International Electronic Journal of Algebra
Volume 9 (2011) 38-60

A CLASS OF RINGS FOR WHICH THE LATTICE OF PRERADICALS IS NOT A SET

Rogelio Fernández-Alonso, Silvia Gavito and Henry Chimal-Dzul

Received: 12 December 2009; Revised: 12 October 2010
Communicated by Dolors Herbera

Abstract. In this paper we define Z-coinitial rings, where Z is an integral domain, and prove some of their properties. In particular, we characterize commutative noetherian domains and discrete valuation domains which are Z-coinital. We define radical modules and radical rings, and we prove that every countable Z-coinitial and right hereditary ring is a right radical ring. We give some examples of rings satisfying these conditions. Finally, we prove that the lattice of preradicals of every right radical ring is not a set.

Mathematics Subject Classification (2000): primary 16S90; secondary 16E60
Keywords: preradical, Z-coinitial ring, radical ring, hereditary ring, Dedekind domain

1. Introduction

A general problem in the theory of preradicals is to describe the big lattice R-pr of preradicals over a ring R. This has been done in [7] for semisimple rings and in [9] for local artinian uniserial rings (and as a consequence, for artinian principal rings). In all these cases the lattice R-pr is a finite set. We can ask in general for which rings R-pr is a (finite) set, or on the other hand, when R-pr is not a set.

In [6] Fay, Oxford and Walls presented some results of Mines (see [15]) that derive in a construction of a sequence of abelian groups and a chain of radicals over $\mathbb{Z}$ which is in one-to-one correspondence with the class of all ordinals. In particular, this proves that the class of all preradicals, in fact the class of all radicals over the ring of integers, is not a set. In this paper, in Section 4, we define a class of rings R which satisfy the hypothesis from which such construction is derived, namely, that there exists a radical $\sigma \neq 1$ over R and an R-module M such that $R = \sigma(M)$. We call those rings radical rings, and in Section 6 we prove that for these rings the same construction can be done; we conclude that the class of preradicals R-pr is not a set. We also find a class of rings which are radical rings. For this aim, for an integral domain Z we define in Section 3 a right Z-coinitial ring, namely, a ring R which is not a division ring, has as a subring of its center a copy of Z, and every non-zero right ideal contains an ideal of the form nR for some $n \in Z\backslash\{0\}$. We characterize commutative noetherian domains and discrete valuation domains which are Z-coinital. In Section 5 we prove that every Z-coinitial right hereditary countable ring R is a right radical ring. This result enables us to have a list of examples which are right radical rings, so that R-pr is not a set, and we present them in Section 7.

2. Preliminaries

In this paper all rings are associative and have identity. Let R be such a ring. $\mathcal{I}(R)$, $\mathcal{I}({}_RR)$ and $\mathcal{I}(R_R)$ will denote respectively the set of ideals, left ideals and right ideals of R. Whenever we use the term *ideal*, instead of *right ideal*, we mean a two-sided ideal. R-Mod and Mod-R denote respectively the categories of all unital left R-modules and all unital right R-modules. In this paper we work mostly with right R-modules.

As usual, for each $M, N \in$ Mod-R, we denote by $Hom_R(M, N)$ the abelian group of all R-homomorphisms $f : M \to N$.

Also, as in [1], for $M \in R$-Mod, we denote by $End^r(M)$ the ring of endomorphisms of M operating on the right; in that case we write $(ax)f = a((x)f)$ for each endomorphism $f : M \to M$, for each $x \in M$ and $a \in R$. Similarly, for $N \in$ Mod-R we denote by $End^l(N)$ the ring of endomorphisms of N operating on the left; in that case we write $g(xa) = (g(x))a$ for each endomorphism $g : N \to N$, for each $x \in N$ and $a \in R$. Let us denote by ${}_RR$ and R_R the regular module R as a left and right R-module, respectively. There are ring isomorphisms $\lambda : R \to End^l(R_R)$ such that $(\lambda(a))(x) = ax$ and $\rho : R \to End^r({}_RR)$ such that $(x)(\rho(a)) = xa$.

Throughout this paper, when we refer to an integral domain we assume that it is commutative. If R is a non-commutative domain we say that R has not zero divisors (for example in Proposition 3.4).

2.1. Preradicals. The terminology and basic concepts used here about preradicals can be found in [7], [8] and [21, VI.1]. A *preradical* over the ring R is a subfunctor of the identity functor on Mod-R. Denote by R-pr the class of all preradicals over R. There is a natural partial order in R-pr given by $\sigma \preceq \tau$ if $\sigma(M) \leq \tau(M)$ for every $M \in$ Mod-R.

Proposition 2.1. [3, Proposition 1(ii)] *For each $\sigma \in R$-pr, if $M, N \in$ Mod-R are such that $N \leq M$, then $(\sigma(M) + N)/N \leq \sigma(M/N)$.*

There are four operations in R-pr, denoted respectively '$\wedge$', '$\vee$', '$\cdot$' and ':', which are defined as follows, for σ, $\tau \in R$-pr and $M \in$ Mod-R.

(1) $(\sigma \wedge \tau)(M) = \sigma(M) \cap \tau(M)$.
(2) $(\sigma \vee \tau)(M) = \sigma(M) + \tau(M)$.
(3) $(\sigma \cdot \tau)(M) = \sigma(\tau(M))$.
(4) $(\sigma : \tau)(M)$ is such that $(\sigma : \tau)(M)/\sigma(M) = \tau(M/\sigma(M))$.

The first two operations can be defined for arbitrary classes of preradicals, as in [7]. This makes sense because for any right R-module M, a sum (or an intersection) of submodules of M which is indexed by a class can also be indexed by a set. With the partial order described above, R-pr results a complete *big* lattice, that is, a class (not necessarily a set) having joins and meets for arbitrary subclasses. The smallest and largest elements of R-pr, which are the zero functor and the identity functor on Mod-R, are denoted respectively by $\widehat{0}$ and $\widehat{1}$. The operation '$\cdot$' is called *product*; we write $\sigma\tau$ for $\sigma \cdot \tau$. The operation ':' is called *coproduct*. Some preradicals are of particular interest, such as *idempotent preradicals*, *radicals*, *left exact preradicals* and *t-radicals*. For basic definitions and results on these classes of preradicals, see [3] or [7].

Let us denote by $\mathcal{OR}$ the class of all ordinals. For $\gamma \in \mathcal{OR}$ and $\sigma \in R$-pr, the preradical σ^γ is defined recursively as follows: $\sigma^0 = \widehat{1}$, $\sigma^{\gamma+1} = \sigma\sigma^\gamma$, and $\sigma^\eta = \bigwedge\{\sigma^\gamma \mid \gamma \in \mathcal{OR}, \gamma < \eta\}$ if η is a limit ordinal. Notice that if γ and η are ordinals such that $\gamma < \eta$, then $\sigma^\eta \preceq \sigma^\gamma$. It results that $\widehat{\sigma} = \bigwedge\{\sigma^\gamma \mid \gamma \in \mathcal{OR}\}$ is the greatest idempotent preradical less than or equal to σ. Similarly we define $\sigma^{(\gamma)}$ as follows. $\sigma^{(0)} = \widehat{0}$, $\sigma^{(\gamma+1)} = (\sigma^{(\gamma)} : \sigma)$, and $\sigma^{(\eta)} = \bigvee\{\sigma^{(\gamma)} \mid \gamma \in \mathcal{OR}, \gamma < \eta\}$ if η is a limit ordinal. It results that $\overline{\sigma} = \bigvee\{\sigma^{(\gamma)} \mid \gamma \in \mathcal{OR}\}$ is the least radical greater than or equal to σ.

The following proposition describes the behavior of preradicals under direct sums and products.

Proposition 2.2. [3, Proposition 2] *Let $\sigma \in R$-pr and $\{M_\alpha\}_{\alpha \in I} \subseteq$ Mod-R, then:*

(1) $\sigma\left(\bigoplus_{\alpha \in I} M_\alpha\right) = \bigoplus_{\alpha \in I} \sigma(M_\alpha)$.
(2) $\sigma\left(\prod_{\alpha \in I} M_\alpha\right) \leq \prod_{\alpha \in I} \sigma(M_\alpha)$.

A preradical σ is a *radical* if, and only if, $\sigma(M/\sigma(M)) = 0$ for each $M \in$ Mod-R. The following property of radicals is straightforward.

Proposition 2.3. [3, Proposition 7] *Let $\sigma \in R$-pr be a radical and let $N \in$ Mod-R such that $N \leq \sigma(M)$. Then $\sigma(M/N) = \sigma(M)/N$.*

Remark 2.4. *It follows from the previous proposition that if $\sigma \in R$-pr is a radical, then σ^{γ} is also a radical for each ordinal γ.*

As in [7], for any $\sigma \in R$-pr, let $\overline{\mathbb{T}}_{\sigma} = \{\, \sigma(M) \mid M \in \text{Mod-}R \,\}$.

Remark 2.5. *Notice that $\overline{\mathbb{T}}_{\sigma}$ is closed under isomorphisms. Also, by Proposition* 2.2(1)*, $\overline{\mathbb{T}}_{\sigma}$ is always closed under direct sums. Moreover, by Proposition* 2.3*, if σ is a radical, $\overline{\mathbb{T}}_{\sigma}$ is closed under quotients.*

The following types of preradicals have special importance.

Definition 2.6. [7, Definition 4] Let N be a fully invariant submodule of M. The preradicals α_N^M and ω_N^M are defined as follows. Let $K \in$ Mod-R.

$$\alpha_N^M(K) = \sum\{f(N) \mid f \in Hom_R(M,K)\}.$$
$$\omega_N^M(K) = \bigcap\{f^{-1}(N) \mid f \in Hom_R(K,M)\}.$$

Any preradical of the form α_N^M is called an *alpha* preradical, and any preradical of the form ω_N^M is called an *omega* preradical.

Remark 2.7. [7, Proposition 5] *Let $\sigma \in R$-pr and $M, N \in$ Mod-R. Then $\sigma(M) = N$ if, and only if, N is a fully invariant submodule of M and $\alpha_N^M \preceq \sigma \preceq \omega_N^M$.*

The previous remark implies that, for a fully invariant submodule N of M, α_N^M is the least preradical (and ω_N^M is the greatest preradical) that assigns N to M.

We have also that if $K \leq N \leq M$, with K and N fully invariant in M, then $\alpha_K^M \preceq \alpha_N^M$ and $\omega_K^M \preceq \omega_N^M$.

Some types of preradicals can be described in terms of alpha and omega preradicals, as follows.

Proposition 2.8. [8, Proposition 2.1] *Let $\sigma \in R$-pr. Then:*

(1) *α_M^M is idempotent for each $M \in$ Mod-R. Moreover, σ is idempotent if, and only if, $\sigma = \bigvee\{\alpha_M^M \mid M \in$ Mod-R and $\sigma(M) = M\}$.*
(2) *ω_0^M is a radical for each $M \in$ Mod-R. Moreover, σ is radical if, and only if, $\sigma = \bigwedge\{\omega_0^M \mid M \in$ Mod-R and $\sigma(M) = 0\}$.*

The following property characterizes the identity functor in Mod-R as a preradical.

Proposition 2.9. *Let $\sigma \in R$-pr. Then $\sigma(R) = R$ if, and only if, $\sigma = \widehat{1}$.*

Proof. Suppose that $\sigma(R) = R$ and let $M \in$ Mod-R. Notice that $\alpha_{\sigma(R)}^R(M) = \sum\{f(\sigma(R)) \mid f \in Hom_R(R,M)\} = M\sigma(R)$. Then, by Remark 2.7, $M = MR =$

$M\sigma(R) = \alpha^R_{\sigma(R)}(M) \leq \sigma(M)$. On the other hand, we always have $\sigma(M) \leq M$. Therefore $\sigma(M) = M$, i.e., $\sigma = \widehat{1}$. The converse is clear. $\square$

Let R be a ring which is the direct product of a finite number of rings R_i. The following result states the relation between the lattices R-pr and R_i-pr.

Theorem 2.10. [4, Proposition I.9.1] *Let $R = \prod_{i=1}^n R_i$. Then R-pr is lattice isomorphic to the product of the lattices R_i-pr.*

The following result states that two rings which are Morita equivalent have the same lattice of preradicals.

Theorem 2.11. [4, Proposition I.9.2] *Let R and S be Morita equivalent rings. Then R-pr and S-pr are lattice isomorphic.*

Corollary 2.12. *Let R be a ring and for $n > 1$ let $S = \mathbb{M}_n(R)$ be the ring of $n \times n$ matrices with entries in R. Then R-pr and S-pr are lattice isomorphic.*

Theorem 2.13. [4, Exercise I.8.E7] *Let $S \to R$ be a surjective ring homomorphism. Then there is an injective order morphism R-pr $\to S$-pr.*

2.2. Ext functor for hereditary rings. The following terminology for Hom and Ext functors can be found in [14, I.6, III.1, III.2]. As usual, for $M \in$ Mod-R we denote by $Hom_R(M, \_) :$ Mod-$R \to$ Mod-$\mathbb{Z}$ the covariant Hom functor that sends each $N \in$ Mod-R to $Hom_R(M, N)$, and each $f \in Hom_R(N, N')$ to the induced homomorphism $f_* = Hom_R(M, f) : Hom_R(M, N) \to Hom_R(M, N')$, such that for each $h \in Hom_R(M, N)$, $f_*(h) = fh$. Here composition is denoted in the usual way; in this case h acts first. Similarly, for $M \in$ Mod-R we denote by $Hom_R(\_, M) :$ Mod-$R \to$ Mod-$\mathbb{Z}$ the contravariant Hom functor that sends each $N \in$ Mod-R to $Hom_R(N, M)$, and each $f \in Hom_R(N, N')$ to $f^* = Hom_R(f, M) : Hom_R(N', M) \to Hom_R(N, M)$, such that for each $h \in Hom_R(N', M)$, $f^*(h) = hf$.

For each $M, N \in$ Mod-R we will denote by $Ext_R(M, N)$ the abelian group of all equivalence classes of extensions of N by M, with the Baer sum. If E is such an extension we write $[E]$ for its equivalence class. If $\alpha \in Hom_R(N, N')$, then for each $K \in$ Mod-R, $(\alpha)_* : Ext_R(K, N) \to Ext_R(K, N')$ will denote the induced $\mathbb{Z}$-homomorphism such that $(\alpha)_*([E]) = [\alpha E]$ for each $[E] \in Ext_R(K, N)$, where αE is the extension that results by adjoining by the left α to E. Similarly, if $\gamma \in Hom_R(M', M)$, then for each $K \in$ Mod-R, $(\gamma)^* : Ext_R(M, K) \to Ext_R(M', K)$ will denote the induced $\mathbb{Z}$-homomorphism such that $(\gamma)^*([E]) = [E\gamma]$ for each

$[E] \in Ext_R(M, K)$, where $E\gamma$ is the extension that results by adjoining by the right γ to E. Recall that Ext functors, usually denoted by $Ext^n_R(M, \_)$ for the covariant, or $Ext^n_R(\_, N)$ for the contravariant, can be defined for each $n \geq 1$ as right derived functors of the covariant (respectively contravariant) Hom functor, and that $Ext^1_R(M, N) \cong Ext_R(M, N)$ (see [19, VI, VII and IX]).

Let $\mathcal{Z}(R)$ denote the center of R. For each $z \in \mathcal{Z}(R)$ and $M \in$ Mod-R we will denote by $\mu^M_z : M \to M$ the R-endomorphism of M such that for each $x \in M$, $\mu^M_z(x) = xz$, and we will refer to it as *multiplication* by z. If S is any subring of $\mathcal{Z}(R)$, then $Ext_R(M, N)$ is an S-module as follows. If $[E] \in Ext_R(M, N)$ and $z \in S$, then $[E]z$ is the equivalence class of $\mu^N_z E \equiv E\mu^M_z$. In other words, $[E]z = (\mu^N_z)_*([E]) = (\mu^M_z)^*([E])$. See [19, VII] for details.

Recall that a ring R is *right hereditary* if each $J \in \mathcal{I}(R_R)$ is projective. The following theorem describes the behavior of Ext functors on this kind of rings. As in [19], for $M \in$ Mod-R we will denote by $pd(M)$ the *projective dimension* of M, and by $rpD(R)$ the *right projective global dimension* of R, i.e., $rpD(R) = \sup\{pd(M) \mid M \in$ Mod-$R\}$.

Theorem 2.14. [19, Exercise 9.3, Theorem 9.5, Corollary 9.6] *For any ring R the following conditions are equivalent:*

(a) *R is right hereditary.*
(b) $rpD(R) \leq 1$.
(c) *$Ext^n_R(M, N) = 0$ for all $M, N \in$ Mod-R and all $n \geq 2$.*

As a consequence of Theorem 2.14 we have the following result.

Corollary 2.15. *Let R be a right hereditary ring. Let $M \in$ Mod-R, S a subring of $\mathcal{Z}(R)$ and $z \in S$. If $\mu^M_z : M \to M$ is a monomorphism, then for each $N \in$ Mod-R, $(\mu^M_z)^* : Ext_R(M, N) \to Ext_R(M, N)$ is an epimorphism of S-modules. In particular, if S is an integral domain and $\mu^M_z : M \to M$ is a monomorphism for each $z \in S$ (i.e., M is torsionfree as an S-module), then $Ext_R(M, N)$ is a divisible S-module for any $N \in$ Mod-R.*

2.3. Dedekind domains. In this subsection we assume that R is a commutative ring. More detailed information about the basic concepts included in this subsection is contained in [2,13]. Let P be a prime ideal of R and let R_P be the localization of R at P. Recall that the elements of R_P can be written as r/s, where $r \in R$ and $s \in S = R \setminus P$; in particular, the element $1/1$ is the identity of R_P. Recall also that R_P is a local ring with maximal ideal $\mathfrak{m} = PS^{-1} = \{r/s \mid r \in P, s \in S\}$. The quotient ring $R_P/\mathfrak{m}$, which is a field, is called the *residue field* of R_P. We have a natural ring homomorphism $f_P : R \to R_P$ defined by $f_P(r) = r/1$. Clearly, $I = f_P^{-1}(\mathfrak{m})$ is a prime ideal of R such that $P \subseteq I$. Therefore, if R has Krull dimension one then $I = P$.

An integral domain D is called a *discrete valuation ring* if it is a noetherian local ring of Krull dimension one and the maximal ideal $\mathfrak{m}$ is principal. In a discrete valuation ring D with maximal ideal $\mathfrak{m} = (t)$, every proper non-zero ideal of D is equal to $\mathfrak{m}^n = (t^n)$ for some integer $n \geq 1$. The following are two examples of discrete valuation rings.

Example 2.16. *Let p be a prime element of the ring of integers $\mathbb{Z}$. Then (p) is a prime ideal and the localization $\mathbb{Z}_{(p)}$ is a discrete valuation domain with maximal ideal $\mathfrak{m} = (p/1)$. The residue field of $\mathbb{Z}_{(p)}$ is the finite field with p elements.*

Example 2.17. [2, VII] *Let $R = F[x]$ be the polynomial ring in the variable x over an algebraically closed field F of zero characteristic. Let $p(x)$ be an irreducible polynomial over F. Then $P = (p(x))$ is a prime ideal of R and R_P is a discrete valuation ring with maximal ideal $\mathfrak{m} = (p(x)/1)$ and residue field isomorphic to F.*

Recall that a noetherian integral domain D of Krull dimension one is a *Dedekind domain* if for every non-zero prime ideal P of D the localization D_P is a discrete valuation ring. Recall also that in a Dedekind domain every non-zero ideal of D can be uniquely expressed as a product of prime ideals of D.

The hereditary integral domains are precisely the Dedekind domains, as is stated in the following result.

Theorem 2.18. [13, Theorem 8.3.4] *Let D be an integral domain. Then D is a Dedekind domain if, and only if, D is an hereditary ring.*

2.4. Slender modules. Suppose in this subsection that Z is an integral domain. For each $n \in \mathbb{N}$, let $\overline{e_n} = (r_1, r_2, \ldots) \in Z^{\mathbb{N}}$ be such that $r_n = 1$ and $r_i = 0$ if $i \neq n$. As in [5], we call a torsionfree Z-module M *slender* if, for every Z-homomorphism $\eta : Z^{\mathbb{N}} \to M$ we have $\eta(\overline{e_n}) \neq 0$ for only a finite number of $n \in \mathbb{N}$.

The following proposition provides an equivalent condition for a module to be slender.

Proposition 2.19. *A torsionfree Z-module M is slender if, and only if, for each countable family of Z-modules $\{N_i\}_{i=1}^{\infty}$ and each Z-homomorphism $\eta : \prod_{i=1}^{\infty} N_i \to M$ there exists $j \geq 1$ such that $\eta(\prod_{i=j}^{\infty} N_i) = 0$.*

Proof. We follow the proof of a similar result for abelian groups due to Rychkov and Thome (see [20, Proposition 1.1]). Assume that M is slender and suppose that there exist a family of Z-modules $\{N_i\}_{i=1}^{\infty}$ and a Z-homomorphism $\eta : \prod_{i=1}^{\infty} N_i \to M$ such that $\eta(\prod_{i=j}^{\infty} N_i) \neq 0$ for all $j \geq 1$. Thus there is a sequence $\{y_j\}_{j\geq 1}$ of pairwise different elements such that $y_j \in \prod_{i=j}^{\infty} N_i$ and $\eta(y_j) \neq 0$ for all $j \geq 1$. Notice that each y_j can be viewed as an element of $\prod_{i=1}^{\infty} N_i$ by setting $y_j = (y_j^{(1)}, y_j^{(2)}, \ldots)$ with $y_j^{(k)} = 0$ if $k < j$. Define a Z-homomorphism $\varphi : Z^{\mathbb{N}} \to \prod_{i=1}^{\infty} N_i$ such that $\varphi(r_1, r_2, \ldots) = (\sum_{i=1}^{\infty} r_i y_i^{(1)}, \sum_{i=1}^{\infty} r_i y_i^{(2)}, \ldots)$ for each $(r_1, r_2, \ldots) \in Z^{\mathbb{N}}$. Being M slender, it follows that $\eta(y_j) = (\eta\varphi)(\overline{e_j}) = 0$ for all but a finite number of $j \in \mathbb{N}$, which contradicts the choice of the elements y_j. The other implication is clear. □

Observe that submodules of slender modules are slender. Observe also that injective modules are not slender, hence slender modules are reduced. See also [11, XIV.7]. The following result states that for countable torsionfree modules the other implication is also true.

Proposition 2.20. *A countable torsionfree Z-module is slender if, and only if, it is reduced.*

Proof. The only if part is stated in the previous observations. For the if part, we follow the proof of the analogous result in abelian groups, due to Sasiada (see [10, Proposition 94.2]). Let M be reduced and countable Z-module, and suppose that M is not slender. We can assume that $\eta : Z^{\mathbb{N}} \to M$ is a homomorphism such that $\eta(\overline{e_n}) \neq 0$ for all $n \in \mathbb{N}$. Since M is reduced, we have $\bigcap_{r\in Z\setminus\{0\}} rM = 0$. Let $k_1 = 1$. Since $\eta(\overline{e_1}) \neq 0$, there exists $k_2 \in Z$ such that $\eta(k_1\overline{e_1}) \notin k_2 M$. Now $k_1 k_2 \eta(\overline{e_2}) \neq 0$, since M is torsionfree. Therefore there exists $k_3 \in Z$ such that $\eta(k_1 k_2 \overline{e_2}) \notin k_3 M$. In this way we can construct a sequence $k_1, k_2, \ldots$ of elements in Z such that $\eta(k_1 k_2 \cdots k_n \overline{e_n}) \notin k_{n+1} M$, for each $n \in \mathbb{N}$. Let A be the set of elements $(r_1, r_2, \ldots) \in Z^{\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $r_n = 0$ or $r_n = k_1 k_2 \cdots k_n$. Then $|A| = 2^{\aleph_0}$, so there are $a_1, a_2 \in A$ such that $a_1 \neq a_2$ and $\eta(a_1) = \eta(a_2)$. Let $a = a_1 - a_2$. Then $a \neq 0$ and if $a = (s_1, s_2, \ldots)$, then for each $n \in \mathbb{N}$ we have $s_n = 0$ or $s_n = \pm k_1 k_2 \cdots k_n$. Let n_0 be the first index such that $s_{n_0} \neq 0$. Then $\eta(s_{n_0}\overline{e_{n_0}}) \notin k_{n_0+1} M$. On the other hand, $\eta(s_{n_0}\overline{e_{n_0}}) = \eta(a) - \eta(0, \ldots, s_{n_0+1}, \ldots) = -\eta(0, \ldots, s_{n_0+1}, \ldots) \in k_{n_0+1} M$, which is a contradiction. We conclude that M is slender. □

Proposition 2.21. *Let R be a ring and Z a subring of $\mathcal{Z}(R)$. If M is a slender Z-module and $\eta : R^{\mathbb{N}} \to M$ is a Z-homomorphism such that $\eta(R^{(\mathbb{N})}) = 0$, then $\eta = 0$. In other words, $Hom_Z(R^{\mathbb{N}}/R^{(\mathbb{N})}, M) = 0$.*

Proof. Following the proof of Lemma 2 in [5], suppose that $\eta \neq 0$. Let $x = (x_n)_{n\geq 1} \in R^{\mathbb{N}}$ such that $\eta(x) \neq 0$. Define $\varphi : R^{\mathbb{N}} \to R^{\mathbb{N}}$ by $\varphi((y_n)_{n\geq 1}) = (x_i \sum_{k=1}^{i} y_k)_{i\geq 1}$. Then φ is a Z-homomorphism. For each $n \in \mathbb{N}$, let ϵ_n denote the element of $R^{\mathbb{N}}$ whose n-th coordinate is 1 and whose other coordinates are 0. Then $\eta\varphi : R^{\mathbb{N}} \to M$ is a Z-homomorphism such that $\eta\varphi(\epsilon_n) = \eta(0, \dots, 0, x_n, x_{n+1}, \dots) \neq 0$ for each $n \geq 1$. Letting $R_i = R$ for all $i \geq 1$, it follows that $\eta(\prod_{i=n}^{\infty} R_i) \neq 0$ for each $n \geq 1$, which contradicts Proposition 2.19. □

3. Z-coinitial rings

3.1. Definition and basic properties. Let Z be an integral domain and let $\chi : Z \to \mathcal{Z}(R)$ be an injective ring homomorphism. To simplify notation, for each $n \in Z$ we will write n instead of $\chi(n)$. For each $n \in Z$, $nR = \{nr \mid r \in R\}$ is a principal ideal of R generated by the element n. Let $\mathcal{N} = \{nR \mid n \in Z\}$.

Recall that if $\langle P, \leq \rangle$ is a poset, a subset Q of P is said to be *coinitial* if for each $x \in P$ there exists $y \in Q$ such that $y \leq x$.

Definition 3.1. Let Z be an integral domain which is not a division ring. Let R be a ring which is not a division ring. R is *left (right) Z-coinitial* if there exists an injective ring homomorphism $\chi : Z \to \mathcal{Z}(R)$ such that the set $\mathcal{N}\backslash\{0\}$ is coinitial in the poset $\mathcal{I}({}_RR)\backslash\{0\}$ ($\mathcal{I}(R_R)\backslash\{0\}$). We say that R is *Z-coinitial* if it is left and right Z-coinitial.

The following are obvious equivalent conditions to the previous definition.

Proposition 3.2. *For a ring R and an integral domain Z which is a subring of $\mathcal{Z}(R)$ the following conditions are equivalent:*

(a) *R is right Z-coinitial.*
(b) *For each $0 \neq I_R \leq R$ we have $Z \cap I \neq 0$.*
(c) *For each $x \in R\backslash\{0\}$ there exist $a \in R$ and $n \in Z\backslash\{0\}$ such that $xa = n$.*
(d) *For each $0 \neq I_R \leq R$, R/I is a Z-torsion module.*

Remark 3.3. *Every integral domain Z which is not a division ring is Z-coinitial.*

Now we prove some properties of right Z-coinitial rings.

Proposition 3.4. *Let Z be an integral domain and let R be a right Z-coinitial ring. Then:*

(1) *R is a right uniform R-module.*
(2) *R has not zero divisors.*

(3) *If R is a subring of the ring B such that the nonzero elements of Z are invertible in B, then all nonzero elements of R are invertible in B. In particular, Z has a nonzero element which is not invertible in R.*

(4) *R is a torsionfree Z-module.*

(5) *$\bigcap_{n\in Z\backslash\{0\}} nR = 0$. In particular, R is reduced as a Z-module.*

(6) *If R is countable then R is a slender Z-module.*

Properties $(2)-(6)$ *also hold if R is a left Z-coinitial ring.*

Proof. (1). Let $0 \neq\ I_R,\ J_R \leq R$. Being R right Z-coinitial, there exist $n, m \in Z\backslash\{0\}$ such that $nR \leq I$ and $mR \leq J$. Therefore $0 \neq nmR \leq nR \cap mR \leq I \cap J$.
(2). Let $x, y \in R\backslash\{0\}$. Then there exist $a, b \in R$ and $n, m \in Z\backslash\{0\}$ such that $xa = n$ and $yb = m$. Therefore $(xy)(ba) = x(yb)a = xma = mxa = mn \neq 0$, so that $xy \neq 0$.
(3). Let $x \in R\backslash\{0\}$. Then there exist $a \in R$ and $n \in Z\backslash\{0\}$ such that $xa = n$. Being n invertible in B, it follows that x is right invertible in B, that is, there is $b = an^{-1} \in B\backslash\{0\}$ such that $xb = 1$. Thus $(bx - 1)b = 0$, which implies that $(ax - n)a = 0$. Therefore x is invertible in B, since R has not zero divisors, by (2). For the second part of the statement, take $B = R$.
(4). This is a consequence of (2).
(5). Let $I = \bigcap_{n\in Z\backslash\{0\}} nR$ and suppose that $I \neq 0$. Then there is $n_0 \in Z\backslash\{0\}$ such that $n_0R \leq I$, so that $n_0R = I$. Let $m \in Z\backslash\{0\}$. Then $n_0mR = n_0R$, and so $n_0 = n_0mx$ for some $x \in R$. Therefore $n_0(mx - 1) = 0$ and since, by (2), R has not zero divisors we conclude that m is invertible in R, which contradicts (3). Therefore $\bigcap_{n\in Z\backslash\{0\}} nR = 0$.
(6). It follows from Proposition 2.20 and the fact that R is reduced, by (5). □

Corollary 3.5. *If R is a right Z-coinitial ring for some integral domain Z, then R has not minimal right ideals. In particular, R is not right artinian.*

Proof. If I is a minimal right ideal of R, then, being right uniform, $I \leq J$ for every nonzero right ideal J. Therefore $I \subseteq \bigcap_{n\in Z\backslash\{0\}} nR$, which contradicts Proposition 3.4.5. □

3.2. Commutative noetherian rings and discrete valuation domains. The following two results characterize commutative noetherian domains which are Z-coinitial.

Theorem 3.6. *Let D be a commutative noetherian domain. Let Z be a domain which is a subring of D. The following conditions are equivalent:*

(a) *D is Z-coinitial.*
(b) *For each nonzero prime ideal P of D the localization D_P is a Z-coinitial ring.*
(c) *For each nonzero prime ideal P of D there exists a nonzero prime ideal $\mathfrak{p}$ of Z such that $\mathfrak{p} \subseteq P$.*

Proof. $(a) \Rightarrow (b)$. Let $P \neq 0$ be a prime ideal of D and $S = D \backslash P$. Then D_P is a local ring with maximal ideal PS^{-1}. Let $I' \neq 0$ be an ideal of D_P. Then $I' = IS^{-1}$ for some nonzero ideal I of D. Therefore $nD \subseteq I$ for some $n \in Z \backslash \{0\}$, so that $nD_P \subseteq I'$.
$(b) \Rightarrow (c)$. Let P be a nonzero prime ideal of D. Since D_P is Z-coinitial, there exists $n \in Z \backslash \{0\}$ such that $nD_P \subseteq PS^{-1}$. Therefore $\mathfrak{p} = P \cap Z$ is a nonzero prime ideal of Z, with $\mathfrak{p} \subseteq P$.
$(c) \Rightarrow (a)$. Let I be a non-zero proper ideal of D. If I is a prime ideal, by hypothesis, there exist a prime ideal $\mathfrak{p} \neq 0$ of Z such that $\mathfrak{p} \subseteq I$. If I is not a prime ideal, then D/I is a noetherian ring, which therefore has (see [12, Theorem 3.4]) a finite number of minimal prime ideals $\bar{P}_1, \ldots, \bar{P}_n$ such that $\bar{P}_1 \cdots \bar{P}_n = 0$, where for each $i = 1, \ldots n$, $\bar{P}_i$ is the image of the prime ideal P_i of D under the natural projection. By hypothesis, there exist nonzero prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of Z such that $\mathfrak{p}_1 \subseteq P_1, \ldots, \mathfrak{p}_n \subseteq P_n$. Therefore $\mathfrak{p}_1 \cdots \mathfrak{p}_n \subseteq I$ and we conclude that D is Z-coinitial. □

Corollary 3.7. *Let D be a discrete valuation domain. Let Z be a domain which is a subring of D and let $\mathfrak{m}$ be the maximal ideal of D. The following conditions are equivalent:*

(a) *D is Z-coinitial.*
(b) *There exists a nonzero prime ideal $\mathfrak{p}$ of Z such that $\mathfrak{p} \subseteq \mathfrak{m}$.*
(c) *$Z \cap \mathfrak{m} \neq 0$.*
(d) *The residue field of D is a torsion Z-module.*

3.3. Z-orders. Here we present some additional examples of Z-coinitial rings. In particular, we shall give an example of a noncommutative one. Throughout this subsection, let Z be a Dedekind domain with field of fractions K and let A be a finite dimensional K-algebra. Following [18], a subring R of A is a *Z-order* in A if R is a finitely generated Z-module such that $KR = A$. A *maximal Z-order* in A is a Z-order which is not properly contained in any other Z-order in A.

Let R be a Z-coinitial ring which is also a Z-order in A. By Proposition 3.4.3 and since $KR = A$, A has to be a division ring (observe that every element of A

is of the form an^{-1} for $a \in R$ and $n \in Z$). Therefore, for our purposes in this subsection, when we refer to a Z-order in A we will always assume that A is a division ring with center K such that A is finite dimensional over K.

Remark 3.8. *Notice that Z-orders are both left and right noetherian, being finitely generated over the noetherian domain Z.*

Let A be a division ring finite dimensional over its center K. Then a Z-order in A can always be constructed, as it is shown in [18, page 109]. The following are useful results about maximal Z-orders.

Proposition 3.9. [18, Corollary 10.4] *Every Z-order in A is contained in a maximal Z-order in A. In particular, there exists a maximal Z-order in A.*

Proposition 3.10. [18, Theorem 21.4] *Every maximal Z-order in A is left and right hereditary.*

Example 3.11. [18, Exercise 10.2] *Let $A = \mathbb{Q} \oplus \mathbb{Q}\mathbf{i} \oplus \mathbb{Q}\mathbf{j} \oplus \mathbb{Q}\mathbf{k}$ be the division ring of hamiltonian quaternions over $\mathbb{Q}$. Let $\Lambda_0 = \Lambda \oplus \mathbb{Z}a$, where $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\mathbf{i} \oplus \mathbb{Z}\mathbf{j} \oplus \mathbb{Z}\mathbf{k}$ and $a = (1 + \mathbf{i} + \mathbf{j} + \mathbf{k})/2$. Then Λ_0 is a maximal $\mathbb{Z}$-order in A. It follows from the previous proposition that Λ_0 is a left and right hereditary ring.*

Proposition 3.12. [12, Proposition 9.1] *Let S be a commutative ring. Let R be an S-algebra finitely generated as an S-module. If R is prime and S is noetherian, then every essential left or right ideal of R contains a nonzero central element of the form $s1$.*

Corollary 3.13. *Let R be a Z-order in A. Then:*

(1) *R is a Z-coinitial ring.*
(2) *If R is a maximal Z-order in A, then R is a hereditary Z-coinitial ring.*

Proof. (1). Z is noetherian and there is a ring monomorphism $\chi : Z \to \mathcal{Z}(R)$ given by $\chi(n) = n1_R$. Since R has not zero divisors, R is prime, and every nonzero left and right ideal is essential. To see the last assertion, let $a, b \in R\backslash\{0\}$, then there exists $x \in A$ such that $xa = b$. Since $x = yn^{-1}$ for $y \in R$ and $n \in Z$, it follows that $ya = nb$, so that $Ra \cap Rb \neq 0$. Similarly, $aR \cap bR \neq 0$. Therefore, if I is a left or right ideal of R it is essential, and by Proposition 3.12, $Z \cap I \neq 0$. Therefore R is Z-coinitial.
(2). It follows immediately from (1) and from Proposition 3.10. □

As a consequence, the ring Λ_0 in Example 3.11 is a noncommutative $\mathbb{Z}$-coinitial ring.

4. Radical modules and rings

In this section we introduce the concepts of radical module and radical ring. In section 6 we shall prove that radical rings constitute a class of rings R for which R-pr is not a set.

Definition 4.1. Let $M \in$ Mod-R and $N \leq M$. We say that N is a *radical submodule* of M if there exists $\sigma \in R$-pr, $\sigma \neq \widehat{1}$, such that σ is a radical and $N = \sigma(M)$. We call M a *radical module* if there exists $L \in$ Mod-R such that M is a radical submodule of L. A ring R is called a *right (left) radical ring* if the regular module R_R $({}_RR)$ is a radical module.

The following are equivalent conditions for a proper submodule to be a radical submodule.

Proposition 4.2. *Let $M \in$ Mod-R and N a proper fully invariant submodule of M. Let $i : N \to M$ be the natural inclusion and $p : M \to M/N$ be the natural projection. The following conditions are equivalent.*

(a) *N is a radical submodule of M.*
(b) $N = \omega_0^{M/N}(M)$.
(c) $\alpha_N^M(M/N) = 0$.
(d) *For each $h \in Hom_R(M, M/N)$ we have $h(N) = 0$.*
(e) *The homomorphism $i^* = Hom_R(i, M/N) : Hom_R(M, M/N) \to Hom_R(N, M/N)$ is zero.*
(f) *The homomorphism $p^* = Hom_R(p, M/N) : Hom_R(M/N, M/N) \to Hom_R(M, M/N)$ is an isomorphism.*
(g) $\overline{\alpha_N^M} \preceq \omega_0^{M/N}$.
(h) $N = \overline{\alpha_N^M}(M)$.

Proof. $(a) \Rightarrow (b)$. Suppose that $N = \sigma(M)$ for some radical $\sigma \neq \widehat{1}$. Then we have $\sigma(M/N) = 0$, so that $\sigma \preceq \omega_0^{M/N}$. Therefore $N = \sigma(M) \leq \omega_0^{M/N}(M) \leq N$, where the last inequality follows from Definition 2.6.

$(b) \Leftrightarrow (d)$ and $(c) \Leftrightarrow (d)$ are direct consequences from Definition 2.6.

$(d) \Leftrightarrow (e)$ is obvious.

$(e) \Leftrightarrow (f)$ is immediate, applying the contravariant Hom functor $Hom_R(\_\, , M/N)$ to the exact sequence $0 \to N \to M \to M/N \to 0$.

$(b) \Rightarrow (g)$. By Remark 2.7, the hypothesis implies that $\alpha_N^M \preceq \omega_0^{M/N}$. Since $\omega_0^{M/N}$ is a radical, we have $\overline{\alpha_N^M} \preceq \omega_0^{M/N}$.

$(g) \Rightarrow (h)$. We have $N \leq \alpha_N^M(M) \leq \overline{\alpha_N^M}(M) \leq \omega_0^{M/N}(M) \leq N$, the last inequality by Definition 2.6.

$(h) \Rightarrow (a)$ is obvious. $\square$

Recall that $M \in$ Mod-R is called *quasi-projective* or *M-projective* if for each epimorphism $g : M \to L$ and each homomorphism $k : M \to L$ there exists an endomorphism $h : M \to M$ such that $gh = k$.

Example 4.3. *Let $M \in$ Mod-R and suppose that M is quasi-projective. Then condition (f) of Proposition 4.2 holds. Therefore each fully invariant proper submodule of M is a radical submodule.*

The following proposition gives equivalent conditions for a ring to be right radical.

Proposition 4.4. *For a ring R the following conditions are equivalent.*

(a) *R is right radical.*
(b) *There is a generator G of Mod-R such that G is a radical module.*
(c) *Every $M \in$ Mod-R is a radical module.*

Proof. $(a) \Rightarrow (b)$ and $(c) \Rightarrow (a)$ are obvious.

$(b) \Rightarrow (c)$. By hypothesis, there is a generator $G \in \overline{\mathbb{T}}_\sigma$, for some radical $\sigma \neq \widehat{1}$. By Remark 2.5, $\overline{\mathbb{T}}_\sigma$ is closed under direct sums and epimorphisms, so that $M \in \overline{\mathbb{T}}_\sigma$ for each $M \in$ Mod-R. $\square$

5. A subclass of the class of right radical rings

Throughout this section we assume that Z is an integral domain. Our aim is to prove that every countable Z-coinitial and right hereditary ring R is a right radical ring for any integral domain Z. Following [6], we consider the radical ω_0^R. We will prove that there exists a right R-module M such that $\omega_0^R(M) = R$. The following is a consequence of Definition 2.6 and Remark 2.7.

Remark 5.1. *Let R be a ring and M a right R-module that contains R as a submodule. The following conditions are equivalent.*

(a) $\omega_0^R(M/R) = 0$.
(b) $\omega_0^R \preceq \omega_0^{M/R}$.
(c) *M/R is cogenerated by R.*

Let Z be an integral domain, and suppose that the ring R is a Z-coinitial ring. Among the right R-modules M that satisfy any (and hence all) of the conditions of Remark 5.1, we characterize those such that $\omega_0^R(M) = R$ as the ones for which $Ext_R(M/R, R)$ has a Z-torsionfree element. When R is a countable Z-coinitial and right hereditary ring, we prove the existence of such a Z-torsionfree element when $M/R = R^{\mathbb{N}}$, clearly satisfying the conditions of Remark 5.1. In what follows, $(\_)^*$ will denote the Hom contravariant functor $Hom_R(\_, R)$.

The following results, namely, Proposition 5.2 and Proposition 5.4, generalize those stated by Mines in [15] and proved in [6, Theorem 3.6].

Proposition 5.2. *Let R be a ring. Let $M \in$ Mod-R be such that $R \leq M$ and let $f : R \to M$ be the inclusion map. Assume that $\omega_0^R(M/R) = 0$. The following conditions are equivalent:*

(a) $\omega_0^R(M) = R$.

(b) $f^* : M^* \to R^*$ *is the zero homomorphism.*

Proof. $(a) \Rightarrow (b)$. Suppose that $\omega_0^R(M) = R$. Let $\phi \in M^*$. Then $f^*(\phi) = \phi f = 0$, since $R \leq Ker(\phi)$, by Definition 2.6.
$(b) \Rightarrow (a)$. Suppose that $f^* = 0$. For each $\phi \in M^*$ we have $\phi f = f^*(\phi) = 0$, which means that $R \leq Ker(\phi)$ and we conclude that $R \leq \omega_0^R(M)$. On the other hand, by Proposition 2.3, $\omega_0^R(M)/R = \omega_0^R(M/R) = 0$, being ω_0^R a radical. Therefore $\omega_0^R(M) = R$. □

Remark 5.3. *Notice that $R^* = End^l(R_R)$ and that there is a ring isomorphism $\lambda : R \to End^l(R_R)$. It follows that, if Z is an integral domain, R is left (respectively right) Z-coinitial if, and only if, R^* is left (respectively right) Z-coinitial. Notice also that if $f \in Hom_R(R, M)$, then $\mathrm{Im} f^* = \{k \in End^l(R_R) \mid k = hf,\ h \in M^*\}$ is a left ideal of R^*.*

Proposition 5.4. *Let R be a left Z-coinitial ring. Let $M \in$ Mod-R be such that $R \leq M$ and let $f : R \to M$ be the inclusion map. For an exact sequence $E : 0 \to R \xrightarrow{f} M \to M/R \to 0$ the following conditions are equivalent:*

(a) $f^* : M^* \to R^*$ *is the zero homomorphism.*

(b) $[E] \in Ext_R(M/R, R)$ *is Z-torsionfree.*

Proof. $(a) \Rightarrow (b)$. Suppose that $n[E] = 0$ for some $n \in Z\backslash\{0\}$. The following diagram:

$$\begin{tikzcd}
E: & 0 \arrow[r] & R \arrow[r, "f"] \arrow[d, "\mu_n^R = n1_R"'] & M \arrow[r] \arrow[d, "\alpha"'] & M/R \arrow[r] \arrow[d, equal] & 0 \\
\mu_n^R E: & 0 \arrow[r] & R \arrow[r, "f_1"] & M_1 \arrow[r] & M/R \arrow[r] & 0
\end{tikzcd}$$

commutes and has exact rows for some $M_1 \in$ Mod-R and some R-homomorphisms f_1 and α. Now, $\mu_n^R E$ splits, since $[\mu_n^R E] = n[E] = 0$. It follows that there is an R-homomorphism $g_1 : M_1 \to R$ such that $g_1 f_1 = 1_R$. Since $n \neq 0$, we have $f^*(g_1\alpha) = g_1\alpha f = \mu_n^R \neq 0$. Therefore $f^* \neq 0$.

$(b) \Rightarrow (a)$. Suppose that $f^* \neq 0$. Since $\mathrm{Im} f^*$ is a left ideal of R^*, which is also left Z-coinitial, there exists $n \in Z\backslash\{0\}$ such that $\mu_n^R = n1_R \in \mathrm{Im} f^*$, i.e., $(\mu_n^R)^*(1_R) = \mu_n^R = f^*(\mu')$ for some $\mu' \in M^*$. Consider again a commutative diagram as the one above. By applying the functor $(\_)^*$ to this diagram, and adding the corresponding connecting homomorphisms δ and δ_1 from the long exact sequences we obtain:

$$\begin{tikzcd}
0 \arrow[r] & (M/R)^* \arrow[r] \arrow[d, equal] & {M_1}^* \arrow[r, "{f_1}^*"] \arrow[d, "\alpha^*"'] & R^* \arrow[r, "\delta_1"] \arrow[d, "(\mu_n^R)^*"'] & Ext_R(M/R, R) \arrow[d, equal] \\
0 \arrow[r] & (M/R)^* \arrow[r] & M^* \arrow[r, "f^*"] & R^* \arrow[r, "\delta"] & Ext_R(M/R, R),
\end{tikzcd}$$

Since $\delta_1(1_R) = \delta(\mu_n^R)^*(1_R) = \delta f^*(\mu') = 0$, it follows that $1_R \in Ker(\delta_1) = {f_1}^*(M_1^*)$. Thus there is $g_1 \in {M_1}^*$ such that $g_1 f_1 = {f_1}^* g_1 = 1_R$. Therefore the extension $\mu_n^R E$ splits, that is, $n[E] = 0$. We conclude that $[E]$ is a Z-torsion element in $Ext_R(M/R, R)$. □

Therefore, to conclude that a Z-coinitial ring R is a right radical ring it is enough to construct a Z-torsionfree element in some extension group $Ext_R(P, R)$, where P is cogenerated by R. We will focus in the case when $P = R^{\mathbb{N}}$, which we assume throughout this subsection.

Proposition 5.5. *Let R be a left or right Z-coinitial ring. If R is right hereditary, then $Ext_R(P, R)$ is a divisible Z-module.*

Proof. By Proposition 3.4.4, R and hence P, are torsionfree Z-modules. By Corollary 2.15, we conclude that $Ext_R(P, R)$ is a divisible Z-module. □

In [17, Theorem 8] Nunke describes the structure of the abelian group $Ext_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$. Using some ideas of Nunke's result we prove, under the hypothesis of Proposition 5.5, that $Ext_R(P, R) \neq 0$ when R is countable. Then, using the fact that $Ext_R(P, R)$ is divisible in this case, we construct a Z-torsionfree element.

Proposition 5.6. *Let R be a countable, right Z-coinitial ring and right hereditary ring. Then $Ext_R(P, R) \neq 0$.*

Proof. First we claim that $|P^*| \leq \aleph_0$. Let $\Phi : R^{(\mathbb{N})} \to P^*$ be the function that sends each $y = (y_i)_{i\in\mathbb{N}}$ to the R-homomorphism $\Phi_y : P \to R$ such that, for each $z \in P$, $\Phi_y(z) = \sum_{n\in supp(y)} y_n\pi_n(z)$, where $supp(y)$ denotes the support of y in $R^{(\mathbb{N})}$, and π_n are the natural projections. We will show that Φ is surjective. Let

$\varphi \in P^*$. In particular, φ is a Z-homomorphism, and, by Proposition 3.4(6), R is a slender Z-module. It follows from Proposition 2.19 that the set $\{n \geq 1 \mid \varphi(\epsilon_n) \neq 0\}$ is finite, where ϵ_n denotes the element of P whose n-th coordinate is 1 and whose other coordinates are 0. Let $\{n \geq 1 \mid \varphi(\epsilon_n) \neq 0\} = \{n_1, \ldots, n_r\}$. Let $\eta = \varphi - \sum_{i=1}^{r} \varphi \iota_{n_i} \pi_{n_i}$, where, for each $i \in \{1, \ldots, r\}$, ι_{n_i} denote natural inclusions. Since R is right Z-coinitial, we can apply Proposition 2.21 to conclude that $\eta = 0$. Thus $\varphi = \sum_{i=1}^{r} \varphi \iota_{n_i} \pi_{n_i}$. Moreover, if $z = (z_i)_{i \in \mathbb{N}} \in P$, then for each i we have that, $(\varphi \iota_i \pi_i)(z) = \varphi \iota_i(z_i) = \varphi(\epsilon_i z_i) = \varphi(\epsilon_i) z_i = \varphi(\epsilon_i) \pi_i(z)$. Thus $\varphi(z) = \sum_{i=1}^{r} \varphi(\epsilon_{n_i}) \pi_{n_i}(z)$. Therefore Φ is surjective, so that $|P^*| \leq |R^{(\mathbb{N})}| = |R| = \aleph_0$, which proves our claim.

By Proposition 3.4.5, R is a reduced Z-module, so that it is not divisible. Hence there exists $m \in Z \backslash \{0\}$ such that $mR \neq R$. Moreover, mR is not a direct summand of R because R is a right uniform Z-module. It follows that the short exact sequence $0 \to R \overset{\mu_m^R}{\to} R \to R/mR \to 0$ does not split, which implies that $Ext_R(R/mR, R) \neq 0$. We claim that $|Ext_R(P/mP, R)| \geq 2^{\aleph_0}$. Being R right hereditary, the monomorphism $0 \to (R/mR)^{(\mathbb{N})} \to (R/mR)^{\mathbb{N}}$ induces an epimorphism $Ext_R((R/mR)^{\mathbb{N}}, R) \to Ext_R((R/mR)^{(\mathbb{N})}, R) \to 0$. Since $(Ext_R(R/mR, R))^{\mathbb{N}} \cong Ext_R((R/mR)^{(\mathbb{N})}, R)$, we have $|Ext_R((R/mR)^{\mathbb{N}}, R)| \geq |(Ext_R(R/mR, R))^{\mathbb{N}}| \geq 2^{\aleph_0}$. Now, $P/mP \cong (R/mR)^{\mathbb{N}}$, so we conclude that $|Ext_R(P/mP, R)| \geq 2^{\aleph_0}$.

Now consider in Mod-R the short exact sequence $0 \to P \overset{\mu_m^P}{\to} P \to P/mP \to 0$, which induces the exactness of the sequence $0 \to (P/mP)^* \to P^* \to P^* \to Ext_R(P/mP, R) \to Ext_R(P, R) \to Ext_R(P, R) \to 0$. If $Ext_R(P, R) = 0$, then we would have an epimorphism $P^* \to Ext_R(P/mP, R) \to 0$, but as we have seen before $|Ext_R(P/mP, R)| > |P^*|$, a contradiction. Therefore $Ext_R(P, R) \neq 0$. $\square$

Proposition 5.7. *Let R be a countable, right Z-coinitial and right hereditary ring. Then $Ext_R(P, R)$ has a Z-torsionfree element.*

Proof. We begin by showing that there is an independent and infinite family $\{T_n\}_{n \geq 1}$ of R-submodules of P such that $T_n \cong P$ for each $n \geq 1$. Let $\mathfrak{P} = \{p_1, p_2, \ldots\}$ be the set of prime numbers and, for each $n \geq 1$, let $A_n = \{{p_n}^k : k \geq 1\}$ and $T_n = \{z \in P : supp(z) \subseteq A_n\}$. Then $T_n \cong R^{A_n} \cong P$ for each $n \geq 1$. Let $z \in T_k \cap (\sum_{n \neq k} T_n)$. Then $supp(z) \subseteq A_k$. On the other hand, $z = z_1 + \cdots + z_m$, with $z_i \in T_{n_i}$ and $n_i \neq k$ for all $i \in \{1, \ldots, m\}$. Accordingly, $supp(z) \subseteq \cup_{i=1}^{m} supp(z_i) \subseteq \cup_{i=1}^{m} A_{n_i}$. It follows that $supp(z) \subseteq A_k \cap (\cup_{i=1}^{m} A_{n_i}) = \emptyset$, that is, $z = 0$.

Now we construct a Z-torsionfree element in $(Ext_R(P, R))^{\mathbb{N}}$. By Proposition 5.6, there is $x_1 \neq 0$ in $Ext_R(P, R)$. Since $Z \subseteq R$ is countable, we can assume that $Z\backslash\{0\} = \{z_1 = 1, z_2, \dots\}$. Since $Ext_R(P,R)$ is Z-divisible, there exists $x_2 \in Ext_R(P,R)$ such that $x_2 z_2 = x_1$, and we construct a sequence $x_1 = x, x_2, \dots$ of nonzero elements in $Ext_R(P,R)$ such that for each $n \in \mathbb{N}$, there exists $x_{n+1} \in Ext_R(P,R)$ such that $x_{n+1}z_{n+1} = x_n$. Then $x = (x_1, x_2, \dots) \in (Ext_R(P,R))^{\mathbb{N}}$ is Z-torsionfree.

Finally, consider the exact sequence $0 \to \oplus_{n\geq 1} T_n \xrightarrow{i} P$. Under the hypothesis that R is right hereditary, and considering that:

$$(Ext_R(P,R))^{\mathbb{N}} \cong \prod_{n\geq 1} Ext_R(T_n, R) \cong Ext_R(\oplus_{n\geq 1} T_n, R),$$

we have the exactness of the sequence $Ext_R(P,R) \xrightarrow{(i)^*} (Ext_R(P,R))^{\mathbb{N}} \to 0$. Then, being $(i)^*$ an epimorphism and since $x \in (Ext_R(P,R))^{\mathbb{N}}$ is Z-torsionfree, there is $w \in Ext_R(P,R)$ such that $(i)^*(w) = x$. Then w is a Z-torsionfree element in $Ext_R(P,R)$. □

Now we prove the main theorem of this section. Notice that we use the fact that R is both left and right Z-coinitial.

Theorem 5.8. *Let R be a countable, Z-coinitial and right hereditary ring. Then there exists $M \in$ Mod-R such that $\omega_0^R(M) = R$. Therefore R is a right radical ring.*

Proof. By Proposition 5.7, there exists an exact sequence $E : 0 \to R \xrightarrow{f} M \to P \to 0$ such that $[E]$ is a Z-torsionfree element in $Ext_R(P,R)$. It follows from Proposition 5.4 that $f^* : M^* \to R^*$ is the zero homomorphism. Since $M/R \cong P$ is cogenerated by R, we conclude, by Proposition 5.2, that $\omega_0^R(M) = R$. □

6. R-pr is not a set for right radical rings

In this section we prove that for every right radical ring the lattice of preradicals is not a set. We use the same technique as the one used in [6] for abelian groups.

Definition 6.1. Let $\sigma \in R$-pr, $M \in$ Mod-R, and $x \in M$. We define the *σ-height* of x in M, denoted by $h(x,\sigma,M)$, as the largest ordinal γ such that $x \in \sigma^{\gamma}(M)$, in case such γ exists. Otherwise, we define $h(x,\sigma,M) = \infty$.

As an immediate consequence we have the following properties.

Remark 6.2. *Let $\sigma \in R$-pr, $M, N \in$ Mod-R, $x \in M$ and $\gamma \in \mathcal{OR}$. Then:*

1. *$x \in \sigma^{\gamma}(M)$ if, and only if, $h(x,\sigma,M) \geq \gamma$.*
2. *If $f \in Hom_R(M,N)$ and $h(x,\sigma,M) \geq \gamma$ then $h(f(x),\sigma,N) \geq \gamma$.*

Theorem 6.3. *Let $\sigma \in R$-pr be a radical. Let $\{M_n\}_{n\geq 1}$ be a family in Mod-R such that, for each $n \geq 1$ we have $\sigma(M_{n+1}) = M_n$. Then for each $\gamma \in \mathcal{OR}$ and each $n \geq 1$ there exists $M(\gamma, n) \in$ Mod-R such that $\sigma^\gamma(M(\gamma, n)) = M_n$.*

Proof. We proceed by transfinite induction on γ. For each $n \geq 1$, take $M(0, n) = M_n$. Assume that $M(\gamma, n)$ is defined for each $n \geq 1$ in such a way that $\sigma^\gamma(M(\gamma, n)) = M_n$. For $n \geq 1$ define $M(\gamma+1, n) = M(\gamma, n+1)$. Notice that $\sigma^{\gamma+1}(M(\gamma+1, n)) = \sigma(\sigma^\gamma(M(\gamma+1, n))) = \sigma(\sigma^\gamma(M(\gamma, n+1))) = \sigma(M_{n+1}) = M_n$. Now suppose that η is a limit ordinal and assume that $M(\gamma, n)$ is defined for each ordinal $\gamma < \eta$ and $n \geq 1$. Consider the following short exact sequence:

$$0 \to M_n \hookrightarrow M(\gamma, n) \to M(\gamma, n)/\sigma^\gamma(M(\gamma, n)) \to 0$$

which induces the exactness of the following sequence:

$$0 \to \underset{\gamma<\eta}{\oplus} M_n \hookrightarrow \underset{\gamma<\eta}{\oplus} M(\gamma, n) \to \underset{\gamma<\eta}{\oplus} (M(\gamma, n)/\sigma^\gamma(M(\gamma, n))) \to 0$$

Let $\nabla : \underset{\gamma<\eta}{\oplus} M_n \longrightarrow M_n$ denote the codiagonal map and define $M(\eta, n)$ to be a corresponding pushout such that $M_n \leq M(\eta, n)$ and completes the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \underset{\gamma<\eta}{\oplus} M_n \arrow[r, hook] \arrow[d, "\nabla"'] & \underset{\gamma<\eta}{\oplus} M(\gamma, n) \arrow[r] \arrow[d] & \underset{\gamma<\eta}{\oplus} (M(\gamma, n)/\sigma^\gamma(M(\gamma, n))) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M_n \arrow[r, hook] & M(\eta, n) \arrow[r] & \underset{\gamma<\eta}{\oplus} (M(\gamma, n)/\sigma^\gamma(M(\gamma, n))) \arrow[r] & 0
\end{tikzcd}$$

From Proposition 2.2 and Remark 2.4, it follows that $\sigma^\eta(\underset{\gamma<\eta}{\oplus}(M(\gamma, n)/\sigma^\gamma(M(\gamma, n))))$ $= \underset{\gamma<\eta}{\oplus} \sigma^\eta((M(\gamma, n)/\sigma^\gamma(M(\gamma, n)))) \leq \underset{\gamma<\eta}{\oplus} \sigma^\gamma((M(\gamma, n)/\sigma^\gamma(M(\gamma, n)))) = 0$. By Proposition 2.1, we have $(\sigma^\eta(M(\eta, n)) + M_n)/M_n \leq \sigma^\eta(M(\eta, n)/M_n) = 0$. Therefore $\sigma^\eta(M(\eta, n)) \leq M_n$. In order to prove the opposite inequality, let $x \in M_n = \sigma^\gamma(M(\gamma, n))$. For each ordinal γ such that $\gamma < \eta$ we consider the natural inclusion $\iota_\gamma : M_n \to \underset{\gamma<\eta}{\oplus} M_n$. By Remark 6.2.1, we have $h(\iota_\gamma(x), \sigma, \underset{\delta<\eta}{\oplus} M(\delta, n)) \geq \gamma$ and considering the previous diagram and Remark 6.2.2 we have $h(x, \sigma, M(\eta, n)) \geq \gamma$, since for each ordinal γ, $\nabla(\iota_\gamma(x)) = x$. Therefore $h(x, \sigma, M(\eta, n)) \geq \gamma$ for each ordinal $\gamma < \eta$, so that $h(x, \sigma, M(\eta, n)) \geq \eta$, which means that $x \in \sigma^\eta(M(\eta, n))$. We conclude that $\sigma^\eta(M(\eta, n)) = M_n$ for the limit ordinal η and this concludes the proof. $\square$

Now we are able to construct a class of preradicals which is not a set. First we need the following definition.

Definition 6.4. Let $M \in$ Mod-R and $\sigma \in R$-pr. The σ-*length* of M, denoted by $l(\sigma, M)$, is the least ordinal λ such that $\sigma^{\lambda}(M) = \sigma^{\lambda+1}(M)$. Notice that this ordinal always exists, since $\{\sigma^{\lambda}(M)\}_{\lambda \in \mathcal{OR}}$ is a descending chain of submodules of M, and M is a set.

Theorem 6.5. *Let R be a right radical ring. Let $\widehat{1} \neq \sigma \in R$-pr be a radical and $M \in$ Mod-R be such that $\sigma(M) = R$. Then:*

(1) *For each $\gamma \in \mathcal{OR}$ there exists $N_\gamma \in$ Mod-R such that $l(\sigma, N_\gamma) = \gamma$.*
(2) *There exists a chain of radicals in R-pr which is not a set.*

Proof. To prove (1), by Proposition 4.4, we have that $R \in \overline{\mathbb{T}}_\sigma$ implies that $M \in \overline{\mathbb{T}}_\sigma$ for each $M \in$ Mod-R. Therefore there is a set $\{M_n\}_{n \geq 1}$ in Mod-R such that $M_0 = \sigma(R)$, $M_1 = R$, and $\sigma(M_{n+1}) = M_n$. By Theorem 6.3, for each $n \geq 1$ and for each $\gamma \in \mathcal{OR}$ there exists $M(\gamma, n) \in$ Mod-R such that $\sigma^\gamma(M(\gamma, n)) = M_n$. In particular, for each ordinal γ, $\sigma^\gamma M(\gamma, 1) = M_1 = R$, so that, by Proposition 2.3 and Remark 2.4, $\sigma^\gamma(M(\gamma,1)/R) = \sigma^\gamma(M(\gamma,1))/R = 0$. It follows that $l(\sigma, M(\gamma,1)/R) \leq \gamma$. We claim that $l(\sigma, M(\gamma,1)/R) = \gamma$. If this is not the case, there would be an ordinal $\eta < \gamma$ such that $\sigma^\eta(M(\gamma,1)/R) = 0$, that is, $\sigma^\eta(M(\gamma,1)) = R$. Hence $\sigma(R) = \sigma(\sigma^\eta(M(\gamma,1))) = \sigma^{\eta+1}(M(\gamma,1)) \geq \sigma^\gamma(M(\gamma,1)) = M_1 = R$. By Proposition 2.9, we would have $\sigma = \widehat{1}$. We conclude that $\sigma^\eta(M(\gamma,1)/R) \neq 0$ for all $\eta < \gamma$, which means that $l(\sigma, M(\gamma,1)/R) = \gamma$.

Now we shall prove (2). Let us consider the chain $\{\sigma^\gamma\}_{\gamma \in \mathcal{OR}}$. By (1), for each $\gamma \in \mathcal{OR}$ there exists $N_\gamma \in$ Mod-R such that $l(\sigma, N_\gamma) = \gamma$. Now suppose that $\delta, \eta \in \mathcal{OR}$ with $\delta < \eta$. By Definition 6.4, we have that $\sigma^\delta(N_\eta) \neq \sigma^\eta(N_\eta)$, and this implies that $\sigma^\delta \neq \sigma^\eta$. Therefore $\{\sigma^\gamma\}_{\gamma \in \mathcal{OR}}$ is not a set. □

As an immediate consequence of Theorem 5.8 and Theorem 6.5 we have the following result.

Corollary 6.6. *Let R be a countable, Z-coinitial and right hereditary ring. Then:*

(1) *For each ordinal γ there exists $M \in$ Mod-R with $l(\omega_0^R, M) = \gamma$.*
(2) *There exists a chain of radicals in R-pr which is not a set.*

7. Some rings for which R-pr is not a set.

We start remarking some constructions of rings for which R-pr is not a set, that arise from the results stated at the end of Section 2.1.

Remark 7.1. *Let R be a countable, Z-coinitial and right hereditary ring. For a ring S we have that S-pr is not a set if any of the following conditions hold:*

(1) $S = \prod_{i=1}^{n} R_i$ *and* $R_j = R$ *for some* $j \in \{1, \dots, n\}$.
(2) S *is Morita equivalent to* R, *in particular, if* $S = \mathbb{M}_n(R)$ *for some* $n > 1$.
(3) R *is isomorphic to a factor ring of* S.

Now we present some examples of rings that satisfy the hypothesis of Corollary 6.6, and hence they are such that R-pr is not a set.

Example 7.2. $\mathbb{Z}$ *satisfies all conditions of Corollary 6.6. In fact as Nunke proved in* [17], $Ext_{\mathbb{Z}}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ *has an element of infinite order. Using this fact, Fay, Oxford and Walls proved in* [6] *that* $\omega_0^{\mathbb{Z}}(M) = \mathbb{Z}$. *They also constructed explicitly the chain of radicals which is in correspondence with* $\mathcal{OR}$.

Example 7.3. *Recall Example 2.16. The ring* $\mathbb{Z}_{(p)}$, *the localization of* $\mathbb{Z}$ *by the prime* p *is a countable discrete valuation ring of zero characteristic and its residue field is a finite field, and hence of prime characteristic. In particular, it is a torsion* $\mathbb{Z}$*-module. By Corollary 3.7,* $\mathbb{Z}_{(p)}$ *is a* $\mathbb{Z}$*-coinitial ring. Being a Dedekind domain, it is hereditary.*

Example 7.4. *In general, let* $R = \mathcal{O}_K$ *be the ring of integers of a number field* K. *Then* R *is a Dedekind domain of zero characteristic. By Theorem 2.18, it is hereditary.* R *is countable, being the ring of algebraic integers of a finite extension of the countable field* $\mathbb{Q}$. *This ring* R *has the finite norm property, that is, for every* $0 \neq I \leq R$ *the factor ring* R/I *is finite (see* [16, I.1.4]*), and therefore it is torsion as a* $\mathbb{Z}$*-module. By Proposition 3.2,* R *is* $\mathbb{Z}$*-coinitial. By Theorem 3.6, for every non-zero prime ideal* P *of* R, *the localization* R_P *is also* $\mathbb{Z}$*-coinitial, and it is a countable discrete valuation ring. Therefore* R *and each of its localizations* R_P *satisfy the conditions of Corollary 6.6.*

Example 7.5. *Let* D *be a countable Dedekind domain. By Theorem 2.18,* D *is hereditary and by Remark 3.3, it is* D*-coinitial. In particular, let* F *be a countable field, for example the field* $\mathbb{Q}$ *of rational numbers. Then the ring of polynomials* $F[x]$ *is a countable Dedekind domain.*

Example 7.6. *(See Section 3.3). Let* Z *be a countable Dedekind domain with field of fractions* K. *Let* A *be a division ring finite dimensional over its center* K. *Let* R *be a right hereditary* Z*-order in* A. *Then* R *satisfies the conditions of Corollary 6.6. In particular, this is the case when* R *is a maximal* Z*-order in* A, *for example* $R = \Lambda_0$ *(see Example 3.11).*

Acknowledgement. The authors are grateful to the referee of this paper for all the detailed suggestions and comments, which were very useful when extending the concept of coinitial ring.

References

[1] F. W. Anderson and K. R. Fuller, Rings and Categories of Modules, Spinger-Verlag, New York, 1992.

[2] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley, Reading, Mass., 1969.

[3] L. Bican, P. Jambor, T. Kepka, P. Nemec, *Preradicals*, Comment. Math. Univ. Carolinae, 15(1) (1974), 75–83.

[4] L. Bican, T. Kepka, P. Nemec, Rings, Modules and Preradicals, Marcel Dekker, New York, 1982.

[5] R. Dimitric, *Slender modules over domains*, Comm. Algebra, 11(15) (1983), 1685–1700.

[6] T. H. Fay, E. P. Oxford, and G. L. Walls, *Preradicals in abelian groups*, Houston J. Math, 8(1) (1982), 39–52.

[7] R. Fernández-Alonso, F. Raggi, J. Ríos, H. Rincón, and C. Signoret, *The lattice structure of preradicals*, Comm. Algebra, 30(3) (2002), 1533–1544.

[8] R. Fernández-Alonso, F. Raggi, J. Ríos, H. Rincón, and C. Signoret, *The lattice structure of preradicals II: Partitions*, J. Algebra Appl., 1(2) (2002), 201–214.

[9] R. Fernández-Alonso, S. Gavito, *The lattice of preradicals over local uniserial rings*, J. Algebra Appl., 5(6) (2006), 731–746.

[10] L. Fuchs, Infinite Abelian Groups, Vol. II, Academic Press, New York, 1970.

[11] L. Fuchs and L. Salce, Modules over Valuation Domains, Pure and Applied Mathematics, Marcel Dekker, New York, 1985.

[12] K. R. Goodearl and R. B. Warfield Jr., An Introduction to Noncommutative Noetherian Rings, London Mathematical Society Student Texts 16, Cambridge University Press, Cambridge, 2004.

[13] M. Hazewinkel, N. Gubareni and V.V. Kirichenko, Algebras, Rings and Modules, Kluwer Academic Publishers, Dordrecht, 2004.

[14] S. Mac Lane, Homology, Spinger-Verlag, Berlin, Heidelberg, 1995.

[15] R. Mines, Radicals and Torsion Free Groups, Seminar Notes, New Mexico State University, Las Cruces, 1971.

[16] W. Narkiewicz, Elementary and Analytic Theory of Algebraic Numbers, Springer-Verlag, Berlin, Heidelberg, 2004.

[17] R. J. Nunke, *Slender groups*, Acta Sci. Math. Szeiged, 23 (1962), 67–73.

[18] I. Reiner, Maximal Orders, London Mathematical Society Monographs New series 28, Great Britain, 2003.

[19] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, Orlando, Florida, 1979.

[20] S. V. Rychkov and B. Thome, *Slender groups and related Concepts*, Comm. Algebra, 14(2) (1986), 333–387.

[21] B. Stenström, Rings of Quotients, Springer-Verlag, Berlin, 1975.

Rogelio Fernández-Alonso, Silvia Gavito and Henry Chimal-Dzul
Departamento de Matemáticas
Universidad Autónoma Metropolitana - Iztapalapa
San Rafael Atlixco 186
09340 México, D.F.
e-mails: rfg@xanum.uam.mx (R. Fernández-Alonso)
silvia_gavito@yahoo.com (S. Gavito)
henrychimal@gmail.com (H. Chimal-Dzul)