided* if each of its prime ideals is divided. A divided domain is a quasilocal *going-down* domain; that is, each of its overring extensions has the going-down property [7]. We also consider the *divided rings* of Badawi [1] that are rings in which each (principal) ideal is comparable to any prime ideal. A commutative ring R is called *treed* if two incomparable prime

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In memory of P. Samuel.

ideals of R are coprime. In case R is quasilocal, the treed property for R means that the spectrum $\mathrm{Spec}(R)$ of R is linearly ordered under inclusion. A going-down integral domain is a treed domain.

We give below a short survey of the main results of the paper. The reader is thereby invited to look at the definitions and notation involved in the different sections and, in particular, at the end of this section.

We are aiming to exhibit new results on divided integral domains. As a factor ring of a divided ring is a divided ring, we are lead to give results on divided rings and also on quasilocal treed rings, when possible. This provides us results on ideals and conductor overrings of divided domains. Key tools are multiplicatively closed subsets (mcs), arising from elements that become units through a ring morphism.

Section 2 introduces two kinds of mcss and gives useful technical results. If I is an ideal of a ring R, the elements of the mcs $\Lambda(I)$ are $x \in R$ that becomes a unit in $\mathrm{Tot}(R/I)$. Another one mcs is $\mathcal{U}(I) = \{x \in R \mid xI = I\}$, widely used in the valuation domain theory [29]. The sets $\mathrm{Ass}_f(I)$ of all *Bourbaki associated prime ideals* of I and $\mathrm{Ass}(I)$ of all *Krull associated prime ideals* of I are closely linked to these mcss. A first observation is that $\Lambda = \mathcal{U}$ on a treed quasilocal domain R if and only if R is a divided domain (see Section 4).

In Section 3, we consider the properties of the mcs $S_f = \{s \in A \mid f(s) \in \mathcal{U}(B)\}$ associated to a GD ring morphism $f : A \to B$, in case A is a quasilocal treed ring and in particular a divided ring. We get a key result; that is, $P_f := A \setminus S_f = \cup [A \cap N \mid N \in \mathrm{Max}(B)] \in \mathrm{Spec}(A)$, $\mathrm{Spec}(B \mid A) = (P_f)^{\downarrow}$ and there is a factorization $A \to A_{P_f} \to B$, where $A_{P_f} \to B$ has the LO and GD properties. If, in addition, R is a divided ring, then $P_f = \cap [As \mid s \in S_f]$. In case R is an integral domain and f is a flat epimorphism, then $B = A_{P_f}$ (see Theorem 3.3). As a consequence, when A is an integral domain, $A \to A_{P_f}$ is the Morita maximal flat epimorphic subextension of $A \to B$ [39]. These results are a powerful tool in the sequel. For instance, we show that if R is a divided ring, then the maximal flat epimorphic extension of R is $R \to \mathrm{Tot}(R)$. When R is a divided domain, we also give information on the Prüfer hull $P(R,T)$ associated to an extension of integral domain $R \subseteq T$ by Knebusch and Zhang [36]. For instance, R is a valuation subring of $P(R,T)$ and $P(R,T) = R_{\mathfrak{p}}$ for some $\mathfrak{p} \in \mathrm{Spec}(R)$. Applying this result to the Prüfer hull of R gives pleasant results for $P(R)$. In particular, $x \in P(R) \setminus R \Rightarrow x^{-1} \in R$.

This section also contains results on ideals of a divided ring. Clearly, each nonzero ideal of a divided ring R is primal and an ideal $I \neq 0$ of R is primary if and only if $I^{\lambda} := R \setminus \Lambda(I) = \sqrt{I}$. Moreover, if R is an integral domain, $IR_P = I$ for each prime ideal $P \supseteq I^\lambda$, whence $IR_{\Lambda(I)} = I$. We show that a treed quasilocal ring R is such that $\mathrm{Tot}(R)$ is a quasilocal Kasch ring. Among many consequences deduced for factor rings, we get that an ideal, whose annihilator is zero, contains a regular element, so that $\mathrm{Tot}(R) = \mathrm{Q}(R)$. One main result is that $\mathrm{Ass}(I) = \mathrm{Ass}_f(I) = \mathcal{V}(I) \cap (I^\lambda)^\downarrow$ for an ideal $I \neq 0$ of a divided ring R.

Section 4 is concerned with conductor overrings $(I : I)$ associated to a nonzero ideal I of an integral domain R. We recover results known for valuation domains. Actually, altough we get many results valid for quasilocal treed domains, we give here them for divided domains (R, M). Let π be the natural map $R \to (I : I)$ and $I^\sharp := R \setminus \mathcal{U}(I)$, then $I^\sharp = I^\lambda = P_\pi$ and $R \to R_{\Lambda(I)}$ is the maximal flat epimorphic subextension of π. If I is a fg-ideal, then $I^\lambda = M$ and if not, $(R : I) = (I : I)$. When $P \neq 0$ is a prime ideal, $I \not\subseteq P \Rightarrow (I : I) \subseteq (P : P)$. Hence the inclusion defines a linear order over the family of overrings $(P : P)$, where $P \in \mathrm{Spec}(R)$, $P \neq 0$. We pause here to claim that if $I_1, \ldots, I_n$ are ideals of a divided ring, containing $\sqrt{0}$ and such that I_1 is a P_1-primary ideal and $\sqrt{I_1} \subseteq \sqrt{I_k}$ for $k = 2, \ldots, n$, then $I_1 \cdots I_n$ is a P_1-primary ideal. Hence any power of a primary ideal containing $\sqrt{0}$ is primary as well as $I\sqrt{I}$ for any nonzero ideal $I \supseteq \sqrt{0}$. Theorem 4.7 is the main result of this section, proving that ${}^t\pi(\mathrm{Max}((I : I))) \subseteq \mathrm{Ass}(I)$. Hence, in case I is a primary ideal, ${}^t\pi(\mathrm{Max}((I : I))) = \{I^\lambda\}$. This generalizes Okabe's result [41, Theorem 2.2] and shows that an integral domain R is divided if and only if each nonzero nonmaximal prime ideal is antesharp (see [22]). We get also that if $(R : I)$ is a ring, then I_v is a primary ideal.

Section 5 deals only with divided domains R. The consideration of rings of sections $\Gamma(X)$ for $X \subset \mathrm{Spec}(R)$ exhibits prime ideals that are intersections of certain families of ideals. It is enough to use $S_X := S_f = \{s \in R \mid X \subseteq \mathcal{D}(s)\}$, where $f : R \to \Gamma(X)$ is the natural map, to get a prime ideal $P_X = \cap[Rs \mid s \in S_X] = \cap[I^n \mid n \in \mathbb{N}, X \subseteq \mathcal{D}(I), I \in \mathcal{I}_f(R)]$. For instance, if $X = \mathcal{D}(I)$ for $I \in \mathcal{I}_f(R)$, we get the Okabe's result: $[I^n \mid n \in \mathbb{N}]$ is a prime ideal P_I, such that $(P_I)^\downarrow = \mathcal{D}(I)$ [40, Corollary 2.7]. In particular, P_I is a prime g-ideal. Actually, we show that Goldman prime (G)-ideals are identical to prime g-ideals. As a consequence, each nonzero nonmaximal prime ideal is divisorial. Moreover, R is an open domain if and only if R is a G-ideal domain. Theorem 5.8 shows that Gilmer's characterization [30, Theorem 17.3] of a nonzero unbranched prime ideal P of a Prüfer domain R is still valid and is equivalent to R_P is fragmented. We give a characterization of the Fontana-Houston divided Ω-domains as QQR-domains in which each nonzero prime ideal is a G-ideal. In that case, R is a propen domain and R_P is a valuation Ω-domain for each nonmaximal prime ideal. This section ends with some calculations of the complete integral closure of a divided domain.

Section 6 deals with some applications. We begin with a descent result of the divided property. Then we focus on divided i-domains and provide conditions for an overring to be divided. We give criteria for the natural map $R \to (I : I)$ to be integral. We characterize APVDs inside the divided context. We show that divided Okabe's PPC-domains (R, M) [41] are characterized by R is an APVD such that $(M : M)$ is the minimal overring of R. We end by giving conditions on the sequence $\{\mathrm{Ass}(I^n)\}_{n>0}$ to be stationary for an ideal I of a divided domain. This question was the subject of many papers in the Noetherian context.

We now give some notation. Let R be an arbitrary commutative unital ring, then $\mathrm{Tot}(R)$ is its total quotient ring, $\mathrm{Q}(R)$ is its complete quotient ring, $\mathcal{U}(R)$ is the set of all its units, $\mathrm{Max}(R)$ is the set of all maximal ideals, $\heartsuit(R) = \mathcal{U}(R) \cup \{0\}$ and ($\mathcal{I}_f(R)$) $\mathcal{I}(R)$ is the set of all its (finitely generated) ideals $I \neq R$. An overring of a ring R is an R-subalgebra of $\mathrm{Tot}(R)$ and R' denotes the integral closure of R in $\mathrm{Tot}(R)$. If $f : R \to S$ is a ring morphism, $\mathrm{Spec}(S \mid R)$ is the image of the spectral map ${}^t f : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. If ${}^t f$ is injective, f is called an *i-morphism*. For $J \subseteq S$, we occasionally write $J \cap R$ instead of $f^{-1}(J)$.

We denote by $P^{\downarrow} := \mathrm{Spec}(R_P \mid R)$ the *generalization* of $P \in \mathrm{Spec}(R)$ and $Q \in P^{\downarrow} \Leftrightarrow Q \subseteq P$ for $Q \in \mathrm{Spec}(R)$. Now $X^{\downarrow} := \cup[P^{\downarrow} \mid P \in X]$ is the *generalization* of $X \subseteq \mathrm{Spec}(R)$ and X is said to be stable under generalization if $X = X^{\downarrow}$. We also set $X^{\uparrow} := \{Q \in \mathrm{Spec}(R) \mid P \subseteq Q$ for some $P \in X\}$ (the *specialization* of X).

For $I \in \mathcal{I}(R)$, we set $\mathcal{V}(I) := \{P \in \mathrm{Spec}(R) \mid I \subseteq P\}$ (a typical Zariski closed subset of $\mathrm{Spec}(R)$), $\mathcal{D}(I) := \mathrm{Spec}(R) \setminus \mathcal{V}(I)$. Now $\mathrm{Z}(I)$ is the set of $x \in R$ that are zero-divisors in R/I, $\mathrm{Min}(I)$ is the set of minimal prime ideals of I and we set $\mathrm{Min}(R) := \mathrm{Min}(0)$ and $\mathrm{Minp}(R) := \{P \in \mathrm{Spec}(R) \mid P \in \mathrm{Min}(Ra)$ for some $a \in R \setminus \{0\}\}$

When I and J are ideals of a ring R, we set $I : J := \{x \in R \mid xJ \subseteq I\}$. For an integral domain R with quotient field K, we set $(I : J) := \{x \in K \mid xJ \subseteq I\}$ if I and J are R-submodules of K.

A multiplicatively closed subset of a ring is termed a mcs and a smcs, when saturated. We abbreviate the words "finitely generated ideal" by fg-ideal, going-down by GD and lying-over by LO.

2. Zero divisors of factor rings and associated mcs

This section deals with technical results on zero divisors in arbitrary factor rings and some associated mcs. Here p_I is the natural map $R \to R/I \to \mathrm{Tot}(R/I)$ for $I \in \mathcal{I}(R)$.

2.1. The smcs Λ and associated prime ideals. Let $I \in \mathcal{I}(R)$, we set $\Lambda(I) := \{x \in R \mid I : x = I\} = \{x \in R \mid p_I(x) \in \mathcal{U}(\mathrm{Tot}(R/I))\}$ and $I^\lambda := R \setminus \Lambda(I) = \mathrm{Z}(I)$ (see our paper [43]). Then $\Lambda(I)$ is a smcs of R and I is called a *primal ideal* if I^λ is a (prime) ideal, and then I is called an I^λ-primal ideal. Note that $\sqrt{I} \subseteq I^\lambda$ and that $\sqrt{I} = I^\lambda \Leftrightarrow I$ is a primary ideal.

Then $\mathrm{Ass}_f(I)$ is the set of all *Bourbaki associated prime ideals* of I. A prime ideal $\mathfrak{P}$ belongs to $\mathrm{Ass}_f(I)$ if $\mathfrak{P} \in \mathrm{Min}(I : r)$ for some $r \in R$. It is well known that $I^\lambda = \cup[\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Ass}_f(I)], \sqrt{I} = \cap[\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Ass}_f(I)]$ and $\mathcal{V}(I) = \mathrm{Ass}_f(I)^\uparrow$ (see [4, Ex. 17, p.165 and Ex. 12, p.169]).

We also introduce the set $\mathrm{Ass}(I)$ of all *Krull associated prime ideals* of I as in [27, Section 2]. A prime ideal $\mathfrak{P}$ of R belongs to $\mathrm{Ass}(I)$ if for each $x \in \mathfrak{P}$ there is some $y \in R$ such that $x \in I : y \subseteq \mathfrak{P}$. In view of [27, Lemma 2.1], $\mathfrak{P} \in \mathrm{Spec}(R)$ is in $\mathrm{Ass}(I)$ if and only if $\mathfrak{P}$ is a union of some elements of $\mathrm{Ass}_f(I)$. It follows that $\mathrm{Ass}_f(I) \subseteq \mathrm{Ass}(I)$ and I is a primal ideal if and only if $I^\lambda \in \mathrm{Ass}(I)$.

Note that $I^\lambda = \cup[\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Ass}(I)], \sqrt{I} = \cap[\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Ass}(I)]$ and $\mathrm{Ass}(I) \subseteq \mathcal{V}(I)$.

Lemma 2.1. *Let R be a ring, $I \in \mathcal{I}(R)$ and $\mathfrak{P} \in \mathrm{Ass}(I)$, then the maximal ideal N of $(R/I)_{\mathfrak{P}}$ consists of zero divisors.*

Proof. Set $P := \mathfrak{P}/I$ and $S := R/I$. If $z = x/s$ is an element of N, where $x \in S$ and $s \notin P$, then x belongs to P. Hence there is some $y \in S$ such that $x \in 0 : y \subseteq P$. It follows easily that $x/1$ is a zero-divisor. □

Proposition 2.2. *The set $\mathcal{I}(R)$ is partially ordered by the relation $\mathcal{R}$ defined by $I\mathcal{R}J \Leftrightarrow$ there is some (s)mcs S of R such that $J = I_S \cap R$ for $I, J \in \mathcal{I}(R)$. Then $I\mathcal{R}J \Rightarrow I \subseteq J$ and $\Lambda(I) \subseteq \Lambda(J)$.*

Proof. The proof is straightforward. □

2.2. The smcs Λ and ring morphisms. If $f : R \to T$ is a ring morphism, the mapping $J \mapsto I := f^{-1}(J)$ defines an application ${}^tf : \mathcal{I}(T) \to \mathcal{I}(R)$, which verifies the following properties:

(λ1): $\mathrm{Ass}_f(I) \subseteq {}^tf(\mathrm{Ass}_f(J))$; whence $f^{-1}(\Lambda(J)) \subseteq \Lambda(I)$ and $I^\lambda \subseteq f^{-1}(J^\lambda)$.

(λ2): Set $K := J_{\Lambda(I)} \cap T$ (so that $J\mathcal{R}K$), then $f^{-1}(\Lambda(K)) = \Lambda(I)$, $f^{-1}(K) = I$ and $f^{-1}(K^\lambda) = I^\lambda$.

(λ3): If ${}^t f$ is injective, then $f^{-1}(\Lambda(J)) = \Lambda(f^{-1}(J))$ for $J \in \mathcal{I}(T)$.

Note that ${}^t f$ is injective if and only if $J = f^{-1}(J)T$ for all $J \in \mathcal{I}(T)$. This property holds in the following cases:

• f is a flat epimorphism [38, Proposition 2.1, p.111] (an example is given by $R \to R_S$, where S is a mcs of R).

• f verifies the condition (C): for each $t \in T$ there are some $r \in R$ and $u \in \mathcal{U}(T)$ such that $t = uf(r)$. A surjective morphism verifies (C).

The proofs of (λ1) and (λ3) may be found in our paper [43] and the others are elementary.

Lemma 2.3. *Let $f : R \to T$ be a ring morphism, $J \in \mathcal{I}(T)$ and $I := f^{-1}(J)$.*

(a) If f is a flat morphism, then $f(\Lambda(I)) = \Lambda(IT) \cap f(R)$.

(b) If I is irreducible, then $I^\lambda \in {}^t f(\mathcal{V}(J) \cap \mathrm{Spec}(T_{\Lambda(J)} \mid T))$.

Proof. (a) Use the following facts. A flat morphism transforms a regular element into a regular element. Moreover, $R/I \to T/IT$ is flat and injective, because $I = f^{-1}(J)$ and then $I = f^{-1}(IT)$.

(b) I is primal, because irreducible and therefore $\mathfrak{P} := I^\lambda \in \mathrm{Ass}(I)$. We can reduce to a quasilocal ring (R, M), where $M = \mathrm{Z}(0)$ and 0 is irreducible in R (consider $\mathrm{Tot}(R/I) = (R/I)_\mathfrak{P} \to (T/J)_\mathfrak{P}$ and use Lemma 2.1). From [31, Proposition 1.2], we derive that $\cup[0 :_R x \mid x \in M \setminus \{0\}] = M$, where the set $\{0 :_R x\}$ is directed under inclusion. If T is an extension ring of R, then $L := \cup[0 :_T x \mid x \in M \setminus \{0\}]$ is an ideal of T such that any minimal prime ideal N of L contracts to M. Then N consists of zero-divisors. Indeed, for an element $z \in N$ there is some $s \notin N$ such that $sz^n \in L$ for a positive integer n. Choose an integer n which is minimum for the preceding property. Then there is some $x \in M \setminus \{0\}$ such that $sz^n x = 0$ and $t := sz^{n-1}x \neq 0$. It follows that $tz = 0$ and z is a zero-divisor of T. Therefore, $N \cap \Lambda(0) = \emptyset$ and $N \cap T \in \mathcal{V}(J) \cap \mathrm{Spec}(T_{\Lambda(J)} \mid T)$ by (a) and (λ3). $\square$

For a mcs S of a ring R and I an ideal of R, we set $I(S) := I_S \cap R$; so that, $I \mathcal{R} I(S)$ and $\Lambda(I) \subseteq \Lambda(I(S)) = \Lambda(I_S) \cap R$ by (λ3) applied to $R \to R_S$ and Proposition 2.2. If $S = R \setminus P$, where $P \in \mathrm{Spec}(R)$, we recover the *isolated P-component* $I(P) := I(S)$ of I. Then $\mathfrak{M}$ is a minimal prime ideal of I if and only if $I(\mathfrak{M})$ is a $\mathfrak{M}$-primary ideal [37, Proposition 6] and $I(\mathfrak{M})$ is the smallest $\mathfrak{M}$-primary ideal containing I.

(λ4): Let $I \in \mathcal{I}(R)$ with $I \neq 0$ and P a prime ideal of R with $I \subseteq P$, then $(I_P)^\lambda \cap R = I(P)^\lambda$. In that case, $I(P) = I$ if $I^\lambda \subseteq P$ and $I \subset I(P)$ if $P \subset I^\lambda$ [27, Lemma 1.3]. If I is a P-primal ideal (hence $P = I^\lambda$), we get $I(P) = I$.

2.3. The smcs $\mathcal{U}$ in a quasilocal treed domain. Let R be a quasilocal treed domain. We introduce a prime ideal linked to an ideal and used in the theory of valuation domains [29, p. 69]. Let $I \neq 0$ be an ideal of R. We set $I^\sharp := \{r \in R \mid rI \neq I\} = R \setminus \mathcal{U}(I)$, where $\mathcal{U}(I) := \{r \in R \mid rI = I\}$ and we also set $0^\sharp = 0$. As $\mathcal{U}(I)$ is a smcs, $I^\sharp$ is a prime ideal. We list some properties of the operation $I \mapsto I^\sharp$, within a quasilocal treed domain R:

($\sharp$1): $I^\sharp$ is a prime ideal of R, containing I.

($\sharp$2): $(I^\sharp)^\sharp = I^\sharp$.

($\sharp$3): $IR_{I^\sharp} = I$.

($\sharp$4): $(rI)^\sharp = I^\sharp$ for $r \in R$, $r \neq 0$.

($\sharp$5): $(IJ)^\sharp \subseteq I^\sharp \cap J^\sharp$.

($\sharp$6): If $\mathfrak{P}$ is a prime ideal of R, then $R_\mathfrak{P} \subseteq (I : I) \Leftrightarrow I^\sharp \subseteq \mathfrak{P}$.

($\sharp$7): If $f : R \to T$ is a ring morphism and I an ideal of R, then $\mathcal{U}(I) \subseteq f^{-1}(\mathcal{U}(IT))$; so that, $f^{-1}((IT)^\sharp) \subseteq I^\sharp$.

($\sharp$8): $\mathcal{U}(I) \subseteq \Lambda(I)$.

For ($\sharp$1) to ($\sharp$3), rework the proof of [29, Lemma 4.3]. Now ($\sharp$6) is a consequence of $R_\mathfrak{P} \subseteq (I : I) \Leftrightarrow 1/s \in (I : I)$ for $s \notin \mathfrak{P}$.

We note here that $(I : I) = R_{I^\sharp} = R_{\Lambda(I)}$, when R is a valuation domain [29, Lemma 4.3] and [23, Lemma 3.1.9]. This result will be extended in a next section to the divided domains context.

3. Properties of quasilocal treed or divided domains

Most of results of this section are derived from the consideration of smcss linked to ring morphisms. More precisely, to a ring morphism $f : A \to B$, we associate the smcs $S_f := \{s \in A \mid f(s) \in \mathcal{U}(B)\}$. We first look at some properties of smcs.

When (R, M) is a quasilocal ring, the intersection of an empty family contained in $\mathcal{I}(R)$ is M by convention.

Proposition 3.1. *Let (R, M) be a quasilocal treed ring and S a mcs of R such that $0 \notin S$. Then $R_S = R_P$ for some prime ideal P of R, such that $\cap[Rs \mid s \in S] \subseteq P$, $P \cap S = \emptyset$ and $P^\downarrow = \cap[\mathcal{D}(s) \mid s \in S]$.*

Proof. We can assume that S is a smcs and then $R \setminus S = \cup[\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Spec}(R), \mathfrak{P} \cap S = \emptyset]$. As $\mathrm{Spec}(R)$ is linearly ordered, $P := R \setminus S \in \mathrm{Spec}(R)$. Set $I := \cap[Rs \mid s \in S \cap M]$. Then $I \subset P$ holds if $S \cap M = \emptyset$ since $I = M$ by convention. We can assume that $S \cap M \neq \emptyset$. Let $x \in \cap[Rs \mid s \in S \cap M]$ and assume that $x \notin P$. Then $x^2 \in S$ and $x = rx^2$ for some $r \in R$. Hence rx is an idempotent of R. As R is connected, either $rx = 0$ or $rx = 1$. Then either $x = 0$ or $x \in \mathcal{U}(R)$ is a contradiction and $x \in P$. Now $\mathrm{Spec}(R_S \mid R) = \mathrm{Spec}(R_P \mid R)$ gives the last statement. $\square$

Okabe proved the following result [41, Corollary 2.7]. Let $I \in \mathcal{I}_f(R)$, where R is a divided domain, then $P := \cap[I^n \mid n \in \mathbb{N}] \in \mathrm{Spec}(R)$. This result is generalized below and in a next section. If I is a principal ideal, Okabe's result is a consequence of the next corollary.

Corollary 3.2. *Let (R, M) be a quasilocal ring.*

(a) If R is a divided ring and S a mcs of R with $0 \notin S$, then $R_S = R_P$, where $P := \cap[Rs \mid s \in S \setminus \mathcal{U}(R)] \in \mathrm{Spec}(R)$ is such that $P^{\downarrow} = \cap[\mathcal{D}(s) \mid s \in S]$.

In particular, let $a \in R \setminus \heartsuit(R)$, then $R_a = R_P$, where $P := \cap[Ra^n \mid n \in \mathbb{N}] \in \mathrm{Spec}(R)$ is such that $P^{\downarrow} = \mathcal{D}(a)$.

(b) If for each $a \in R \setminus \heartsuit(R)$, there is a prime ideal $P \subseteq Ra$, such that $R_a = R_P$, then R is a divided ring.

Proof. (a) It is enough to show that $P \subseteq \cap[Rs \mid s \in S]$. But from $s \notin P$ for $s \in S$, we deduce that $P \subseteq Rs$ because R is divided.

Assume that the hypotheses of (b) hold. Let $Q \in \mathrm{Spec}(R)$ and $a \in R \setminus Q$. If a is a unit, then $Q \subseteq Ra = R$. If not, there is some $P \in \mathrm{Spec}(R)$, such that $R_a = R_P$ and $P \subseteq Ra$. Then $Q \subseteq P \Rightarrow Q \subseteq Ra$. Hence, R is a divided ring. $\square$

Let $X \neq \emptyset$ be a subset of $\mathrm{Spec}(A)$, where A is an integral domain. The *ring of global sections* over X is $\Gamma(X) := \cap[A_P \mid P \in X]$. We will consider *epimorphisms* of the category of commutative rings and in particular flat epimorphisms (see [38, Chapter 4]). They do not need to be surjective maps.

The following theorem is a (the) key result of this paper.

Theorem 3.3. *Let $f : A \to B$ be a going-down ring morphism, where A is a quasilocal treed ring.*

(a) S_f is a smcs of A, $P_f := A \setminus S_f \in \mathrm{Spec}(A)$ is such that $\cap[As \mid s \in S_f] \subseteq P_f = \cup[A \cap N \mid N \in \mathrm{Max}(B)]$ and $\mathrm{Spec}(B \mid A) = (P_f)^{\downarrow}$. If in addition, A is a divided ring, then $\cap[As \mid s \in S_f] = P_f$.

(b) There is a factorization $A \to A_{P_f} \to B$, where $A_{P_f} \to B$ has the lying-over property and the going-down property.

(c) In case A is an integral domain, f is a flat epimorphism and $X := \mathrm{Spec}(B \mid A)$, then $B \simeq \Gamma(X)$, $P_f = \cup[P \mid P \in X]$ and $B = A_{P_f}$.

Proof. (a) Clearly $S := S_f$ is a smcs, $P_f := A \setminus S \in \mathrm{Spec}(A)$ and $P_f = \cup[f^{-1}(Q) \mid Q \in \mathrm{Spec}(B)]$. Then $P_f = f^{-1}(Q_f)$ for some $Q_f \in \mathrm{Spec}(B)$. To see this, we observe that $P_f B \neq B$; for if not, $1 = p_1 b_1 + \cdots + p_n b_n$, where $p_i \in P_i = f^{-1}(Q_i)$, with $Q_i \in \mathrm{Spec}(B)$ and $b_i \in B$. There is some P_j such that $P_i \subseteq P_j$ for each i and $1 \in P_j B \subseteq Q_j$, an absurdity. From $P_f B \neq B$, we deduce that $P_f B \subseteq N$, for some maximal ideal N of B. Set $M := f^{-1}(N)$, then $P_f \subseteq M \Rightarrow P_f = f^{-1}(Q_f)$ for some $Q_f \in \mathrm{Spec}(B)$ by the going-down property of f and $\mathrm{Spec}(B \mid A) \subseteq (P_f)^{\downarrow}$. The reverse inclusion is again deduced from the going-down property of f and $P_f = f^{-1}(Q_f)$.

(b) The factorization exists because $s \in S \Rightarrow f(s) \in \mathcal{U}(B)$ and its properties follow from $(P_f)^{\downarrow} = \mathrm{Spec}(B \mid A)$.

(c) $A \to B$ identifies to $A \to \Gamma(X)$ by [47, Proposition 4.7]. In view of [38, Corollaire 3.2, p.114], $A_S \to B$ is a flat epimorphism with the lying-over property, whence is faithfully flat. It follows from [38, Lemme 1.2] that $A_S \to B$ is an isomorphism. □

Morita defines for an arbitrary injective ring morphism $f : R \to T$ the *maximal flat epimorphic subextension* $g : R \to \mathrm{E}(f)$ of $R \to T$ [39]. We recall that a going-down morphism $A \to B$ is injective when A is an integral domain, because ${}^t f(\mathrm{Min}(B)) \subseteq \mathrm{Min}(A)$.

Proposition 3.4. *Let $f : A \to B$ be an (injective) going-down ring morphism, where A is a quasilocal treed domain. Then $A \to A_{P_f}$ is the maximal flat epimorphic subextension of $A \to B$.*

Proof. Let g be the map $A \to E(f)$, then we have $S_g \subseteq S_f$ and $E(f) = A_{P_g}$ by Theorem 3.3(c). It follows that $E(f) \subseteq A_{P_f} \subseteq E(f)$ and $A \to A_{P_f}$ is the maximal flat epimorphic subextension of $A \to B$. □

The above result applies to an extension of integral domains $A \subseteq B$, where A is a GD-domain (such extensions have the GD-property).

Corollary 3.5. *Let (A, M) be a divided integral domain and $f : A \to B$ an extension of integral domains, then $A \to A_{P_f}$ is the maximal flat epimorphic subextension of $A \to B$ and $P_f = \cap[As \mid s \in S_f]$.*

Proof. Each extension of integral domains $A \subseteq B$ has the GD-property [8, Theorem 1]. □

Corollary 3.6. *Let R be a divided domain and $f : R \to T$ an injective ring i-morphism, where T is an integral domain. Then T is a divided domain. In particular, each overring of a divided i-domain is a divided i-domain, (i.e. R is a strong divided domain).*

Proof. Observe that $\mathrm{Spec}(T) \to \mathrm{Spec}(R_{P_T})$ is a homeomorphism. Then use [48, Proposition 4.1]. $\square$

Proposition 3.7. *Let R be a divided (quasilocal) ring and $I \in \mathcal{I}(R)$.*

(a) $I^\lambda \in \mathrm{Ass}(I)$ and I is a I^λ-primal ideal.

(b) I is a primary ideal if and only if $I^\lambda = \sqrt{I}$.

(c) Either $J \subseteq I^\lambda$ or $I \subseteq J$ holds for any ideal J of R.

(d) If R is an integral domain, $IR_P = I$ for each $P \in \mathcal{V}(I^\lambda)$.

Proof. (a) and (b) are known and are written here for further references.

(c) Assume that $J \not\subseteq I^\lambda$, then $I \subseteq I^\lambda \subseteq J$ follows from $I^\lambda \in \mathrm{Spec}(R)$.

(d) Let x/s with $x \in I$ and $s \in \Lambda(I)$, then $s \notin I^\lambda \Rightarrow I^\lambda \subseteq Rs$. From $I \subseteq I^\lambda$, we deduce $x = rs$ and then $r \in I$, because $s \in \Lambda(I)$. Hence, $x/s = r \in I$ completes the proof of $IR_{\Lambda(I)} = I$. Then $IR_P = I$ for $I^\lambda \subseteq P$ is clear because $R_{\Lambda(I)} \supseteq R_P$. $\square$

An ideal I of a ring R is called a *divided ideal* if the above statement (c) holds. In case R is an integral domain, (c) is equivalent to $IR_{\Lambda(I)} = I$. Dobbs proved part of (d) in a particular case [12, Proposition 2.2].

We recall that a commutative ring R is called a *Kasch ring* if its maximal ideals are of the form $0 : I$ for some $I \in \mathcal{I}(R)$. A ring is said to have *few zero divisors* if $\mathfrak{m} := 0^\lambda = \mathrm{Z}(0)$ is a finite union of prime ideals. Then R has few zero-divisors if and only if $\mathrm{Tot}(R)$ is semilocal Kasch [19, Theorem]. This corrects a wrong statement frequently asserted in the literature.

Proposition 3.8. *Let (R, M) be a treed quasilocal ring and $K := \mathrm{Tot}(R)$.*

(a) $\mathfrak{m} \in \mathrm{Spec}(R)$ and $K = R_\mathfrak{m}$ is a quasilocal Kasch ring.

(b) For each $I \in \mathcal{I}(R)$, each overring of R/I has few zero divisors.

(c) $\mathfrak{m} = 0 : x$ for some $x \in R$, whence $\mathfrak{m} \in \mathrm{Ass}_f(0)$.

(d) Each ideal I of R such that $0 : I = 0$ contains a regular element, whence $\mathrm{Tot}(R) = \mathrm{Q}(R)$.

(e) Let $\mathfrak{P} \in \mathrm{Spec}(R)$ and $I \in \mathcal{I}(R)$, then $\mathfrak{P} \in \mathrm{Ass}(I) \Leftrightarrow I(\mathfrak{P})^\lambda = \mathfrak{P}$.

(f) Let $I \in \mathcal{I}(R)$, then $I^\lambda = I : x$ for some $x \in R$ and $I^\lambda \in \mathrm{Ass}_f(I)$. If $I \notin \mathrm{Spec}(R)$, then $I^\lambda = I : x \Rightarrow x \in I^\lambda$ and $x^2 \in I$.

(g) Let $I, J \in \mathcal{I}(R)$, then $I : J = I \Leftrightarrow J \cap \Lambda(I) \neq \emptyset$.

Proof. (a) That K is a Kasch ring follows from [19, Theorem], because $\mathfrak{m}$ is a prime ideal. Then (b) is a consequence of (a), because each R/I is a treed quasilocal ring.

(c) Set $N = \mathfrak{m}R_\mathfrak{m}$. Since K is a Kash ring, $N = 0 : L$ for some ideal L of K. Since L is nonzero, there is some $k \in L$ such that $0 : k \neq K$ and then $N = 0 : (x/1)$, where $x \in R$. It follows that $\mathfrak{m} = 0 : x$, because $0 : (x/1) \cap R = 0 : x$.

(d) This is [18, Lemma 1.3].

(e) The statement is [37, Proposition 1].

(f) Since R/I is a treed quasilocal ring, apply (c). If I is not a prime ideal, $I^\lambda = I : x$ and $I^\lambda \neq I \Rightarrow x \notin \Lambda(I)$; so that $x \in I^\lambda$, $x^2 \in I$.

(g) Replace J with $I + J$ containing I and use (d) in the ring R/I. □

Lemma 3.9. *Let (R, M) be a divided (quasilocal) ring.*

(a) M consists of zero divisors $\Leftrightarrow M = \mathfrak{m} \Leftrightarrow M \in \mathrm{Ass}_f(0)$.

(b) If $\mathfrak{P} \subset M \in \mathrm{Ass}_f(0)$ is a prime ideal, then $\mathfrak{P} \in \mathrm{Ass}_f(0)$.

(c) $\mathfrak{P} \in \mathrm{Ass}_f(0) \Leftrightarrow \mathfrak{P} = 0(\mathfrak{P}) : y$ for some $y \in R$.

Proof. (a) Assume that M consists of zero divisors, then $M \subseteq \mathfrak{m} \Rightarrow M = \mathfrak{m} \in \mathrm{Ass}_f(0)$ by Proposition 3.8(c). The converse is clear.

(b) Check that the clever proof of [31, Lemma 1.1] works in our context, because we only need the comparability of prime ideals with arbitrary ideals. Hence $\mathfrak{P}R_\mathfrak{P}$ consists of zero divisors in $S := R_\mathfrak{P}$ and by (a) $\mathfrak{P}R_\mathfrak{P} \in \mathrm{Ass}_f(0_S)$. Conclude by using [4, Exercise 17(d),p.166].

(c) $\mathfrak{P}R_\mathfrak{P} \in \mathrm{Ass}_f(0)$ consists of zero divisors. By Proposition 3.8, we get $\mathfrak{P}R_\mathfrak{P} = 0 : (y/1)$. Taking inverse images in R of this equation completes the proof. □

We generalize [27, Proposition 2.7] to divided rings.

Theorem 3.10. *Let R be a divided ring and $I \in \mathcal{I}(R)$.*

(a) $\mathrm{Ass}(I) = \mathrm{Ass}_f(I) = \mathcal{V}(I) \cap (I^\lambda)^\downarrow$ is (Zariski) compact.

(b) $\mathfrak{P} \in \mathrm{Ass}(I) \Leftrightarrow \mathfrak{P} = I(\mathfrak{P}) : y$ for some $y \in R$.

Proof. (a) Set $N := I^\lambda/I$. Then $S := (R/I)_N$ is a divided ring with maximal ideal M, which consists of zero divisors by Lemma 2.1. Then $P \in \mathrm{Ass}(I)$ verifies $I \subseteq P \subseteq I^\lambda$. By Lemma 3.9(b), $(P/I)S \in \mathrm{Ass}_f(0)$ and then $P \in \mathrm{Ass}_f(I)$. It follows that $\mathrm{Ass}(I) = \mathrm{Ass}_f(I)$. The same reasoning shows that $\mathcal{V}(I) \cap (I^\lambda)^\downarrow \subseteq \mathrm{Ass}_f(I)$. As the reverse containment is clear, the proof is complete. The compactness of $\mathrm{Ass}_f(I)$ follows, because it is the intersection of two patches of $\mathrm{Spec}(R)$ and a patch is Zariski compact.

(b) Apply Lemma 3.9(c) to the ring $(R/I)_{I^\lambda}$. □

Remark 3.11. Let R be a divided ring and $f : R \to T$ a ring morphism.

(a) For $J \in \mathcal{I}(T)$ and $I := f^{-1}(J)$, then $\mathrm{Ass}(I) \subseteq {}^t f(\mathrm{Ass}_f(J))$. This is [43, Corollaire, p. 89].

(b) If in addition R is a domain, T an overring of R and $\mathfrak{P} \in \mathrm{Spec}(R)$ is such that $\mathfrak{P}T \neq T$, then $\mathfrak{P} = (\mathfrak{P}T)^\lambda \cap R$ and $\{\mathfrak{P}\} = {}^t f(\mathrm{Ass}(\mathfrak{P}T))$. Indeed, we have

$\{\mathfrak{P}\} = {}^t f(\mathrm{Ass}_f(\mathfrak{P}T))$ by [17, Proposition 2.1], since a divided integral domain is straight.

Proposition 3.12. *Let (R, M) be a divided integral domain, $I \in \mathcal{I}(R)$ and $P \in \mathcal{V}(I)$ (if $I \supset P$, then $I(P) = R$ and $\mathrm{Ass}(I) = \emptyset$).*

(a) If $I^\lambda \subset P$, then $\mathrm{Ass}(I(P)) = \mathrm{Ass}(I)$ and $I(P)^\lambda = I^\lambda$.

(b) If $I^\lambda = P$, then $I(P) = I$.

(c) If $I^\lambda \supset P \supseteq I$, then $\mathrm{Ass}(I(P)) = P^\downarrow \cap \mathcal{V}(I)$ and $I(P)^\lambda = P$.

In any case, $\sqrt{I(P)} = \sqrt{I}$.

Proof. We read in [37, Proposition 5] that $\mathrm{Ass}(I(P)) = P^\downarrow \cap \mathrm{Ass}(I)$. To complete the proof, use Theorem 3.10(a). □

D. Lazard defined the *maximal flat epimorphic extension* of a ring R [38] as an injective flat epimorphism $R \to E$ that can be factored by any injective flat epimorphism $R \to T$.

Proposition 3.13. *Let R be a divided ring, with total quotient ring $K = R_{\mathfrak{m}}$. Then the maximal flat epimorphic extension of R is $R \to K$.*

Proof. Let $R \to E$ be the maximal flat epimorphic extension. It is enough to show that $g : K \to K \otimes_R E$ is an isomorphism. Since g is an injective flat epimorphism, it is enough to show that the spectral map ${}^t g$ is surjective, because a faithfully flat epimorphism is an isomorphism [38, Lemme 1.2, p.109]. In view of Remark 3.11(a), $\mathrm{Spec}(K \mid R) = \mathfrak{m}^\downarrow = \mathrm{Ass}(0) \subseteq \mathrm{Spec}(E \mid R)$. By a well known property of tensor products, ${}^t g$ is surjective. □

We recall that for a nontrivial ring extension $R \hookrightarrow T$ and $\mathfrak{q} \in \mathrm{Spec}(R)$, the pair $(R, \mathfrak{q})$ is a *valuation pair* of T if for each $x \in T \setminus R$ there is some $c \in \mathfrak{q}$ with $cx \in R \setminus \mathfrak{q}$ (see [36]).

Proposition 3.14. *Let (R, M) be a divided domain and $(R, \mathfrak{q})$ a valuation pair for an overring $T \neq R$ of R.*

(a) Let $x \in T \setminus R$, then $x \in \mathcal{U}(T)$ and $x^{-1} \in \mathfrak{q} \subseteq R$.

(b) (R, M) is a Manis pair of T.

Proof. (a) Let $x \in T \setminus R$, there is some $c \in \mathfrak{q}$ with $cx \in R \setminus \mathfrak{q}$. Since R is divided, we get $c \in \mathfrak{q} \subseteq Rcx$. Hence there is some $r \in R$ such that $c = rcx$. As $cx \neq 0 \Rightarrow c \neq 0$, we have $1 = rx$ with $r \in R$. Moreover, $c = cxr \in \mathfrak{q} \Rightarrow r \in \mathfrak{q}$.

(b) In view of [36, Theorem 2.5], (a) entails that (b) holds. □

We now intend to give some information about the *Prüfer hull* $P(R,T)$ associated to a ring extension $R \hookrightarrow T$ (Knebusch and Zhang [36, Chapter I]). Let $R \hookrightarrow S$ be a ring extension, then R is called a S-Prüfer ring if each subextension of $R \hookrightarrow S$ defines a flat epimorphism. Clearly, an integral domain R is Prüfer if and only if R is Prüfer in its quotient field. A ring extension $R \hookrightarrow T$ admits a unique subextension $R \hookrightarrow P(R,T)$, such that R is Prüfer in $P(R,T)$ and $P(R,T)$ contains every T-overring of R in which R is Prüfer [36, Ch. I,Theorem 5.15].

Proposition 3.15. *Let $f : R \hookrightarrow T$ be a going-down extension of integral domains, where (R,M) is a quasilocal treed domain.*

(a) There is $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $P(R,T) = R_{\mathfrak{p}}$.

(b) For $x, y \in P(R,T)$ such that $xy \in R$, then either $x \in R$ or $y \in R$.

(c) R is a valuation subring of $P(R,T)$.

Proof. (a) Since $g : R \to P(R,T)$ is a flat epimorphism, $P(R,T) = R_{S_g}$ by Theorem 3.3(c). Then take $\mathfrak{p} := R \setminus S_g$.

(b) The (u, u^{-1})-Lemma for an arbitrary ring extension $A \hookrightarrow B$ generalizes as follows. Let $P \in \mathrm{Spec}(A)$ and $x, y \in B$ be such that $xy \in A$. There exists either $P_x \in \mathrm{Spec}(A[x])$ or $P_y \in \mathrm{Spec}(A[y])$ lying over P. Set $A := R$, $B := P(R,T)$ and $P := M$, there is for instance a prime ideal of $R[x]$ lying over M. Because $R \to R[x]$ is a faithfully flat epimorphism, $R = R[x]$ by [38, Ch.IV, Lemme 1.2].

(c) Use [36, Ch. I, Proposition 5.1](iii) and (b). □

For an integral domain R with quotient field K, we set $P(R) := P(R,K)$.

Corollary 3.16. *Let (R,M) be a divided integral domain.*

(a) The Prüfer hull $P(R)$ is of the form $P(R) = R_{\mathfrak{p}}$, where $\mathfrak{p} \in \mathrm{Spec}(R)$.

(b) The conductor of $R \to P(R)$ is $\mathfrak{p}$, R is a valuation subring of $P(R)$ and R is integrally closed in $P(R)$.

(c) $P(R) \setminus R \subseteq \mathcal{U}(R)$ and $x \in P(R) \setminus R \Rightarrow x^{-1} \in R$.

(d) Every subextension of $R \hookrightarrow P(R)$ is of the form R_Q, where $Q \in \mathrm{Spec}(R)$.

Proof. (a) is already proved.

(b) Use Proposition 3.15 and observe that $\mathfrak{p}$ is a divided ideal for the statement about the conductor.

(c) Use Proposition 3.14. □

Remark 3.17. The Prüfer hull of an arbitrary ring R is its Prüfer hull $P(R)$ in $\mathrm{Q}(R)$. It may happen that a ring R is Prüfer-closed; that is, $R = P(R)$. We give here an example without proofs. Let X be a topological space and $R := \mathcal{C}(X)$ its ring of continuous functions $f : X \to \mathbb{R}$. Then R is seminormal, reduced, locally integral and with divided factor domains. If, in addition, X is a connected metric space, then $\mathrm{Tot}(R)$ is von Neumann regular and R is Prüfer-closed. Hence R is a locally divided ring, which is Prüfer-closed.

4. Conductor overrings of divided domains

Fossum proved that if $P \subset Q$ are ideals of an integral domain R with $P \in \mathrm{Spec}(R)$, then $(Q : Q) \subseteq (R : Q) \subseteq (P : P)$ [26, Lemma 3.7]. This property is considered below in the treed quasilocal domains context.

Proposition 4.1. *Let (R, M) be a quasilocal treed domain, $0 \neq I \in \mathcal{I}(R)$. For the natural map $\pi : R \to (I : I)$, we set $S := S_\pi$, $P := R \setminus S$.*

The following statements hold:

(a) $I^\lambda \subseteq I^\sharp = \cup[N \cap R \mid N \in \mathrm{Max}((I : I))] = P$. Hence, $R \to R_{I^\sharp}$ is the maximal flat epimorphic subextension of $R \to (I : I)$ and $R_{I^\sharp} \to (I : I)$ has the lying-over and the going-down properties.

(b) If in addition R is divided, then $I^\lambda = \cap[Rs \mid s \in \Lambda(I) \cap M]$, $IR_{\Lambda(I)} = I$, $R_{\Lambda(I)} \subseteq (I : I)$ and $I^\sharp = P = I^\lambda = R \cap I^\lambda$ (the second I^λ is relative to $(I : I)$). In particular, $\mathfrak{P}^\sharp = \mathfrak{P}$ for $\mathfrak{P} \in \mathrm{Spec}(R)$.

(c) If $I \in \mathcal{I}_f(R)$, then $I^\sharp = M$.

(d) If $I \notin \mathcal{I}_f(R)$, then $\cup[r^{-1}I \mid Rr \supset I] \subseteq (R : I)I \subseteq I^\sharp$ and $(R : I) = (I^\sharp : I)$. Moreover, $I^\lambda \subseteq \cup[r^{-1}I \mid r \notin I]$ holds.

Proof. (a) The following logical equivalences hold: $r \in R$ is a unit in $(I : I) \Leftrightarrow 1/r \in (I : I) \Leftrightarrow I \subseteq rI \Leftrightarrow r \notin I^\sharp$. Moreover, let $r \in R \setminus I^\sharp$, then $I : r = rI : r \subseteq I \Rightarrow r \in \Lambda(I)$, whence $I^\lambda \subseteq I^\sharp$.

(b) Assume that R is divided, then $I^\lambda = \cap[Rs \mid s \in \Lambda(I) \cap M]$ by Corollary 3.2 and $IR_{\Lambda(I)} \subseteq I$ by Proposition 3.7(d). It follows from (a) that $P = I^\sharp \subseteq I^\lambda$, since $R_{\Lambda(I)} \subseteq (I : I) \Rightarrow R_{I^\lambda} \subseteq R_{I^\sharp}$ and $I^\lambda \subseteq I^\sharp$. Then use ($\lambda$2) for the last equality.

(c) Assume that $I \in \mathcal{I}_f(R)$ and let $r \in \mathcal{U}(I)$. Then $r \in M$ implies that $I = rI = MI$ and, by the Nakayama Lemma, we get $I = 0$, a contradiction. It follows that $M \subseteq I^\sharp$, whence $M = I^\sharp$.

(d) Assume that $I \notin \mathcal{I}_f(R)$ and let $x \in (R : I)I \setminus I^\sharp$. We can write $x = k_1y_1 + \cdots + k_ny_n$, where $k_iI \subseteq R$ and $y_i \in I$. From $I = xI$, we deduce that $I \subseteq (y_1, \ldots, y_n)$ and I is generated by $\{y_1, \ldots, y_n\}$, a contradiction. Therefore we have proved that $(R : I)I \subseteq I^\sharp$. Now if x belongs to $\cup[r^{-1}I \mid Rr \supset I]$, then $x = r^{-1}y$ with $y \in I$. From $I \subset Rr$, we get $r^{-1} \in (R : I)$ and $x \in (R : I)I$.

Therefore, $\cup[r^{-1}I \mid Rr \supset I] \subseteq (R:I)I$ holds. Now consider $x \in I^{\lambda}$. From $I \subset I:x$, we get some $y \notin I$, such that $x \in y^{-1}I$. □

Proposition 4.2. *Let (R,M) be a quasilocal treed domain. The following statement holds:*

(a) Let $I,J \in \mathcal{I}(R)\setminus\{0\}$ be such that $I\cap(R\setminus J^\sharp)\neq\emptyset$. Then $(I:I)\subseteq(R:I)\subseteq(J:J)$, $(I:I)\neq(J:J)$ and $J^\sharp\subset I^\sharp$ hold.

(b) Let $I\in\mathcal{I}(R)\setminus\{0\}$ be such that there is some $J\in\mathcal{I}(R)$ verifying $J\cap(R\setminus I^\sharp)\neq\emptyset$, then $(M:M)\subseteq(I:I)$ holds.

If in addition R is a divided domain, the following statements hold:

(c) If I and $P\in\mathrm{Spec}(R)$ are nonzero ideals of R, then

(1) $P\in\mathcal{D}(I)\Leftrightarrow P\subset I\Rightarrow(I:I)\subset(P:P)$.

(2) $R_P\subseteq(I:I)\Leftrightarrow P\in\mathcal{V}(I^\lambda)$.

(3) $R_P\subseteq(R:I)\Leftrightarrow P\in\mathcal{V}(I)$.

(d) If $I,J\in\mathrm{Spec}(R)\setminus\{0\}$, then $I\supset J$ implies $(I:I)\subset(J:J)$. Hence the inclusion defines a linear order over the family of overrings $\mathcal{C}(R):=\{(I:I)\mid I\in\mathrm{Spec}(R)\}$ with minimum member $(M:M)$. If $(M:M)$ is a valuation domain, each element of $\mathcal{C}(R)$ is a valuation domain.

Proof. (a) Let $r\in I\cap\mathcal{U}(J)$, then $(R:I)r\subseteq R$ entails that $(R:I)Jr\subseteq J$ and, since $r\in\mathcal{U}(J)$, we get $(R:I)\subseteq(J:J)$; so that, $(I:I)\subseteq(R:I)\subseteq(J:J)$. It follows that $J^\sharp\subseteq I^\sharp$ by Proposition 4.1(a). If $(I:I)=(J:J)$, then $I^\sharp=J^\sharp$ because $R_{I^\sharp}=R_{J^\sharp}$ by (b). This is a contradiction, because $I\cap(R\setminus J^\sharp)=\emptyset$. It follows that $(I:I)\subseteq(R:I)\subset(J:J)$ and $J^\sharp\subset I^\sharp$. To complete the proof, observe that $\mathrm{Spec}(R)$ is linearly ordered.

(b) If $J\cap(R\setminus I^\sharp)\neq\emptyset$ and $J\neq R$, then $M\cap(R\setminus I^\sharp)\neq\emptyset$ and we can use (a).

(c) (1) and (2) are clear, because $\mathcal{U}(I)=\Lambda(I)$ for any ideal I and because of ($\sharp$6). We show (3). Let $P\supseteq I$ be a prime ideal, then $IR_P\subseteq PR_P=P\subseteq R$ because P is divided. Therefore, $R_P\subseteq(R:I)$ holds. Conversely, assume that $IR_P\subseteq R$ holds. If $P=M$, there is nothing to show. If not, assume that $I\not\subseteq P$. Then $R_P\subseteq R\Rightarrow P=M$ leads to a contradiction.

(d) is a consequence of (c)(1). □

We defined in our paper [43] the Λ-topology on $Y:=\mathcal{I}(R)$ as follows. Set $Y_r:=\{I\in Y\mid r\in\Lambda(I)\}$ for $r\in R$. Then the set $\{Y_r\mid r\in R\}$ is a basis of open subsets on Y and defines the Λ-topology on Y. It induces the Zariski-topology on $\mathrm{Spec}(R)$. Let $f:R\to T$ be a ring morphism, then ${}^tf:\mathcal{I}(T)\to\mathcal{I}(R)$ is Λ-continuous if and only if $f^{-1}(\Lambda(J))=\Lambda(f^{-1}(J))$ for each $J\in\mathcal{I}(T)$. If ${}^tf$ is injective, then ${}^t f$ is continuous. We could write a non-integral domain version for the next result.

Proposition 4.3. *Let R be a treed quasilocal domain and $Y := \mathcal{I}(R)$. There exist two maps $\lambda, \sharp : \mathcal{I}(R) \to \mathrm{Spec}(R)$ defined respectively by $\lambda(I) = I^\lambda$ and $\sharp(I) = I^\sharp$. Then R is a divided domain if and only $\lambda = \sharp$. In that case the map $\lambda = \sharp$ is a surjective open continuous map and Y_r is a compact open subset of Y for each $r \in R$.*

Proof. Proposition 4.1(b) shows that R is divided implies $\lambda = \sharp$. Assume that $\lambda = \sharp$ and let $I \in \mathcal{I}(R)$. Then we have $I = IR_{I^\sharp} = IR_{I^\lambda}$ by ($\sharp$3), whence $I = IR_{\Lambda(I)}$ and R is divided. In that case λ verifies the properties claimed above, essentially because the identity map of $\mathrm{Spec}(R)$ can be factored $\mathrm{Spec}(R) \hookrightarrow \mathcal{I}(R) \to \mathrm{Spec}(R)$. The compactness assertion is [43, Corollaire 1,p.86], because each ideal $I \neq R$ of R is primal. $\square$

Let (R, M) be a divided ring. We set $\mathfrak{n} := \sqrt{0}$. If I is an ideal of R, then either $I \subseteq \mathfrak{n}$ or $\mathfrak{n} \subset I$. Note that $\mathfrak{n} \subset I \Leftrightarrow \mathfrak{n} \notin \mathrm{Ass}(I)$. We look at the behavior of such ideals, generalizing [12, Proposition 2.2].

Proposition 4.4. *Let (R, M) be a divided ring and $\mathfrak{n} := \sqrt{0}$.*

(a) Let an ideal $I \in \mathcal{I}(R)$ be such that $\mathfrak{n} \notin \mathrm{Ass}(I)$, then $\Lambda(I) = \mathcal{U}(I)$.

(b) $\Lambda(I_k) \subseteq \Lambda(I_1 \cdots I_n)$ for $k = 1, \ldots, n$ when $I_1, \ldots, I_n \supset \mathfrak{n}$.

(c) Let $I_1, \ldots, I_n \supset \mathfrak{n}$ be ideals of R, such that I_1 is a primary ideal and $P_1 := \sqrt{I_1} \subseteq \sqrt{I_k}$ for each $k = 1, \ldots, n$, then:

(i) $I_1 \cdots I_n$ is a P_1-primary ideal.

(ii) If $\Lambda(I_1) = \cdots = \Lambda(I_n)$, then $I_1 \cap \cdots \cap I_n$ is a P_1-primary ideal.

(d) If Q is a P-primary ideal and $Q \supset \mathfrak{n}$, then $Q = Qx$ for each $x \in R \setminus P$. If $Q \in \mathcal{I}_f(R)$, then $P = M$.

Proof. (a) Set $S := R/\mathfrak{n}$ and $J := I/\mathfrak{n}$. Then $\Lambda(J) = \mathcal{U}(J)$ follows from Proposition 4.1. Let $p : R \to S$ the natural map. In view of (λ3) in Section 2.2, $p^{-1}(\Lambda(J)) = \Lambda(I)$. We show that $p^{-1}(\mathcal{U}(J)) = \mathcal{U}(I)$. This follows from $p(r) \in \mathcal{U}(J) \Leftrightarrow p(r)J = J \Leftrightarrow I \subseteq rI + \mathfrak{n}$. Since $rI \subseteq \mathfrak{n} \Rightarrow I \subseteq \mathfrak{n}$ is a contradiction, we get $\mathfrak{n} \subseteq rI$ and then $I = rI$. Hence $\Lambda(I) = \mathcal{U}(I)$ is proved.

(b) is a consequence of (a) because $\Lambda(I) = \mathcal{U}(I)$ if $\mathfrak{n} \notin \mathrm{Ass}(I)$.

(c)(i) This is a consequence of Proposition 3.7(c), because $\sqrt{I_1 \cdots I_n} = P_1 = I_1^\lambda \supseteq (I_1 \cdots I_n)^\lambda \Rightarrow (I_1 \cdots I_n)^\lambda = \sqrt{I_1 \cdots I_n}$.

(c)(ii) Use $\Lambda(I_1) \cap \cdots \cap \Lambda(I_n) \subseteq \Lambda(I_1 \cap \cdots \cap I_n)$.

(d) Mimic the proof of [30, Theorem 17.3(a)]. The only change is as follows. For $x \notin P$, we have $x \notin \mathfrak{n}$ and x^{-1} exists in the total quotient ring $R_{\mathfrak{n}}$ of R. It is enough to choose $A = Qx^{-1}$. In case $Q \in \mathcal{I}_f(R)$, use Proposition 4.1(c), because $\sqrt{Q} = Q^\sharp$ by Proposition 3.7(c). □

In particular, I^n is a primary ideal for each primary ideal $I \supset \mathfrak{n}$ of R and each positive integer n. This was proved by Dobbs for the powers of a prime ideal of a divided integral domain [12, Proposition 2.2(a)]. Moreover, $I\sqrt{I}$ is a $\sqrt{I}$-primary ideal for $I \in \mathcal{I}(R)$ such that $\mathfrak{n} \subset I$.

Let R be an integral domain, with quotient field K. An ideal I of R is called *K-irreducible* if $I = J_1 \cap J_2$, where J_1, J_2 are R-submodules of K implies either $I = J_1$ or $I = J_2$. In the same way, the *complete K-irreducibility* of I is defined by considering infinite families of R-submodules of K.

Proposition 4.5. *Let R be a divided domain and $P \in \operatorname{Spec}(R)$. Then P is K-irreducible if and only if R_P is a valuation domain. In that case, $P = PR_P$ is K-completely irreducible in R_P. A nonzero ideal $I \in \mathcal{I}(R)$ is K-irreducible if $R_{I^\sharp}$ is a valuation domain.*

Proof. It is enough to apply [28, Corollary 20.2.8]. For the second part, observe that a proper ideal of a valuation domain is irreducible, whence K-irreducible by [28, Corollary 20.2.7(ii)]. In view of [28, Lemma 20.2.3(iii)], we get that I is K-irreducible because $IR_{I^\sharp} = I$. □

Okabe proved that a quasilocal domain R is divided if and only if $\operatorname{Max}((P : P))$ contracts to P in R for each $P \in \operatorname{Spec}(R)$ [41, Theorem 2.2]. We further generalize this result to an arbitrary ideal. A nonzero prime ideal P of an integral domain R is called *antesharp* in [22] if $\operatorname{Max}((P : P)) \cap \mathcal{V}(P)$ contracts to P in R.

Proposition 4.6. *Let (R, M) be a quasilocal domain. The following statements are equivalent:*

(1) R is a divided domain;

(2) Each nonzero nonmaximal prime ideal of R is antesharp;

(3) $P + Rr$ is a principal ideal for each $P \in \operatorname{Spec}(R) \setminus \{M\}$ and for each $r \notin P$.

Proof. (1) ⇒ (2) is a consequence of Okabe's result and (2) ⇒ (3) by [22, Proposition 2.3]. Assume that $P + Rr = Rs$ for $r \notin P$. Then we have $s = p + rx$ and $r = sy$, where $p \in P$, $x, y \in R$. We draw from these relations $s(1 - xy) \in P$ and $s \notin P$. It follows that $1 - xy \in M$ and xy is a unit, whence $Rr = Rs$. Therefore, $P \subseteq Rr$ and P is divided. Hence, (3) ⇒ (1). □

We give below information about the factorization $R \to R_{\Lambda(I)} \to (I:I)$, when R is a divided integral domain and generalize Okabe's result [41, Theorem 2.2] to arbitrary ideals. We keep the notation of Proposition 4.1 and set $I_v = (R:(R:I))$.

Theorem 4.7. *Let (R, M) be a divided domain and $I \in \mathcal{I}(R)$, $I \neq 0$. The following statements hold.*

(a) $\operatorname{Spec}((I:I) \mid R) = (I^\lambda)^\downarrow$, ${}^t\pi(\operatorname{Max}((I:I))) \subseteq \mathcal{V}(I) \cap (I^\lambda)^\downarrow = \operatorname{Ass}_f(I) = \operatorname{Ass}(I)$ *and* $I^\lambda \in {}^t\pi(\operatorname{Max}((I:I)))$.

(b) $\operatorname{Rad}((I:I)) \cap R \in \operatorname{Ass}(I)$ *and* $\sqrt{I} = \sqrt{J} \cap R$ *for some ideal* $J \subseteq \operatorname{Rad}((I:I))$.

(c) If I is a primary ideal, i.e. $\sqrt{I} = I^\lambda$, *then* ${}^a\pi(\operatorname{Max}((I:I))) = \{I^\lambda\}$, $\pi^{-1}(I^\lambda) = \mathcal{V}(I)$, $\operatorname{Max}((I:I)) \subseteq \mathcal{V}(I)$ *and* $\sqrt[(I:I)]{I} \subseteq \operatorname{Rad}((I:I))$.

Hence, $\{\pi^{-1}(I^\lambda), \mathcal{D}(I)\}$ *defines a partition of* $\operatorname{Spec}((I:I))$ *and the mapping* $Q \mapsto (Q:I)$ *is a bijection* $\mathcal{D}(I) = \sqrt{I}^\downarrow \setminus \{\sqrt{I}\} \to \mathcal{D}(I)$, *with inverse* $Q' \mapsto Q' \cap R$. *Moreover,* $R_{I^\lambda} \to (I:I)$ *has the lying-over and going-down properties.*

(d) $I^\lambda = \sqrt{I}$ *in case* $(R:I) = (I:I)$ *and then I is a primary ideal. In particular, if* $(R:I)$ *is a ring, then* $(R:I) = (R:I_v) = (I_v:I_v)$ *and* I_v *is a primary ideal.*

(e) If I is not a principal ideal, then $(R:I) = (I^\lambda:I)$. *In particular,* $(R:P) = (P:P)$ *for a non-principal ideal* $P \in \operatorname{Spec}(R) \setminus \{0, M\}$.

Proof. (a) $\operatorname{Spec}((I:I) \mid R) = (I^\lambda)^\downarrow$ by Proposition 4.1(a)(b). Let M be a maximal ideal of $(I:I)$ and suppose that $I \nsubseteq M$. Then $M + I = (I:I)$ implies that $1 = m + i$, where $m \in M$ and $i \in I$. Let $x \in R_{\Lambda(I)}$, then $xm = x - xi \in R_{\Lambda(I)} \cap M$ shows that $R_{\Lambda(I)} = R_{\Lambda(I)} \cap M + IR_{\Lambda(I)}$. As $IR_{\Lambda(I)} = I$ by Proposition 4.1(b), we get that $R_{\Lambda(I)} = R_{\Lambda(I)} \cap M + I$ with $I \subseteq I^\lambda = I^\lambda R_{\Lambda(I)}$. In short, we have $R_{\Lambda(I)} = R_{\Lambda(I)} \cap M + I^\lambda R_{\Lambda(I)}$, from which we deduce that $R_{\Lambda(I)} \cap M \nsubseteq I^\lambda R_{\Lambda(I)}$. Then $I^\lambda R_{\Lambda(I)} \subseteq R_{\Lambda(I)} \cap M$, because $R_{\Lambda(I)}$ is a divided domain and consequently, $R_{\Lambda(I)} = R_{\Lambda(I)} \cap M$, an absurdity. Thus we have proved that ${}^a\pi(\operatorname{Max}((I:I))) \subseteq \mathcal{V}(I)$. By using Theorem 3.10(a) we complete the proof of the first statement of (a). Since I^λ is lain over by $Q \in \operatorname{Spec}((I:I))$, pick some $N \in \operatorname{Max}((I:I))$ with $Q \subseteq N$. Then $N \cap R \subseteq I^\lambda$ because $N \cap R \in \operatorname{Ass}(I)$ gives us $N \cap R = I^\lambda$.

(b) Use (a) and the going-down property of $R \to (I:I)$.

(c) Use the known fact that for a ring extension $A \subseteq B$ and a nonzero ideal I shared by A and B, the mapping $Q \mapsto Q \cap A$ induced by $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ defines a bijection $\mathcal{D}(I) \to \mathcal{D}(I)$. Use also [40, Proposition 1.3] and Proposition 3.7(c).

(d) Choose $P = \sqrt{I}$ in Proposition 4.2(c)(3) and assume that $(R:I) = (I:I)$, then $R_P \subseteq (I:I) \Rightarrow I^\lambda \subseteq P$ by Proposition 4.2(c)(2) and $I^\lambda = \sqrt{I}$ follows. In view of [33, Proposition 2.2], we have $(R:I) = (I_v:I_v)$ if $(R:I)$ is a ring. Then $(I:I) \subseteq (R:I)$ gives $(I:I) \subseteq (I_v:I_v)$; so that, $(I_v)^\lambda \subseteq I^\lambda$.

(e) It is enough to show that $(R : I) \subseteq (I^{\lambda} : I)$. Assume the contrary. There are $x \in K$, $a \in I$ and $b \in \Lambda(I)$ such that $xa = b$ and $xI \subseteq R$. It follows that $c = a/b \in IR_{\Lambda(I)} = I$ and $xc = 1$. This is absurd, because $xI = R \Rightarrow I$ is a principal ideal. Now by [9, Corollary 2.4], a nonzero principal prime ideal of a going-down domain is a maximal ideal. $\square$

5. Goldman prime ideals and divided open rings

Let R be a ring. For $X \neq \emptyset$, $X \subseteq \mathrm{Spec}(R)$, we set $\mathfrak{U}(X) := \cup[P \mid P \in X]$, $\mathfrak{R}(X) := \cap[P \mid P \in X]$ and $S_X := \{s \in R \mid X \subseteq \mathcal{D}(s)\}$. If I is an ideal of R, we set $S_I = S_{\mathcal{D}(I)}$. Then S_X is a smcs of R, $0 \notin S_X$, $R_X := R_{S_X}$ is called the *localization* of R at X and $X^u := \mathrm{Spec}(R_X \mid R) = \{P \in \mathrm{Spec}(R) \mid P \subseteq \mathfrak{U}(X)\}$ is stable under generalizations and compact [47, Remark 2.8(4)]. If $f : R \to T$ is a ring morphism such that $\mathrm{Spec}(T \mid R) \subseteq X^{\downarrow}$, there is a factorization $R \to R_X \to T$.

Then $Y := \mathrm{Spec}(R)$ is endowed with the *flat topology* F, whose closed sets are the Zariski compact subsets of Y that are stable under generalization. This topology was introduced by M. Hochster under another name. We proved that its closed subsets are of the form $\mathrm{Spec}(T \mid R)$, where $R \to T$ is a flat morphism [45, Section IV]. If $\overline{X}^F$ is the F-closure of $X \subseteq Y$ and $\mathcal{F}_X$ is the family of all elements $I \in \mathcal{I}_f(R)$ such that $X \subseteq \mathcal{D}(I)$, we have $\overline{X}^F = \cap[\mathcal{D}(I) \mid I \in \mathcal{F}_X]$ and $S_X = S_{\overline{X}^F}$.

We defined in "collective" form the *g-ideal rings* in [44, Section V] as rings in which each prime ideal is a g-ideal. Fontana and Houston in [21, Proposition 1.8] give a characterization of prime g-ideals, which is essentially extracted from our paper. We recall that a prime ideal P is called a g-ideal in case $P^{\downarrow}$ is an open subset of $\mathrm{Spec}(R)$, necessarily of the form $\mathcal{D}(a)$, where $a \in R$ is nonzero. Actually, P is a g-ideal if and only if $R_P = R_a$ for some $a \in R \setminus \{0\}$: See also [2, page 77].

The next proposition generalizes and completes Okabe's result about the set intersection of the power of a fg-ideal [41, Corollary 2.7].

Proposition 5.1. *Let (R, M) be a divided integral domain and $X \neq \emptyset$ a subset of* $\mathrm{Spec}(R)$.

(a) $S_X = \{s \in R \mid s \in \mathcal{U}(\Gamma(X))\}$, $P_X := \mathfrak{U}(X) = R \setminus S_X = \cap[Rs \mid s \in S_X]$ is a prime ideal of R and $X^u = (P_X)^{\downarrow}$.

(b) $P_X = P_{\overline{X}^F}$, $\overline{X}^F = (P_X)^{\downarrow}$ and $P_X = \cap[I^n \mid n \in \mathbb{N}, I \in \mathcal{F}_X]$.

(c) Let $I, J \in \mathcal{I}(R)$, $I \neq 0$ and $J = (a_1, \ldots, a_n) \in \mathcal{I}_f(R)$ be such that $\mathcal{D}(I) = \mathcal{D}(J)$. Then $P_I := \cap[J^n \mid n \in \mathbb{N}] = \cap[Rs \mid s \in S_I]$ is a prime ideal of R and $\mathcal{D}(I) = (P_I)^{\downarrow}$. In particular, P_I is a prime g-ideal and $R_{P_I} = R_a$, where $a \in \{a_1, \ldots, a_n\}$.

Proof. (a) Use Theorem 3.3 with $f : R \to R_X$ and $S_f = S_X$.

(b) We can assume that X is compact and stable under generalizations. From $X = \cap[\mathcal{D}(I) \mid I \in \mathcal{F}_X]$, we deduce that $P_X \in \mathcal{D}(I)$ for each $I \in \mathcal{F}_X$. Indeed, $I \subseteq P_X = \mathfrak{U}(X) \Rightarrow I \subseteq P$ for some $P \in X$, because $I \in \mathcal{I}_f(R)$ and X is linearly ordered. We are lead to the contradiction $P \in X \subseteq \mathcal{D}(I)$. Therefore, P_X belongs to X; so that, $(P_X)^{\downarrow} \subseteq X \subseteq X^u = (P_X)^{\downarrow}$ by the first part of the proof. The last part of (b) is a consequence of (c), because $P_X^{\downarrow} = \cap[P_I^{\downarrow} \mid I \in \mathcal{F}_X]$ and the set intersection of the P_Is is a prime ideal.

(c) $X := \mathcal{D}(I) = \mathcal{D}(J)$ is F-closed and then $X = (P_X)^{\downarrow}$ by (b). We have to show that $H := \cap[J^n \mid n \in \mathbb{N}] = P_X$. From $P_X \in X$ we get that $P_X \subseteq H$, because R is divided. Now if $s \in J^n$ for each n and $s \notin P_X$, we get $s \in S_X$ and then $J \subseteq \sqrt{Rs}$. Since J is a fg-ideal, there is some positive integer k such that $Rs \subseteq J^{k+p} \subseteq Rs$ for each integer p. Therefore, $J^k = (J^k)^2$ and J^k is an idempotent fg-ideal of the integral domain R and $J^k = 0$, a contradiction. Therefore, $H \subseteq P_X$. □

Corollary 5.2. *Let (R, M) be a divided ring and $J \supset \mathfrak{n}$ a f.g. ideal of R, then $\cap[J^n \mid n \in \mathbb{N}] = \cap[Rs \mid s \in S_I]$ is a prime ideal of R.*

Proof. Consider $J/\mathfrak{n}$ in $R/\mathfrak{n}$. □

Remark 5.3. For the notions involved in this remark, we refer to a paper by Badawi and Houston [3]. They proved that if I is a proper *powerful ideal* of an integral domain R, then $\cap[I^n \mid n \in \mathbb{N}]$ is a (strongly) prime ideal [3, Proposition 1.8]. This is a consequence of the following facts: a power of a powerful ideal is a powerful ideal and if J is an ideal of R, then either $J \subseteq I$ or $I^2 \subseteq J$. If R is an APVD, each $P \in \mathrm{Spec}(R)$ is strongly primary and then P^3 is a powerful ideal [3, Corollary 2.6]. It follows that $\cap[I^n \mid n \in \mathbb{N}]$ is a prime ideal for each $I \in \mathcal{I}(R)$. Actually, an APVD is a divided domain (R, M). For an ideal $I \subseteq M$, we have $I^3 \subseteq M^3$ with M^3 powerful and by [3, Proposition 1.4], I^3 is a powerful ideal.

If I is an ideal of an integral domain R, many authors call the ring of global sections $\Gamma(\mathcal{D}(I))$ over $\mathcal{D}(I)$ the *Kaplansky transform* of the ideal I, using the notation $\Omega(I)$ (see for instance [23, Chapter III]). Then $\mathcal{D}(I)$ endowed with the sheaf induced by the scheme $\mathrm{Spec}(R)$ is called an *affine open subset* if it is a scheme. An affine open subset is a quasi-compact subset but the converse does not generally hold. The reader is referred to [32], in order to get information.

Theorem 5.4. *Let (R, M) be a divided integral domain, $I \neq 0$ an ideal of R and $Y := \mathcal{D}(I)$.*

(a) There is a factorization $R \to R_{P_I} \to \Gamma(Y)$, where $P_I = R \setminus S_I = \cap[Rs \mid s \in S_I]$ is a prime ideal.

(b) Y is an affine open subset if and only if there is some $J \in \mathcal{I}_f(R)$ such that $\sqrt{I} = \sqrt{J}$ and $R_{P_I} = \Gamma(\mathrm{D}(I))$. In that case, $R \to R_{P_I}$ is a flat epimorphism of finite presentation (as an R-algebra) of the form $R \to R_a$, where a belongs to a set of generators of J.

Proof. Consider the natural map $f : R \to \Gamma(Y) = \cap[R_P \mid P \in Y]$. It is easy to check that $S_f = S_I$ and (a) follows from Proposition 5.1. By [47, Proposition 4.16], Y is an affine open subset if and only if $R \to \Gamma(Y)$ is a flat epimorphism. In view of Theorem 3.3(c), this condition holds if and only if $\Gamma(Y) := R_{P_I}$. Then $R \to R_{P_I}$ is of finite presentation, because $Y \to X$ is an open immersion [46, Lemme 4.9]. □

Recall from the Kaplansky's book [35] the following notation and results. An integral domain R with quotient field K is called a *G-domain* if $\{0\}$ is an open subset of $\mathrm{Spec}(R)$ ($\Leftrightarrow R_a = K$ for some $a \in R \setminus \heartsuit(R)$). For a ring R and $P \in \mathrm{Spec}(R)$, the *pseudo-radical* of P is $\pi(P) := \cap[Q \in \mathrm{Spec}(R) \mid Q \supset P]$. Now $P \in \mathrm{Spec}(R)$ is called a *G-ideal* if R/P is a G-domain; that is $\pi(P) \neq P$.

An integral domain is called a *G-ideal domain* if each prime ideal of R is a G-ideal.

Note that a ring R is divided if and only if its prime G-ideals are divided, since any prime ideal of R is an intersection of prime G-ideals.

For an arbitrary ring it is known that P is a G-ideal $\Leftrightarrow P = M \cap R$ for some $M \in \mathrm{Max}(R[X])$. We have a better result in a divided ring.

Proposition 5.5. *Let R be a divided ring and $P \supset \mathfrak{n}$ a prime G-ideal. There is $M \in \mathrm{Max}(R[X])$, such that $P^n = M^n \cap R$ for each $n \in \mathbb{N}$.*

Proof. For a prime G-ideal P, there is a maximal ideal $M = P[X] + (aX - 1)$ of $R[X]$, where $a \notin P$, and such that $P = M \cap R$. We need only to show that $M^n \cap R \subseteq P^n$. Let $r = (p(X) + q(X)(aX - 1))^n \in R$, where $p(X) \in P[X]$ and $q(X) \in R[X]$. We can write $r = p(X)^n + (aX - 1)s(X)$ where $s(X) \in R[X]$. Consider this equation in $R_P[X]$ and substitute $1/a$ to X as $a \in \mathcal{U}(R_P)$. We get $r = p(1/a)^n$ and then $a^t r \in P^n$ for some positive integer t. Since P^n is P-primary by Proposition 4.4, we see that $r \in P^n$. □

Papick introduced *propen* domains R [42] (such that $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a Zariski-open map for each overring $S \neq K$ of R). An integral domain R is called *open* if R is propen and $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ is open (*i.e.* R is a propen G-domain). A propen domain is a going-down domain. For all these facts, see [42].

Theorem 5.6. *Let (R, M) be a divided (quasilocal) integral domain.*

(1) Let $P \in \mathrm{Spec}(R) \setminus \{0\}$, then P is a G-ideal $\Leftrightarrow$ P is a g-ideal.

(2) If $P \neq 0, M$ is a prime g-ideal, $R_P = R_r$ for some $r \in R \setminus \heartsuit(R)$ and $P = \cap\,[Rr^n \mid n \in \mathbb{N}]$ for $P \neq M$.

(3) Let $r \in R \setminus \heartsuit(R)$, then $\cap[Rr^n \mid n \in \mathbb{N}]$ is a prime g-ideal $P \neq M$ and $P = \cap[Rs^n \mid n \in \mathbb{N}]$ for some $s \in R \setminus \heartsuit(R)$ is equivalent to $\sqrt{Rs} = \sqrt{Rr}$.

(4) Each $P \in \mathrm{Spec}(R) \setminus \{0, M\}$ is an intersection of g-ideals and is divisorial.

(5) $P^{\downarrow} \setminus \{P\} = \mathcal{D}(P)$ is an open subset for each $P \in \mathrm{Spec}(R)$.

(6) R is an open domain if and only if R is a G-ideal domain and also, if and only if R is a g-ideal domain.

In particular, a finite dimensional divided domain is an open domain.

Proof. (1) A g-ideal is clearly a G-ideal. Consider a G-ideal P. To prove our claim, we can assume that $P \neq M$ and P is nonzero. In light of [32, 0.1.3.3], we have $R_P = \varinjlim R_a$, where a varies in $R \setminus P$. Then [32, Proposition 0.3.4.10] provides us the relation $(\star)$: $P^{\downarrow} = \cap\,[\mathcal{D}(a) \mid a \notin P]$, where we can assume that $a \notin \mathcal{U}(R)$, for if not, $\mathcal{D}(a) = \mathrm{Spec}(R)$ is surperfluous. There is at least a nonunit $a \notin P$, because $P \neq M$. In view of Corollary 3.2, there is a prime ideal $P(a) = \cap\,[Ra^n \mid n \in \mathbb{N}]$ such that $\mathrm{D}(a) = P(a)^{\downarrow}$. Then the relation $(\star)$ implies that $P \subseteq P(a)$. Assume that $P^{\downarrow}$ is not equal to any open subset $\mathcal{D}(r)$, then $P = P(a)$ implies $P^{\downarrow} = \mathrm{D}(a)$, a contradiction; whence $P \subset P(a)$. The inclusion $P \subseteq \cap\,[P(a) \mid a \in R \setminus P]$ is clear. Let $x \neq 0$ be in the intersection of all the $P(a)$, then $x \notin \mathcal{U}(R)$ and $x \in P$; deny, then $x \notin P$ shows that $x \in P(x) = \cap\,[Rx^n \mid n \in \mathbb{N}]$. In that case $x = rx^2$ for some $r \in R$ and x is a unit, an absurdity. Therefore, $P = \pi(P)$, an absurdity since P is a G-ideal. Thus $P^{\downarrow} = \mathcal{D}(r)$ for some $r \in R$ and $R_P = R_r$. Set $Q := \cap\,[Rr^n \mid n \in \mathbb{N}]$, then $R_r = R_Q$ by Corollary 3.2. It follows that $P = Q$. Hence, (1) is proved.

(2), (3) Use Corollary 3.2.

(4) It is well-known that a prime ideal of an arbitrary ring is equal to a set intersection of G-ideals.

(5) A divided prime ideal P is comparable to each prime ideal.

(6) The proof is an easy consequence of [42, Proposition 3.2], because a divided domain is a going-down domain. If R is finite dimensional, let $P \in \mathrm{Spec}(R)$ be such that $P \neq M$, then $\pi(P) \neq P$ shows that P has a prime ideal $\pi(P)$ right above P; so that, P is a G-ideal. $\square$

Dobbs calls an integral domain R either *pointwise non-Archimedean* or a *power-Ahmes domain* if $\cap\,[Rr^n \mid n \in \mathbb{N}] \neq 0$ for all $r \in R \setminus \{0\}$ [10, Theorem 2.4]. We recover below [10, Theorem 2.6] with a complement.

Proposition 5.7. *Let R be a divided domain. The following statements are equivalent:*

(a) R is a power-Ahmes domain;

(b) R is not a G-domain;

(c) Each nonzero prime ideal of R has infinite height.

Proof. If R is a power-Ahmes domain, assume that 0 is a G-ideal. There is some $a \in R \setminus \heartsuit(R)$, such that $\{0\} = \mathcal{D}(a)$. In view of Corollary 3.2 (a), we get $\cap\, [Ra^n \mid n \in \mathbb{N}] = 0$, a contradiction. Hence R is not a G-domain. Conversely, if R is not a G-domain, let $a \neq 0$ in R. If a is a unit, then $Ra^n = R$ and $R \neq 0$. If not, $P := \cap\, [Ra^n \mid n \in \mathbb{N}]$ is a prime G-ideal, which is nonzero and R is a power-Ahmes domain. Hence (a) is equivalent to (b) and (a) $\Leftrightarrow$ (c) by [10, Theorem 2.6]. $\square$

An integral domain R is called *fragmented* if for each $r \in R \setminus \heartsuit(R)$, there exists $s \in R \setminus \heartsuit(R)$ such that $r \in \cap\, [Rs^n \mid n \in \mathbb{N}]$. We generalize a result of Dobbs [11, Theorem 2.5] and recover Gilmer's results on unbranched prime ideals of Prüfer domains [30, Theorem 17.3].

Theorem 5.8. *Let (R, M) be a divided integral domain and $P \in \mathrm{Spec}(R) \setminus \{0\}$. The following statements are equivalent:*

(1) R_P is fragmented;

(2) P is the union of all (some) prime ideals $Q \subset P$;

(2') P is the union of all (some) prime ideals $P' \subseteq Q \subset P$ for each $P' \subseteq P$;

(3) If I is an ideal of R such that $\sqrt{I} = P$, then $I = P$;

(4) $P \neq \sqrt{Rr}$ for each $r \in R \Leftrightarrow P \notin \mathrm{Minp}(R)$;

(5) If Q is a P-primary ideal, then $Q = P$;

(6) For $x \in P \setminus \{0\}$, there is a strictly ascending chain $\mathcal{C} := \{P_n\}_{n\in\mathbb{N}} \subseteq P^{\downarrow}$ with $x \in P_0$.

If one of the preceding equivalent conditions holds, P is called a (Gilmer) unbranched prime ideal and in that case $\mathrm{ht}(P) = \infty$.

Proof. (1) $\Leftrightarrow$ (2) In light of [11, Theorem 2.5], P is the union of all $Q \in \mathrm{Spec}(R)$ with $Q \neq P$ if and only if R_P is fragmented.

(2) $\Rightarrow$ (3). Suppose that (2) holds and let I be an ideal with $\sqrt{I} = P$, then $I \subseteq P$. If $I \neq P$ and if $Q \subseteq I$ for all $Q \in \mathrm{Spec}(R)$ with $Q \subset P$, then $P \subseteq I \subset P$, a contradiction. If there is some $Q \subset P$ with $I \subset Q$, then $P = \sqrt{I} \subseteq Q$, a contradiction. Hence, $I = P$.

(3) $\Rightarrow$ (4). Suppose that $\sqrt{Rr} = P$ and that (3) holds. Then $P = \sqrt{Rr^2}$ and then $P = Rr = Rr^2 \Rightarrow r \in \{0, 1\}$, a contradiction.

(4) $\Rightarrow$ (2) Assume that (4) holds and let $P' = \cup[Q \in \mathrm{Spec}(R) \mid Q \subset P] \in \mathrm{Spec}(R)$. If $P' \subset P$, let $r \in P \setminus P'$; so that $\sqrt{Rr} \subseteq P$ and actually $\sqrt{Rr} \subset P$. Therefore, $\sqrt{Rr} = Q \subset P$ and $r \in P'$ leads to a contradiction. Hence $P = P'$.

(3) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (2). Let $P' = \cup[Q \in \mathrm{Spec}(R) \mid Q \subset P] \in \mathrm{Spec}(R)$ and suppose that $P' \subset P$. The prime ideals P' and P are adjacent. Set $S = R/P'$ and $N = P/P'$, then (S, N) is a one-dimensional integral domain. Let $\bar{r} \in S \setminus N$. Then $S\bar{r}$ and $S\bar{r}^2$ are N-primary ideals, which gives two P-primary ideals $P' + Rr \neq P' + Pr^2$, a contradiction. Hence $P = P'$ and (2) holds.

(1) $\Leftrightarrow$ (6) This is [5, Corollary 2.10]. $\square$

It follows that $P = P^2$ if P is unbranched ($\Leftrightarrow P \notin \mathrm{Minp}(R)$) by Proposition 4.4). The same property holds for a P-primary ideal I. We say that $P \in \mathrm{Spec}(R)$ is *strongly unbranched* if for an ideal I of R, $I^\lambda = P \Rightarrow P = I$. Then P is strongly unbranched implies that P is unbranched, because $\sqrt{I} = I^\lambda$ if I is primary.

Remark 5.9. We refer the reader to [23, Section 5.1] for the definitions and results on *localizing systems*. Let R be a divided domain, $\mathcal{F}$ a localizing system and the natural map $f : R \to R_{\mathcal{F}} = \cup[(R : I) \mid I \in \mathcal{F}]$. We observe that $R \setminus P_f := S_f = \{r \in R \mid Rr \in \mathcal{F}\}$. We set $\mathfrak{P} = \cup[Q \in \mathrm{Spec}(R) \mid Q \notin \mathcal{F}]$.

(a) Suppose that $\mathfrak{P} \in \mathcal{F}$; so that $\mathfrak{P}$ is unbranched and $\mathfrak{P} = \mathfrak{P}^2$. Then we have $\mathfrak{P} \subseteq \cap[I \mid I \in \mathcal{F}] \subseteq \cap[Rr \mid r \in S_f] = P_f$, with $\mathfrak{P}, P_f \in \mathrm{Spec}(R)$. Setting $\overline{\mathcal{F}}_{\mathfrak{P}} := \{I \mid \mathfrak{P} \subseteq I\}$, we get that $\overline{\mathcal{F}}_{\mathfrak{P}} = \mathcal{F}$. We claim that $\mathfrak{P} = \cap[I \mid I \in \mathcal{F}] = P_f$. If $\mathfrak{P}$ is not principal, $(R : \mathfrak{P}) = (\mathfrak{P} : \mathfrak{P})$ by Theorem 4.7(e). Now by the Fossum's result [26, Lemma 3.7] applied to $\mathfrak{P} \subset I$, we get that $(R : I) \subseteq (\mathfrak{P} : \mathfrak{P})$. It follows that $R_{\mathcal{F}} = (R : \mathfrak{P}) = (\mathfrak{P} : \mathfrak{P})$ and $P_f = \mathfrak{P} = \cap[I \mid I \in \mathcal{F}]$. Now if $\mathfrak{P}$ is principal, then $P_f \subseteq \mathfrak{P}$ gives the same result.

(b) What happens when $\mathfrak{P} \notin \mathcal{F}$? This holds for a localizing system $\mathcal{F}_S$ associated to a smcs S of a divided domain but we have an answer by Corollary 3.2. For instance, let I be an ideal of a ring R and the localizing system $\mathcal{F}_I := \{J \mid I : J = I\}$, which by Proposition 3.8(g) is nothing but $\mathcal{F}_{\Lambda(I)}$ and in this case $\mathfrak{P} \notin \mathcal{F}_I$. Let X be a subset of $\mathrm{Spec}(R)$, where R is a divided integral domain. Consider the localizing system $\tilde{\mathcal{F}}_X = \{I \mid X \subseteq \mathcal{D}(I)\}$. Proposition 5.1 gives a result and $\mathfrak{P} \notin \tilde{\mathcal{F}}_X$ if X is compact. Note that the question is completely solved in [23, Section 5.1] for valuation domains.

(c) We come back to the maximal flat epimorphic subextensions of Section 3. Let A be a treed quasilocal domain, $f : A \to B$ an injective ring morphism and $\mathcal{F} := \{I \mid IB = B\}$ the fg-localizing system associated to f. In light of [47, Lemma

2.5], $A_{\mathcal{F}} = \{b \in B \mid A : b \in \mathcal{F}\}$. The maximal flat epimorphic subextension of $A \to B$ can be gotten by transfinite induction, the first step being $A_{\mathcal{F}}$ [39, p.36]. In view of [20, Corollary 1.15, Lemma 1.11], there is some $P \in \mathrm{Spec}(A)$, such that $\mathcal{F} = \mathcal{F}_P$ because $\mathrm{Spec}(A)$ is linearly ordered. Hence $A \to A_{\mathcal{F}} = A_P$ is a flat epimorphism factoring $A \to B$ and is the maximal flat epimorphic subextension of $A \to B$. Proposition 3.4 shows that $P = P_f$.

Fontana and Houston define an Ω-*domain* as an integral domain R whose overrings are of the form $\Omega(I) := \Gamma(I)$ for some ideal I of R [21].

Proposition 5.10. *Let R be a divided domain and $P \in \mathrm{Spec}(R) \setminus \{0\}$.*

(1) If $R_P = \Gamma(I)$ for some ideal I of R such that $I \subseteq P$, then P is unbranched, $I = P$ and $\sqrt{I} = \sqrt{J}$ where $J \in \mathcal{I}_f(R)$.

(2) P is a prime g-ideal $\Leftrightarrow$ $R_P = \Gamma(I)$ for some (necessarily fg)-ideal I of R.

Proof. (1) Since $f : R \to \Gamma(I)$ is a flat epimorphism, $\mathcal{D}(I)$ is an affine open subset [47, Proposition 4.16] or [25, Theorem 2.4] and then $\sqrt{I} = \sqrt{J}$ where J is a fg-ideal and $S_f = R \setminus P = R \setminus \cup[Q \mid Q \in \mathcal{D}(I)]$. It follows that $P = \cup[Q \mid Q \in \mathcal{D}(I)]$. Moreover, $Q = P \Rightarrow I \subseteq Q$ for $Q \in \mathcal{D}(I)$ is absurd, whence $Q \subset P$. Thus $P = \cup[Q \mid Q \subset I]$ is unbranched and $P \subseteq I$.

(2) Since prime g-ideals and prime G-ideals coincide, we deduce from [21, Lemma 2.4] that $R_P = \Gamma(I) \Rightarrow R_P = \Gamma(\pi(P))$. If P is not a prime g-ideal, taking $I := \pi(P) \subseteq P$, we draw from (1) that P is unbranched and $P = \sqrt{J}$, where $J \in \mathcal{I}_f(R)$. Then $P = \sqrt{J} = \cup[Q \mid Q \subset P]$ leads to the contradiction $P \subseteq Q$ for some $Q \subset P$. Hence P is a g-ideal. For the converse, use that a prime g-ideal P is such that $R_P = R_r = \Gamma(Rr)$ for some $r \in R$. □

The following result completes [21, Theorem 3.12], where only one divided prime ideal is involved.

Theorem 5.11. *Let (R, M) be a divided domain. Then R is an Ω-domain if and only if R is a QQR-domain and each nonzero prime ideal is a G-ideal (g-ideal). In that case, R is a propen domain and R_P is a valuation Ω-domain for each prime ideal $P \neq M$.*

Proof. An Ω-domain R is clearly a QQR-domain; that is, each overring is of the form $\cap_{i \in I} R_{P_i}$ with $\{P_i\}_{i \in I} \subseteq \mathrm{Spec}(R)$. Proposition 5.10 implies that each nonzero prime ideal is a G-ideal. Assume that the preceding conditions hold. It is enough to show that $R_P = \Gamma(\pi(P))$ for $P \in \mathrm{Spec}(R)$ because $\cap_\alpha \Gamma(I_\alpha) = \Gamma(\Sigma_\alpha I_\alpha)$. If P

is a G-ideal, then $\mathcal{D}(\pi(P)) = P^{\downarrow}$ and then $R_P = \Gamma(\pi(P))$. Moreover, $R_0 = \Gamma(0)$. Then R_P is a valuation domain by [21, Theorem 3.12](1). □

We denote by $R^\star$ and R^+ the *complete integral closure* and the Swan's *seminormalization* of a domain R with quotient field K [49]. Set $\pi := \pi(0)$ and $\pi^+ := \pi(0)$ for the pseudo-radicals of 0 in R and R^+. A *conducive domain* is a domain R, whose overrings $\neq K$ admit a nonzero conductor [14]. Completely integrally closed is shorten into cic.

Proposition 5.12. *Let (R, M) be a divided domain.*

(a) If $P \in \mathrm{Spec}(R) \setminus \{0\}$, then $R_P \subseteq (P : P) \subseteq R^\star$. Hence $\dim(R) \leq 1$ if R is cic.

(b) If R is not a G-domain, then $R^\star = K$.

(c) If R is a G-domain, $\pi = \sqrt{Rr}$ for some $r \in R$, $\mathrm{ht}(\pi) = 1$, $R_\pi \subseteq (\pi : \pi) \subseteq R^\star = (R_\pi)^\star$, $R^\star$ is integrally closed and $R^{\star\star}$ is cic.

(d) If R is a G-domain, then $(R^+)^\star = (\pi^+ : \pi^+)$ is cic and then $R^{\star\star} \subseteq (R^+)^\star$.

(e) If R is a conducive G-domain, then $R^\star \subseteq K$ is a minimal extension; so that $R^\star = (R^+)^\star$ is cic and seminormal.

Proof. (a) Use $R^\star = \cup\, [(I : I) \mid I \in \mathcal{I}(R) \setminus \{0\}]$ and the fact that P is divided.

(b) Observe that R is a power-Ahmes domain by Proposition 5.7, because R is not a G-domain. It follows easily that $R^\star = K$.

(c) We have $\pi = \sqrt{Rr}$ for some $r \in R$ by Theorem 5.8. Then $\mathrm{ht}(\pi) = 1$ and $R_\pi \subseteq (\pi : \pi) \subseteq R^\star$ are clear. Let $x = b/a \in R^\star$. There is some nonzero $u \in R$ such that $ux^n \in R$ for all $n > 0$. It follows that $1/a \in (R_b)^\star$. In view of Corollary 3.2, $P := \cap\, [Rb^n \mid n \in \mathbb{N}]$ is a prime g-ideal such that $R_b = R_P$. Since R is not a G-domain, P is nonzero. It follows that $R^\star \subseteq \cup\, [(R_P)^\star \mid P \neq 0, P \text{ g-ideal}] \subseteq (R_\pi)^\star$, because $\pi \subseteq P$ and then $R^\star = (R_\pi)^\star$. The last statements are proved in [50].

(d) Since $R \subseteq R^+$ is subintegral [49], $\mathrm{Spec}(R^+) \to \mathrm{Spec}(R)$ is a homeomorphism. We deduce from [48, Proposition 4.1] that the G-domain R^+ is divided. Hence we can assume that R is seminormal. Let I be a nonzero ideal of R. If $\pi \subseteq I$, then $(I : I) \subseteq (\pi : \pi)$ [26, Lemma 3.7]. If $I \subset \pi$, then $\sqrt{I} = \pi$ and $(I : I) \subseteq (\pi : \pi)$ since R is seminormal [24, Theorem 3.3]. It follows that $R^\star = (\pi : \pi)$. Then $R^\star = (\pi : \pi)$ is cic by [15, Corollary 2.12].

(e) is a consequence of [14, Proposition 4.3]. □

Remark 5.13. Let (R, M) be a divided G-domain, such that R_P is a valuation domain for each $P \in \mathrm{Spec}(R) \setminus \{M\}$.

(a) If $\dim(R) \geq 2$, we get that $R^\star = R_\pi = (\pi : \pi)$, because a one-dimensional valuation domain is completely integrally closed. This property holds for either APVD's or divided Ω-domains whose dimension is ≥ 2 (see [6, Lemma 3.1] and Theorem 5.11).

(b) Let I be a nonzero non-Archimedean ideal of R (such that $I^\lambda \neq M$). By reworking the proofs of [12, Theorem 2.3] and [12, Corollary 2.4], we may get that I is R-flat, but there is an easier proof as follows. Observe that $IR_{\Lambda(I)} = I$ is torsion-free over the valuation domain $R_{\Lambda(I)}$, whence is flat. By transitivity of flatness, I is R-flat. In particular, a non-maximal prime ideal of R is flat when R is either an APVD or a divided Ω-domain. Dobbs observed that in case (R, M) is a coherent divided domain each non-maximal prime ideal is flat if and only if R_P is a valuation domain for each $P \in \mathrm{Spec}(R) \setminus \{M\}$ [12, Remark 2.10].

6. Some applications

We first give a descent result.

Proposition 6.1. *Let $f : R \hookrightarrow T$ be an extension of integral domains, where T is a divided domain and with respective quotient fields K and L, such that $R = K \cap T$.*

(a) An ideal I of R is divided and primal if there is some $J \in \mathcal{I}(T)$, such that $I = f^{-1}(J)$.

(b) If in addition R is a quasilocal treed domain and $R \to T$ has the going-down property, then R is a divided integral domain.

Proof. (a) Let $I = f^{-1}(J)$, in view of ($\lambda 2$) in Section 2.2, we can suppose that $f^{-1}(\Lambda(J)) = \Lambda(I)$. Consider $x \in I_{\Lambda(I)}$, then $x \in J_{\Lambda(J)} = J$, because $\Lambda(I) \subseteq \Lambda(J)$. It follows that $x \in R_{\Lambda(I)} \cap J \subseteq K \cap J \cap T = I$. Thus $I = I_{\Lambda(I)}$ follows. Moreover, $f^{-1}(J^\sharp) = R \setminus \Lambda(I)$ is a prime ideal.

(b) With the notation of Theorem 3.3, for $P := P_f$ we have $T_P = T$ and $K \cap T = R_P$, whence $R = R_P$. To complete the proof, apply (a) to the ring morphism $R = R_P \to T$ defined in Theorem 3.3, since this morphism has the lying-over property. $\square$

The above result shows that if either $R \to T$ is faithfully flat or R is going-down and $R \to T$ is pure and if T is a divided integral domain, then so is R.

In the sequel, we focus on divided i-domains.

Proposition 6.2. *Let (R, M) be a quasilocal i-domain, with integral closure (V, N), a valuation domain. If $I \neq 0, R$ is an ideal with $\sqrt{I} = P$, then $(I^n : I^n) \subseteq V_P$ for each positive integer n. In particular, if $M = \sqrt{I}$, then $(I^n : I^n)$ is integral over R.*

Proof. Reworking the proof of [17, Proposition 6.1(a)], we first consider the case $\sqrt{I} = M$. Setting $J := I^nV$, we know that $(J : J) = V_{J^\lambda}$ by [23, Lemma 3.1.9]. Then $I^n \subseteq J \subseteq J^\lambda$ gives $M \subseteq J^\lambda$ and $M = R \cap J^\lambda$. Since R is an i-domain, it follows that $J^\lambda = N$ and $I^n : I^n \subseteq V_N = V$ since $R \subseteq V$ is an INC-extension. Now if $\sqrt{I} = P$, it is enough to consider R_P instead of R, because $(I^n : I^n) \subseteq (I_P^n : I_P^n)$. □

Corollary 6.3. *Let (R, M) be a divided i-domain with integral closure (V, N) a valuation domain and $I \in \mathcal{I}(R)$, $I \neq 0$ a P-primary ideal. Then V_P is the integral closure of $(I^n : I^n) \supseteq R_P$.*

Proof. Observe that $I^nR_P = I^n$ and that I^n is P-primary. Use also Theorem 4.7. □

The above result holds for $I\sqrt{I}$ for an ideal $I \neq R, 0$ (see Proposition 4.4). Next we generalize an Okabe's result [41, Corollary 3.13] and clarify Corollary 6.3.

Proposition 6.4. *Let R be a divided integral domain and I a nonzero P-primary ideal of R. Then* $\mathrm{Spec}((I : I)) \to \mathrm{Spec}(R_P)$ *is a homeomorphism if and only if* $\sqrt[(I:I)]{I}$ *is a (the unique) maximal ideal of $(I : I)$. In that case, each overring of R, between $(I : I)$ and R_P, is a divided domain.*

Proof. The equivalence claimed is a consequence of Theorem 4.7(c) because a GD morphism $A \to B$, whose spectral map is bijective, is such that $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a homeomorphism. Then $(I : I)$ is divided by [48, Proposition 4.1]. □

An ideal I of an integral domain R, with quotient field K, is called *strongly primary* if for $x, y \in K$, the relation $xy \in K \Rightarrow$ either $x \in I$ or $y^n \in I$ for some positive integer n. Then $I \in \mathcal{I}(R) \setminus \{0\}$ is strongly primary if and only if $(I : I)$ is a valuation domain and $\sqrt[(I:I)]{I} \in \mathrm{Max}((I : I))$ [3, Theorem 2.11].

Badawi and Houston introduced *almost pseudo-valuation domains* (APVDs) in [3]. An integral domain R is an APVD if (R, M) is quasilocal and M is strongly primary. An APVD (R, M) is a divided domain and R_P is a valuation domain for each prime ideal $P \neq M$ (see [6, Lemma 3.1]). In the next result, we examine some (partial) converse.

An ideal I of a treed quasilocal domain (R, M) is termed *Archimedean* if $I^\sharp = M$. Actually, an ideal I of a valuation domain is Archimedean if and only if $(I : I) = R$, since $(I : I) = R_{I^\sharp}$ [29, p.71].

Assume that (R, M) is a treed quasilocal domain. By Proposition 4.1(c) a fg-ideal of R is Archimedean. Observe that $I^\sharp$ is Archimedean in $R_{I^\sharp}$, because $I^\sharp R_{I^\sharp} = I^\sharp$.

An ideal I of R is Archimedean if $(I : I) = R$, since $R_{I^\sharp} \subseteq (I : I)$. An M-primary ideal I is Archimedean.

Proposition 6.5. *Let (R, M) be a quasilocal integral domain.*

(a) If R is a treed domain and $I \neq 0$ is a strongly primary ideal of R, then $I^\sharp = \sqrt{I}$ (I is a primary ideal). Hence, I is Archimedean $\Leftrightarrow \sqrt{I} = M$.

(b) R is an APVD if and only if R is divided, $\mathrm{Spec}((M : M)) \to \mathrm{Spec}(R)$ *is injective (a homeomorphism) and $(M : M)$ is a valuation domain. In that case, each nonzero prime ideal $P \neq M$ is non-Archimedean.*

(c) Suppose that R is treed. Let $I \neq R$ be an ideal of R, such that $R_{I^\sharp}$ is a valuation domain, then $(I : I)$ is a valuation domain and I is comparable with any prime ideal of R.

(d) Suppose that R_P is a valuation domain for each prime ideal $P \neq M$ and that R is treed. Then any non-Archimedean ideal $I \neq R$ is comparable to any prime ideal of R.

(e) In particular, if M is strongly unbranched, R is treed and R_P is a valuation domain for each prime ideal $P \neq M$, then R is divided.

Proof. (a) Let $x \in R \setminus \sqrt{I}$, in view of [3, Lemma 2.3] we have $xI = I$, whence $x \notin I^\sharp$. It follows that $I \subseteq I^\sharp \subseteq \sqrt{I}$ and $I^\sharp = \sqrt{I}$.

(b) Use [3, Theorem 3.4], Proposition 6.2 and that an APVD is divided [3, Proposition 3.2]. Moreover, each nonzero prime ideal is strongly primary. Hence $P^\sharp = M$ implies $P = M$.

(c) Since $R_{I^\sharp}$ is a valuation domain and $R_{I^\sharp} \subseteq (I : I)$, we get that $(I : I)$ is a valuation domain. It follows from [34, Lemma 1] that I is comparable with any prime ideal of R.

(d) An ideal I which is not Archimedean is such that $I^\sharp \neq M$, whence $R_{I^\sharp}$ is a valuation domain. To conclude, use (b). □

Okabe defines a *PPC-domain* as an integral domain R with quotient field K such that each overring $S \neq K$ is of the form $(P : P)$ for some $P \in \mathrm{Spec}(R)$ [41]. Any PVD (R, M) is a PPC-domain if $(M : M)$ is a minimal overring of R. The following result is proved by Okabe, in case R is integrally closed [41, Theorem 3.5].

Proposition 6.6. *Let R be an integral domain. Then (R, M) is a divided PPC-domain if and only if R is an APVD and $(M : M)$ is the minimal overring of R.*

Proof. Assume that R is a divided PPC-domain. Observe that $R' = (P : P)$ for some $P \in \mathrm{Spec}(R)$. As $R_P \subseteq (P : P)$, we get that $R_P = R$ because $R \to R_P$ is integral and $P = M$ the maximal ideal of R. Then $Q \subset M$ implies $R' = (M : M) \subset (Q : Q)$ by Proposition 4.2(d). It follows that $R \to R' = (M : M)$ is a minimal morphism and $(M : M)$ is the minimal overring of R. Moreover, R is an i-domain by a result of Gilbert, quoted by Dobbs [13, Theorem 4.1], whence (R', M') is a valuation domain. In view of Proposition 6.5(b), R is an APVD.

Now suppose that (R, M) is an APVD and $(M : M)$ is the minimal overring of R. Then $(M : M)$ is a valuation domain by Proposition 6.5(b). The overrings of R, different from R, are the overrings of the valuation domain $(M : M)$. But an overring of $(M : M)$ is of the form $(P' : P')$, where P' is a prime ideal of $(M : M)$. Since R is divided, $\mathrm{Spec}(R)\setminus\{M\} = \mathrm{Spec}(R')\setminus\{M'\}$ by [16, Proposition 5.6] applied to the minimal morphism $R \to (M : M)$. Therefore, if $P' \neq M'$, such an overring is of the form $(P : P)$ with $P \in \mathrm{Spec}(R)$. Now $(M' : M') = (M : M)_{M'} = (M : M)$ by [23, Lemma 3.1.9]. It follows that R is a divided PPC-domain. $\square$

Proposition 6.7. *Let (R, M) be a divided i-domain such that R_P is a valuation domain for each $P \in \mathrm{Spec}(R) \setminus \{M\}$. If R has finite Krull dimension, then each overring extension $R \to S$ is either integral or of the form $R \to R_P$, where $P = R \cap N$ and N is the maximal ideal of S. In particular, such a domain is strong divided (each of its overrings is divided).*

Proof. In view of [17, Proposition 6.1(b)], we have a factorization $R \to R_P \to S$, with $P = R \cap N$ and $R_P \to S$ is integral. Then $R_P = S$ in case $P \neq M$. If not, $R = R_P$ and then $R \to S$ is integral. $\square$

Let R be an integral domain and $I \in \mathcal{I}(R)$. Then R^I is the union of the sequence of overrings $\{(I^n : I^n)\}_{n>0}$ and defines a map $\beta_I : R \to R^I$. Many authors studied asymptotic prime divisors in Noetherian rings. We give a version for divided rings.

Proposition 6.8. *Let (R, M) be a divided integral domain, $I \in \mathcal{I}(R)$ and $n > 0$ an integer.*

(a) $\sqrt{I} \subseteq (I^{n+1})^\lambda \subseteq (I^n)^\lambda \subseteq I^\lambda$ *and* $\mathrm{Ass}(I^{n+1}) \subseteq \mathrm{Ass}(I^n) \subseteq \mathrm{Ass}(I)$.

(b) If $R/\sqrt{I}$ is a g-ideal domain, the sequence $\{(I^k)^\lambda\}_{k>0}$ is stationary and so is $\{\mathrm{Ass}(I^k)\}_{k>0}$.

Therefore, if R is a divided g-ideal ring, $\{(I^n)^\lambda\}_{n>0}$ and $\{\mathrm{Ass}(I^n)\}_{n>0}$ are stationary for each $I \in \mathcal{I}(R)$.

Proof. (a) From $\mathcal{U}(I^n) \subseteq \mathcal{U}(I^{n+1})$ and $\lambda = \sharp$, we deduce that $\sqrt{I} \subseteq (I^{n+1})^\lambda \subseteq (I^n)^\lambda \subseteq I^\lambda$. In view of Theorem 4.7, $\mathrm{Ass}(I^{n+1}) \subseteq \mathrm{Ass}(I^n) \subseteq \mathrm{Ass}(I)$.

(b) If $R/\sqrt{I}$ is a g-ideal domain, the flat topology on $\mathrm{Spec}(R/\sqrt{I})$ is Noetherian [44, Proposition 4]. As an irreducible closed subset of the flat topology is of the form $Q^{\downarrow}$ for some $Q \in \mathrm{Spec}(R/\sqrt{I})$ ([44]), $\{(I^k)^\lambda\}$ is stationary and so is $\{\mathrm{Ass}(I^k)\}$. $\square$

Note that the g-ideal domain condition is not necessary for the sequence $\{\mathrm{Ass}(I^n)\}$ to be stationary. This occurs if $I \in \mathcal{I}_f(R)$, because $I^\lambda = M$, if I is idempotent and also if R is a valuation domain, because $(IJ)^\sharp = I^\sharp \cap J^\sharp$ [29, Lemma 4.6].

If we set $I^\mu := \cap_{k>0} (I^k)^\lambda$, then $\cap_{k>0} \mathrm{Ass}(I^k) = \mathcal{V}(I) \cap (I^\mu)^\downarrow$ and $S_{\beta_I} = R \setminus I^\mu$.

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