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CENTRAL AUTOMORPHISM GROUPS FIXING THE CENTER ELEMENT-WISE

S. Heydar Jafari

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Abstract. Let G be a finite p-group. We find a necessary and sufficient condition on G such that each central automorphism of G fixes the center element-wise.

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1. Introduction

Let G be a finite group. An automorphism σ of G is said to be central if and only if it induces the identity on $G/Z(G)$, or equivalently, $g^{-1}\sigma(g) \in Z(G)$ for all $g \in G$, where $Z(G)$ is the center of G. The central automorphisms of G, denoted by $Aut_c(G)$, forms a normal subgroup of $Aut(G)$, the group of automorphisms of G. Notice that $Aut_c(G) = C_{Aut(G)}(Inn(G))$. For a group H and an abelian group K, $Hom(H, K)$ denotes the group of all homomorphisms from H to K. Let M and N be two normal subgroups of G. By $Aut^N(G)$ we mean the subgroup of $Aut(G)$ consisting of all automorphisms which induces the identity on G/N. Also by $Aut_M(G)$ we mean the subgroup of $Aut(G)$ consisting all automorphisms which induces identity on M. Let $Aut^N_M(G) = Aut^N(G) \cap Aut_M(G)$. we use the notation $G^{p^n} = \langle g^{p^n} \mid g \in G\rangle$ and C_t for cyclic group of order t.

The group of central automorphisms of a finite groups is of great importance in the investigation of $Aut(G)$, and has been studied by several authors (see e.g., [1-7]). In the article of Attar [2], it is proved for a finite p-group G, $Aut^{Z(G)}_{Z(G)}(G) = Inn(G)$ if and only if G is nilpotent of class 2 and $Z(G)$ is cyclic. M.K.Yadav in [7] obtained some necessary and sufficient conditions for the equality $Aut_c(G) = Aut^{Z(G)}_{Z(G)}(G)$ on p-groups of class 2.

We find a necessary and sufficient condition on G in the general case in order for $Aut_c(G) = Aut^{Z(G)}_{Z(G)}(G)$.

Theorem. *Let G be a finite p-group. Then $Aut_c(G) = Aut^{Z(G)}_{Z(G)}(G)$ if and only if $Z(G)G' \subseteq G^{p^n}G'$ where $exp(Z(G)) = p^n$.*

2. Proofs

Let G be a finite group and M be a central subgroup of G and $\sigma \in Aut^M(G)$. Then we can define a homomorphism f_σ from G into M such that $f_\sigma(x) = x^{-1}\sigma(x)$. On the other hand, given a homomorphism f from G to M, we can always define an endomorphism σ_f of G such that $\sigma_f(x) = xf(x)$. But σ_f is an automorphism of G if and only if for every non-trivial element $x \in M$, $f(x) \neq x^{-1}$.

A finite group G is said to be purely non-abelian if it has no nontrivial abelian direct factor. In [1], Adney and Yen proved that the correspondence $\sigma \mapsto f_\sigma$ defined above is a one-to-one map of $Aut_c(G)$ onto $Hom(G/G^{'}, Z(G))$ for purely non-abelian finite groups.

We recall that if $H \leq Z(G)$ and $K/G^{'}$ is a direct factor of $G/G^{'}$, then any element f of $Hom(K/G^{'}, H)$ induces an element $\bar{f}$ of $Hom(G/G^{'}, H)$ which is trivial on the complement of $K/G^{'}$ in $G/G^{'}$. To simplify the notation in the proof of the main theorem, we will identify f with the corresponding homomorphism from G into H.

The following Lemma can be obtained from Proposition 2.3 of [6] immediately.

Lemma 2.1. [7, Lemma 2.4] *Let G be a finite non-abelian p-group such that $Aut_c(G) = Aut_{Z(G)}^{Z(G)}(G)$. Then G is purely non-abelian.*

Proof. Suppose G has a nontrivial abelian direct factor. Let $G = A \times H$, where A is abelian and H is a non-abelian p-groups. Then $Z(H) \neq 1$ and so $Hom(A, Z(H))$ is nontrivial and by [6, Proposition 2.3] it can be assumed to be a subgroup of $Aut_c(G)$. It is clear that $Hom(A, Z(H)) \nsubseteq Aut_{Z(G)}^{Z(G)}(G)$, which is a contradiction. □

Lemma 2.2. *Let G be a finite abelian p-group and x an element of G. Then, there exist nontrivial elements $x_1, ..., x_t$ such that $G = \langle x_1\rangle \times \cdots \times \langle x_t\rangle$ and $x = x_1^{p^{s_1}} ... x_t^{p^{s_t}}$, where $s_i \in \mathbf{N} \cup \{0\}$.*

Proof. Let $G = \langle y_1\rangle \times \cdots \times \langle y_t\rangle$ and $x = y_1^{r_1} ... y_t^{r_t}$, where $r_i \geq 0$. There exist l_i and s_i such that $r_i = p^{s_i} l_i$ and $(p, l_i) = 1$. We have $\langle y_i\rangle = \langle y_i^{l_i}\rangle$ and hence $G = \langle y_1^{l_1}\rangle \times \cdots \times \langle y_t^{l_t}\rangle$. This completes the proof. □

Lemma 2.3. *Let $f \in Hom(G, Z(G))$ and $\sigma_f \in Aut_c(G)$. Then $\sigma_f \in Aut_{Z(G)}^{Z(G)}(G)$ if and only if $Z(G) \subseteq Ker(f)$.*

Proof. This follows directly from the definition of σ_f. □

Lemma 2.4. *Let G be a finite p-group and $Z(G)G^{'} \subseteq G^{p^n}G^{'}$ where $exp(Z(G)) = p^n$. Then G is purely non-abelian.*

Proof. Let $G = A \times H$ where $A \neq 1$ is abelian. Then $A \subseteq Z(G)$ and A is not a subset of $G^{p^n}G^{'} = H^{p^n}H^{'}$, which is a contradiction. $\square$

Proof of main theorem. By Lemmas 2.1 and 2.4 we can suppose that G is a purely non-abelian p-group. Let $Z(G)G^{'} \subseteq G^{p^n}G^{'}$ and $x \in Z(G)$. Then there exist elements a in G and b in $G^{'}$ such that $x = a^{p^n}b$. For any $f \in Hom(G, Z(G))$, since $exp(Z(G)) = p^n$ and $G^{'} \subseteq Ker(f)$, we have $f(x) = f(a)^{p^n}f(b) = 1.1 = 1$. Hence $Z(G) \subseteq Ker(f)$ and $\sigma_f \in Aut_{Z(G)}^{Z(G)}(G)$.

Conversely, let $Aut_c(G) = Aut_{Z(G)}^{Z(G)}(G)$ and suppose $Z(G)G^{'} \nsubseteq G^{p^n}G^{'}$. So there exists x in $Z(G)$ which is not in $G^{p^n}G^{'}$. By Lemma 2.2, there exist $x_1, ..., x_t$ of G such that $G/G^{'} = \langle x_1G^{'}\rangle \times \cdots \times \langle x_tG^{'}\rangle$ and $xG^{'} = x_1^{p^{s_1}}G^{'}...x_t^{p^{s_t}}G^{'}$. Hence for some j, $x_j^{p^{s_j}}$ is not in G^{p^n}, so $p^{s_j} < p^n$. Select $z \in Z(G)$, where $O(z) = min(O(x_jG^{'}), p^n)$, and define f by $x_jG^{'} \longmapsto z$. Then f can be considered as a homomorphism of $G/G^{'}$ into $Z(G)$ and so $\sigma_f \in Aut_c(G)$. We obtain $f(x) = f(xG^{'}) = f(x_1^{p^{s_1}}G^{'}...x_t^{p^{s_t}}G^{'}) = f(x_j^{p^{s_j}}) = z^{p^{s_j}}$. On the other hand $z^{p^{s_j}}$ is a nontrivial element of $Z(G)$. Therefore σ_f is not in $Aut_{Z(G)}^{Z(G)}(G)$, which is a contradiction.

In particular it is of interest for groups of class 2 with elementary abelian centers, and generalises a result of M. J. Curran [3, Proposition 2.7].

Corollary 2.5. *Let G be a finite non-abelian p-group and $exp(Z(G)) = p$. Then $Aut_c(G) = Aut_{Z(G)}^{Z(G)}(G)$ if and only if $Z(G) \subseteq \phi(G)$, where $\phi(G)$ is frattini subgroup of G.*

Lemma 2.6. *Let $G = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_t}}$ where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_t$ and $H \leq G^{p^n}$ for some integer n. If $n > a_k$ for some $k \in \{1, ..., t\}$ then, G/H and G have equal rank, and $G/H \cong (C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_{k-1}}})/H \times (C_{p^{a_k}} \times \cdots \times C_{p^{a_t}})$. Moreover $(C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_{k-1}}})/H \cong C_{p^n} \times C_{p^n} \times \cdots \times C_{p^n}$ if and only if $H = G^{p^n}$.*

The proof immediately follows from the fact that $H \cap (C_{p^{a_k}} \times \cdots \times C_{p^{a_t}}) = 1$.

Let G be a finite p-group of class 2. Then $G/Z(G)$ and $G^{'}$ have equal exponent p^c (say). Let $G/Z(G) = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_r}}$, where $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_r > 0$. Let k be the largest integer between 1 and r such that $a_1 = a_2 = \cdots = a_k = c$. Let $\overline{M} = M/Z(G) = C_{p^{a_1}} \times C_{p^{a_2}} \times \cdots \times C_{p^{a_k}}$. Let $G/G^{'} = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_s}}$, where $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_s > 0$, be a cyclic decomposition of $G/G^{'}$ such that $\overline{M}$ is isomorphic to a subgroup of $\overline{N} = N/G^{'} = C_{p^{b_1}} \times C_{p^{b_2}} \times \cdots \times C_{p^{b_k}}$. Using the above terminology, we in particular obtain the main theorem of Yadav [7].

Corollary 2.7. *Let G be a finite non-abelian p-group of class 2. Then the following are equivalent.*

(a) $Aut_c(G) = Aut^{Z(G)}_{Z(G)}(G)$.

(b) $Z(G) = G^{p^n}G'$.

(c) $r = s$, $(G/Z(G))/\overline{M} \cong (G/G')/\overline{N}$ *and the exponent of* $Z(G)$ *and* G' *are equal.*

Proof. Since $exp(G/Z(G)) = exp(G') \leq exp(Z(G))$, we have $G^{p^n} \subseteq Z(G)$. Therefore $G^{p^n}G' \subseteq Z(G)$, and by main theorem of article (a) and (b) are equivalent.

Let (b) hold. We first show $exp(G') = exp(Z(G))$. Let $G' < Z(G)$. Then $Z(G)/G' = G^{p^n}G'/G'$ is nontrivial subgroup of abelian group G/G', so $exp((G/G')/(Z(G)/G')) = exp(G/Z(G)) = p^n$. On the other hand $exp(G/Z(G)) = exp(G') \leq exp(Z(G))$. Thus $exp(G') = exp(Z(G))$. The case $Z(G) = G'$, is trivial. Finally (b) and (c) are equivalent by Lemma 2.6. □

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S. Heydar Jafari
Department of Mathematics
Shahrood University of Technology
P. O. Box 3619995161-316, Shahrood, Iran
e-mail: shjafari55@gmail.com