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CANCELLATION PROPERTIES IN IDEAL SYSTEMS OF MONOIDS

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Abstract. We pursue the work by M. Fontana, K.A. Loper and R. Matsuda [2]. Let D be an integral domain, let $\mathrm{F}(D)$ (resp., $\mathrm{f}(D)$) be the set of non-zero (resp., finitely generated) fractional ideals of D, let $\star$ be a semistar operation on D. They showed that if $\star$ satisfies $(FF_1)^\star = (FF_2)^\star$ implies $F_1^\star = F_2^\star$ for every $F, F_1, F_2 \in \mathrm{f}(D)$, then $\star$ need not satisfy $(FG_1)^\star = (FG_2)^\star$ implies $G_1^\star = G_2^\star$ for every $F \in \mathrm{f}(D)$ and every $G_1, G_2 \in \mathrm{F}(D)$. In this paper, we show its analogy for monoids.

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1. Introduction

Let D be an integral domain with quotient field $K = \mathrm{q}(D)$. Let $\bar{\mathrm{F}}(D)$ be the set of non-zero D-submodules of K, let $\mathrm{F}(D)$ be the set of non-zero fractional ideals G of D, i.e., $G \in \bar{\mathrm{F}}(D)$ and $dG \subset D$ for some $d \in D \setminus \{0\}$, and let $\mathrm{f}(D)$ be the set of non-zero finitely generated D-submodules of K. A semistar operation on D is a mapping $\star : \ \bar{\mathrm{F}}(D) \longrightarrow \bar{\mathrm{F}}(D), E \longmapsto E^\star$, such that the following properties hold for every $x \in K \setminus \{0\}$ and every $E, H \in \bar{\mathrm{F}}(D)$: $(xE)^\star = xE^\star; E \subset E^\star; (E^\star)^\star = E^\star; E \subset H$ implies $E^\star \subset H^\star$. Recently, M. Fontana, K.A. Loper and R. Matsuda [2] showed that if a semistar operation $\star$ on D satisfies $(FF_1)^\star = (FF_2)^\star$ implies $F_1^\star = F_2^\star$ for every $F, F_1, F_2 \in \mathrm{f}(D)$, then $\star$ need not satisfy $(FG_1)^\star = (FG_2)^\star$ implies $G_1^\star = G_2^\star$ for every $F \in \mathrm{f}(D)$ and every $G_1, G_2 \in \mathrm{F}(D)$.

A subsemigroup S, with $0 \in S$, of a torsion-free abelian additive group is called a grading monoid (or, a g-monoid). It is known that we can define on S the notions of ideal, valuation, integral closure, (Krull) dimension, etc., and we have an ideal theory on S similar to the classical one on integral domains. In this paper, we pursue [2] and, in Section 3, we show that if a semistar operation $\star$ on a g-monoid S satisfies $(F + F_1)^\star = (F + F_2)^\star$ implies $F_1^\star = F_2^\star$ for every $F, F_1, F_2 \in \mathrm{f}(S)$, then $\star$ need not satisfy $(F + G_1)^\star = (F + G_2)^\star$ implies $G_1^\star = G_2^\star$ for every $F \in \mathrm{f}(S)$

and every $G_1, G_2 \in \mathrm{F}(S)$. Section 2 is a review, and in Section 4 we give some semigroup versions of some propositions in M. Fontana and K.A. Loper [1].

2. A Review

A subsemigroup S, with $0 \in S$, of a torsion-free abelian additive group is called a g-monoid. Throughout the paper, S denotes a g-monoid with $S \supsetneqq \{0\}$. For example, let L be a torsion-free abelian additive group. Then $L[X] := \{a + kX \mid a \in L$ and $0 \leq k \in \mathbb{Z}$ is a g-monoid called the polynomial semigroup of X over L, where X is an indeterminate. We use the symbol $\subset$ (resp., $\subsetneqq$) for the large inclusion (resp., the strict inclusion). For the general theory of g-monoids, we refer to [4] and [6]. Let S be a g-monoid, the additive group $\{s - s' \mid s, s' \in S\}$ is called the quotient group of S, and is denoted by $\mathrm{q}(S)$. A non-empty subset I of S is called an ideal of S if $S + I \subset I$. If $S \subsetneqq \mathrm{q}(S)$, then S has a unique maximal ideal M, and M is the set of non-units of S. Let $\bar{\mathrm{F}}(S)$ be the set of non-empty subsets E of $\mathrm{q}(S)$ such that $S + E \subset E$. An element E of $\bar{\mathrm{F}}(S)$ is called a fractional ideal of S if $s + E \subset S$ for some $s \in S$. Let $\mathrm{F}(S)$ be the set of fractional ideals of S, and let $\mathrm{f}(S)$ be the set of finitely generated fractional ideals of S.

If a mapping $E \longmapsto E^\star$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ satisfies the following conditions, then $\star$ is called a semistar operation on S: For every $a \in \mathrm{q}(S)$ and every $E, H \in \bar{\mathrm{F}}(S)$, we have $(a+E)^\star = a + E^\star$, $E \subset E^\star$, $(E^\star)^\star = E^\star$, $E \subset H$ implies $E^\star \subset H^\star$. The mapping $E \longmapsto \mathrm{q}(S)$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation on S, and is called the e-semistar operation. The mapping $E \longmapsto E$ from $\bar{\mathrm{F}}(S)$ to $\bar{\mathrm{F}}(S)$ is a semistar operation on S, and is called the d-semistar operation. For every $E \in \bar{\mathrm{F}}(S)$, set $E^{-1} := (S : E) := \{x \in \mathrm{q}(S) \mid x + E \subset S\}$, and set $\emptyset^{-1} = \mathrm{q}(S)$. The mapping $E \longmapsto E^{\mathrm{v}} := (E^{-1})^{-1}$ is a semistar operation on S, and is called the v-semistar operation.

Let $\star$ be a semistar operation on a g-monoid S, set $\mathrm{f} := \mathrm{f}(S)$, $\mathrm{g} := \mathrm{F}(S)$, $\mathrm{h} := \bar{\mathrm{F}}(S)$. Let $\mathrm{x} \in \{\mathrm{f}, \mathrm{g}, \mathrm{h}\}$, and let $E \in \mathrm{x}$. We define that E is $\star$-x.y. cancellative (or, $\star$-x.y.), for $\mathrm{y} \in \{\mathrm{f}, \mathrm{g}, \mathrm{h}\}$, if $(E + E_1)^\star \subset (E + E_2)^\star$ implies $E_1^\star \subset E_2^\star$ for every $E_1, E_2 \in \mathrm{y}$ (cf., M. Fontana and K.A. Loper [1, Section 2]). We define that $\star$ is x.y. cancellative (or, x.y.), with $\mathrm{x}, \mathrm{y} \in \{\mathrm{f}, \mathrm{g}, \mathrm{h}\}$, if E is $\star$-x.y. for every $E \in \mathrm{x}$.

Let Γ be a totally ordered abelian additive group, and let v be a mapping from $\mathrm{q}(S)$ onto Γ. If $v(a + b) = v(a) + v(b)$ for every $a, b \in \mathrm{q}(S)$, then v is called a valuation. Γ is called the value group of v, and the set $V := \{a \in \mathrm{q}(S) \mid v(a) \geq 0\}$ is called the valuation semigroup belonging to v. If $V \supset S$, then V is called a valuation oversemigroup of S.

A semistar operation $\star$ is called a w-semistar operation if there is a set $\{V_\lambda \mid \lambda \in \Lambda\}$ of valuation oversemigroups of S such that $E^\star = \cap_\lambda(E+V_\lambda)$ for every $E \in \bar{\mathrm{F}}(S)$. If $\{V_\lambda \mid \lambda \in \Lambda\}$ is the set of all valuation oversemigroups of S, the w-semistar operation defined by $\{V_\lambda \mid \lambda \in \Lambda\}$ is called the b-semistar operation.

We will review a part of [7] for the convenience.

Remark 2.1. ([7, §5]) (1) Let $\star$ be a semistar operation on S. The following conditions are equivalent: $\star$ is h.h., $\star$ is h.g., $\star$ is h.f., and $\star$ coincides with e.

(2) h.h. implies g.h.

g.h. implies g.g.

g.g. implies g.f.

g.h. implies f.h.

g.g. implies f.g.

g.f. implies f.f.

f.h. implies f.g.

f.g. implies f.f.

Proposition 2.2. ([7, §5]) (1) g.h. *need not imply* h.h.

(2) g.f. *need not imply* g.g.

(3) f.h. *need not imply* g.f.

3. An f.f. semistar operation which is not f.g.

Lemma 3.1. *Let $\mathcal{S}$ be a subset of $\bar{\mathrm{F}}(S)$ with $\mathcal{S} \ni \mathrm{q}(S)$ such that, for every $x \in \mathrm{q}(S)$ and every $E \in \mathcal{S}$, $x + E \in \mathcal{S}$. For every $H \in \bar{\mathrm{F}}(S)$, set $H^\star := \cap\{E \in \mathcal{S} \mid E \supset H\}$. Then the mapping $H \longmapsto H^\star$ is a semistar operation on S.*

The proof is similar to that of [3, (32.4) Proposition] and $\star$ in this case is called the semistar operation defined by $\mathcal{S}$.

Lemma 3.2. (cf., [6, (19.6)]) *Let P be a prime ideal of S. Then there is a valuation oversemigroup V of S such that $P = M \cap S$, where M is the maximal ideal of V.*

Lemma 3.3. ([5, p.163]) *Every* w-*semistar operation on S is an* f.h. *semistar operation.*

Let D be an integral domain. The semigroup ring of S over D is denoted by $D[X;S]$. $D[X;S]$ is the ring of elements $\sum_{\text{finite}} a_i X^{s_i}$ for every $a_i \in D$ and $s_i \in S$. For every element $f = \sum a_i X^{t_i} \in D[X;\mathrm{q}(S)]$ with $a_i \neq 0$ and $t_i \neq t_j$ for $i \neq j$, the fractional ideal $\cup_i(S + t_i)$ of S is denoted by $e_S(f)$ (or, by $e(f)$). S is canonically regarded as a subset of $D[X;S]$.

Lemma 3.4. ([5, Proposition 4]) *Let* $\star$ *be an* f.f. *semistar operation on* S*, and set* $S_\star^D := \{\frac{f}{g} \mid f, g \in D[X;S] \setminus \{0\}$ *with* $e(f)^\star \subset e(g)^\star\} \cup \{0\}$.

(1) $S_\star^D$ *is an extension domain of* $D[X;S]$ *with* $\mathrm{q}(S_\star^D) = \mathrm{q}(D[X;S])$*;* $S_\star^D \cap \mathrm{q}(S) = S^\star$.

(2) $S_\star^D$ *is a Bezout domain.*

(3) *For every* $F \in \mathrm{f}(S)$*,* $(FS_\star^D) \cap \mathrm{q}(S) = F^\star$ *and* $FS_\star^D = F^\star S_\star^D$.

$S_\star^D$ is called the Kronecker function ring of S with respect to $\star$ and D, and is denoted also by $\mathrm{Kr}(S, \star, D)$ (or, by $\mathrm{Kr}(S, \star)$).

Let v be a valuation on $\mathrm{q}(S)$. For every $f = a_1X^{s_1} + \cdots + a_nX^{s_n} \in D[X;S] \setminus \{0\}$ with each $a_i \neq 0$ and $s_i \neq s_j$ for every $i \neq j$, we set $w(f) := \inf_i v(s_i)$. Then w is a valuation on the quotient field $\mathrm{q}(D[X;S])$ of $D[X;S]$ called the trivial extension of v to $\mathrm{q}(D[X;S])$.

Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a set of indeterminates, and let $S[X_\lambda \mid \lambda \in \Lambda]$ be the polynomial semigroup of $\{X_\lambda \mid \lambda \in \Lambda\}$ over a g-monoid S. For every element $f = s + \sum_\lambda k_\lambda X_\lambda \in S[X_\lambda \mid \lambda \in \Lambda]$ with $s \in S$ and every $0 \leq k_\lambda \in \mathbb{Z}$, set $w(f) := v(s)$. Then w is a valuation on the quotient group $\mathrm{q}(S[X_\lambda \mid \lambda \in \Lambda])$ of $S[X_\lambda \mid \lambda \in \Lambda]$, and is called the trivial extension of v to $\mathrm{q}(S[X_\lambda \mid \lambda \in \Lambda])$.

Lemma 3.5. ([5, Proposition 8]) *Let* $\star$ *be an* f.f. *semistar operation on* S*, and let* W *be a valuation overring of* $\mathrm{Kr}(S, \star, D)$*. Then the restriction* V *of* W *to* $\mathrm{q}(S)$ *is a valuation oversemigroup of* S*, and* W *is the trivial extension of* V *to* $\mathrm{q}(D[X;S])$.

Lemma 3.6. ([5, Proposition 9]) *Let* $\{V_\lambda \mid \lambda \in \Lambda\}$ *be a set of valuation oversemigroups of* S*. Let* w *be the* w*-semistar operation on* S *defined by the family* $\{V_\lambda \mid \lambda \in \Lambda\}$*, and let* W_λ *be the trivial extension of* V_λ *to* $\mathrm{q}(D[X,S])$*. Then* $\mathrm{Kr}(S, \mathrm{w}) = \cap_\lambda W_\lambda$.

Let $\star$ be a semistar operation on S. We recall that $\star$ is f.f. if $(F+F_1)^\star = (F+F_2)^\star$ implies $F_1^\star = F_2^\star$ for every $F, F_1, F_2 \in \mathrm{f}(S)$, and $\star$ is f.g. if $(F+G_1)^\star = (F+G_2)^\star$ implies $G_1^\star = G_2^\star$ for every $F \in \mathrm{f}(S)$ and every $G_1, G_2 \in \mathrm{F}(S)$.

Example 3.7. *Let* $u_1, u_2, u_3, \cdots$ *be an infinite set of indeterminates over a torsion-free abelian additive group* L*, let* $S := L[u_1, u_2, u_3, \cdots]$ *be the polynomial semigroup over* L*, that is,* $S = \{a + k_1u_1 + k_2u_2 + \cdots + k_nu_n \mid a \in L, 0 \leq k_i \in \mathbb{Z}$*, and* $0 < n \in \mathbb{Z}\}$*. Then* S *is a* g*-monoid, and* $M := \{a + k_1u_1 + k_2u_2 + \cdots + k_nu_n \mid k_i > 0$ *for some* $i\}$ *is the unique maximal ideal of* S*. Consider the following subset of* $\bar{\mathrm{F}}(S)$*:* $\mathcal{S} := \{F^{\mathrm{b}}, x + M, \mathrm{q}(S) \mid x \in \mathrm{q}(S), F \in \mathrm{f}(S)\}$*, where* b *is the* b*-semistar operation on* S*. Let* $\star$ *be a semistar operation on* S *defined by* $\mathcal{S}$*. We claim that* $\star$ *is an* f.f. *semistar operation which is not an* f.g. *semistar operation.*

Proof. Since S is integrally closed, we have $S^{\mathrm{b}} = S$ ([5, Corollary 10 (1)]). By Lemma 3.1, the set $\mathcal{S}$ defines a semistar operation $\star$ on S. Lemma 3.2 implies that $P^{\mathrm{b}} = P$ for every prime ideal P of S. Let $F \in \mathrm{f}(S)$, and let $x \in \mathrm{q}(S)$ with $F \subset x+M$. Since $F^{\mathrm{b}} \subset (x+M)^{\mathrm{b}} = x+M^{\mathrm{b}} = x+M$, we have $F^\star = F^{\mathrm{b}}$, especially $S^\star = S$. Since the b-semistar operation is an f.h. semistar operation by Lemma 3.3, it follows that $\star$ is an f.f. semistar operation.

Set $I := (u_1, u_2) = (u_1 + S) \cup (u_2 + S)$. Clearly, I is a finitely generated prime ideal of S. Hence $I^\star = I^{\mathrm{b}} = I$. We prove that $(I + M)^\star = I^\star$. This will show that $\star$ is not f.g., because $(I + M)^\star = (I + S)^\star$ but $M^\star = M \neq S = S^\star$. We will prove that any fractional ideal in $\mathcal{S}$ which contains $I + M$ also contains I.

(1) Suppose that $I+M \subset x+M$ for some element $x \in \mathrm{q}(S)$. Then $-x+I+M \subset M$. Set $x = a + l_1u_1 + l_2u_2 + l_3u_3 + \cdots$ with $a \in L$ and every $l_i \in \mathbb{Z}$. Since $-x + u_1 + u_3 \in -x + I + M \subset M$, we have $l_2 \leq 0$ and $l_i \leq 0$ for every $i \geq 4$. Since $-x + u_2 + u_4 \in -x + I + M \subset M$, we have $l_1 \leq 0$ and $l_3 \leq 0$. Hence $l_i \leq 0$ for every i, hence $-x \in S$. There are two possibilities.

(1a) If x is not in S, then $-x$ is in M and so $I \subset S \subset x + M$.

(1b) If x is in S, then $x + M = M$, and so $I \subset x + M$.

(2) Suppose that $F \in \mathrm{f}(S)$ is such that $I+M \subset F^{\mathrm{b}} = F^\star$. We extend everything to the b-Kronecker function ring of S.

We have $I\mathrm{Kr}(S, \mathrm{b})M\mathrm{Kr}(S, \mathrm{b}) \subset F^{\mathrm{b}}\mathrm{Kr}(S, \mathrm{b}) = F\mathrm{Kr}(S, \mathrm{b})$ (Lemma 3.4 (3)). Since $\mathrm{Kr}(S, \mathrm{b})$ is a Bezout domain (Lemma 3.4 (2)) and so both $I\mathrm{Kr}(S, \mathrm{b})$ and $F\mathrm{Kr}(S, \mathrm{b})$ are principal ideals. Then we have $M\mathrm{Kr}(S, \mathrm{b}) \subset F\mathrm{Kr}(S, \mathrm{b})(I\mathrm{Kr}(S, \mathrm{b}))^{-1}$, the latter fractional ideal being principal. There are two possibilities.

(2a) $\mathrm{Kr}(S, \mathrm{b}) \subset F\mathrm{Kr}(S, \mathrm{b})(I\mathrm{Kr}(S, \mathrm{b}))^{-1}$. This implies that $I\mathrm{Kr}(S, \mathrm{b}) \subset F\mathrm{Kr}(S, \mathrm{b})$, which implies that $I = I^{\mathrm{b}} \subset F^{\mathrm{b}}$ by Lemma 3.4 (3).

(2b) $\mathrm{Kr}(S, \mathrm{b}) \not\subset F\mathrm{Kr}(S, \mathrm{b})(I\mathrm{Kr}(S, \mathrm{b}))^{-1}$. Rename the principal fractional ideal $F\mathrm{Kr}(S, \mathrm{b})(I\mathrm{Kr}(S, \mathrm{b}))^{-1}$ as J. We know that $M\mathrm{Kr}(S, \mathrm{b}) \subset J$.

If $J \subset \mathrm{Kr}(S, \mathrm{b})$, then we may assume that $M\mathrm{Kr}(S, \mathrm{b})$ is contained in a proper principal ideal of $\mathrm{Kr}(S, \mathrm{b})$. If $J \not\subset \mathrm{Kr}(S, \mathrm{b})$, then $J \cap \mathrm{Kr}(S, \mathrm{b}) \subsetneqq \mathrm{Kr}(S, \mathrm{b})$. Moreover, it is also finitely generated by [3, Proposition 25.4 (1)], hence principal in $\mathrm{Kr}(S, \mathrm{b})$.

In either case $M\mathrm{Kr}(S, \mathrm{b})$ is contained in a proper principal ideal of $\mathrm{Kr}(S, \mathrm{b})$. Assume that $\varphi \in \mathrm{Kr}(S; \mathrm{b})$ is a non-zero non-unit element such that $M\mathrm{Kr}(S, \mathrm{b}) \subset \varphi\mathrm{Kr}(S, \mathrm{b})$. We have $\varphi \in \mathrm{q}(D[X; S_0])$, where $S_0 = L[u_1, \cdots, u_r]$ for some r. Since φ is a non-unit in $\mathrm{Kr}(S, \mathrm{b})$, there must be a valuation overring W of $\mathrm{Kr}(S, \mathrm{b})$ such that φ is a non-unit in W. By Lemma 3.5, there is a valuation oversemigroup V of S such that W is the trivial extension of V to $\mathrm{q}(D[X; S])$. Let V_0 be the contraction of V to $\mathrm{q}(S_0)$. Note that $\mathrm{q}(S)$ is the quotient group of the polynomial semigroup $\mathrm{q}(S_0)[u_{r+1}, u_{r+2}, \cdots]$. Let V' be the trivial extension of V_0 to $\mathrm{q}(S)$, and let W' be the trivial extension of V' to $\mathrm{q}(D[X;S])$. Clearly, V' is a valuation oversemigroup of S, hence $W' \supset \mathrm{Kr}(S, \mathrm{b})$. Let v (resp., w, v_0, v', w') be the valuation belonging to V (resp., W, V_0, V', W'). Since φ is a non-unit of W, we have $w(\varphi) > 0$. By the definition of W', we have $w'(\varphi) = w(\varphi) > 0$. Let $i > r$. By the definition of V', we have $v'(u_i) = 0$, and hence $w'(X^{u_i}) = 0$. On the other hand, $X^{u_i} \in M\mathrm{Kr}(S, \mathrm{b}) \subset \varphi\mathrm{Kr}(S, \mathrm{b}) \subset \varphi W'$, hence $w'(X^{u_i}) \geq w'(\varphi)$; a contradiction. □

4. A Note on M. Fontana and K.A. Loper [1]

M. Fontana and K.A. Loper [1] studied cancellation properties in ideal systems of integral domains. In this Section, we give some semigroup versions of some propositions in [1].

Proposition 4.1. *Let $F \in \mathrm{f}(S)$ which is* d-f.f.*, where* d *is the* d*-semistar operation on S. Then F is principal.*

Proof. We may assume that $I := F$ is an ideal of S. Suppose that I is not principal. Let M be the maximal ideal of S. If $I + S = I + M$, then there is a finitely generated ideal J with $J \subset M$ such that $I \subset I + J$. Then $S = J$; a contradiction. Hence $I + M \subsetneqq I$. Choose an element $x \in I$ with $x \notin I + M$. Since I is not principal, we have $(x) \subsetneqq I$. Choose an element $y \in I$ with $y \notin (x)$, and put $a := x + y$. Then clearly, we have $a \notin (2x)$. There is a maximal member J in the set of ideals that do not contain a, and then $2x \in J$. Since $J \not\ni a$, and since I is d-f.f., $I + J$ does not contain $I + a$. Hence there is $b \in I$ with $b + a \notin I + J$. The case where $b \in (x)$: Then $b + a \in (y + 2x) \subset I + J$; a contradiction. □

Let $\star$ be a semistar operation on S. We set, for every $E \in \bar{\mathrm{F}}(S)$, $E^{\star_f} := \cup\{F^\star \mid F \in \mathrm{f}(S) \text{ with } F \subset E\}$. If $\star = \star_f$, then $\star$ is called of finite type.

Proposition 4.2. (A semigroup version of [1, Proposition 4]) *Let $\star$ be a semistar operation on S, and let $F \in \mathrm{f}(S)$. The following conditions are equivalent.*

(1) *F is* $\star$-f.f.

(2) *F is* $\star_f$-f.f.

(3) *F is* $\star_f$-f.g.

(4) *F is* $\star_f$-f.h.

Proof. Assume that $\star$ is f.f. of finite type. Let $F \subset (F + H)^\star$ for $F \in \mathrm{f}(S)$ and $H \in \bar{\mathrm{F}}(S)$. We need only to prove that $0 \in H^\star$. There is $F_1 \in \mathrm{f}(S)$ with $F_1 \subset H$ such that $F \subset (F + F_1)^\star$. Then $0 \in F_1^\star \subset H^\star$. □

Let $\star$ be a semistar operation on S. A valuation oversemigroup V of S is called a $\star$-valuation oversemigroup if $F^\star \subset F + V$ for every $F \in \mathrm{f}(S)$, and set $E^{\mathrm{b}(\star)} := \cap\{E + V \mid V$ is a $\star$-valuation oversemigroup of $S\}$ (cf., [1, Section 2]).

Proposition 4.3. (A semigroup version of [1, Proposition 7]) *Let $\star$ be a semistar operation on S. Consider the following five propositions.*

(1) *$\star$ is an* f.f. *semistar operation.*
(2) *$\star$ is an* f.g. *semistar operation.*
(3) *$\star$ is an* f.h. *semistar operation.*
(4) *$\star$ is a* w*-semistar operation.*
(5) *$\star = \mathrm{b}(\star)$.*

Then (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1).

Proof. The only implication which is not trivial is (4) $\Longrightarrow$(3). This is proved in [5, p.163]. □

Example 4.4. (A semigroup version of [1, Example 14])

(1) *There is a* w*-semistar operation which is not of finite type.*
(2) *There is a* w*-semistar operation $\star$ such that* $\mathrm{b}(\star) \neq \star$.

For example, let V be a valuation semigroup with maximal ideal M, set $S := V$, and let $\{P_\lambda \mid \lambda \in \Lambda\}$ be the set of prime ideals P of S with $P \subsetneqq M$, and set $V_\lambda := S_{P_\lambda}$ for every λ. Assume that $M = \cup_\lambda P_\lambda$. Let $\star$ be the w-semistar operation defined by the set $\{V_\lambda \mid \lambda \in \Lambda\}$. Then $V^\star = V$ and $M^\star = V$. We have that $\{V_\lambda \mid \lambda \in \Lambda\} \cup \{V\}$ is the set of $\star$-valuation oversemigroups of S. It follows that $\mathrm{b}(\star) = \mathrm{d}$, $\mathrm{b}(\star) \neq \star$, and that w is not of finite type.

Example 4.5. (A semigroup version of [1, Example 15]) *There is an* f.h. *semistar operation which is not a* w*-semistar operation.*

For example, let V be a 1-dimensional valuation semigroup with maximal ideal M. Assume that M is not finitely generated. Let v be the v-semistar operation on V. We have $V^{\mathrm{v}} = V$ and $M^{\mathrm{v}} = V$. Suppose that v is a w-semistar operation. Then v is the w-semistar operation defined by the set $\{V\}$. Hence $M^{\mathrm{v}} = M$; a contradiction. Therefore v is not a w-semistar operation. Since every $F \in \mathrm{f}(V)$ is principal, v is an f.h. semistar operation.

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