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MULTIPLIERS FOR A DERIVED LIE ALGEBRA, A MATRIX EXAMPLE

Louis A. Levy

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Abstract. Using some common assumptions, we extend the study of the Lie algebra multiplier for the strictly upper triangular matrices to the multiplier of its derived algebra. Even though the algebras are very similar, many new elements and cases appear in the derived algebra that do not exist in the original.

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1. Introduction

We begin with a few definitions from [2]. Suppose L is a finite dimensional Lie algebra over a field with characteristic not equal to two.

Definition 1.1. A pair of Lie algebras (C, M) is called a *defining pair* for L if

(1) $L \cong C/M$
(2) $M \subset Z(C) \cap C^2$.

Definition 1.2. If (C, M) is a defining pair for L, then a C of maximal dimension is called a *cover* for L. Likewise an M of maximal dimension is called a *multiplier*.

A multiplier is the Lie algebra analogue of the Schur multiplier from group theory. Please see [3] for a collection of Schur's contributions to this area. In [1] we notice that if $\dim L = n$, then $\dim M$ has an upperbound of $\frac{1}{2}n(n-1)$. Therefore a finite dimensional L will give both M and C finite dimensional. For M maximal, $\dim M = \frac{1}{2}n(n-1) \Leftrightarrow L$ is abelian. Similarly [2] shows that $\dim C \leq \frac{1}{2}n(n+1)$ and that for Lie algebras (unlike groups) all covers are isomorphic. Since the multiplier is abelian all brackets on M are trivial, making isomorphisms immediate, hence the multiplier is unique and we will denote it by $M(L)$. Accordingly, our interest in classifying these Lie algebras lies in finding their dimensions.

The multiplier is classified in [2] for $L =$ the $n \times n$ strictly upper triangular matrices. In this paper we find the dimension of the multiplier for the derived algebra to be $\dim M(L^2) = 2n^2 - 11n + 16$ for $n \geq 6$. For $n = 5, 4, 3$, we have $\dim M(L^2) = 11, 3, 0$ respectively.

2. Structure

Let L be the Lie algebra of $n \times n$ strictly upper triangular matrices and let (C, M) be a defining pair for L^2, therefore $C/M \cong L^2$ and $M \subset Z(C) \cap C^2$. Let E_{ab} denote the usual matrix units that form a basis for L^2. Notice that $L^2 \subset L$ is the Lie algebra of strictly upper triangular matrices with a superdiagonal of zeros, thus $a + 2 \leq b$. Since $L^2 \cong C/M$ each element in L^2 corresponds to an entire coset in C. For each E_{ab} in the basis of L^2 we choose a representative from C in the corresponding coset, called a transversal element and denote it by F_{ab}. Define $u : L^2 \to C$ to be the map taking each E_{ab} in the basis to its transversal F_{ab}, then extend the map u linearly. As in [2] we can completely describe the bracket operation as

$$[F_{st}, F_{ab}] = \begin{cases} F_{sb} + y(s,t,a,b) & \text{if } t = a \\ y(s,t,a,b) & \text{if } t \neq a \end{cases}$$

where $y(s,t,a,b) \in M$. For brevity we often denote $y(s,t,a,b)$ as y_{stab}. Due to the anti-symmetry of the bracket we can assume that either $s < a$ or $s = a$ and $t < b$.

Following the model of [2], first make a change in the choice of F_{rt}. Notice that $[F_{rs}, F_{st}] = F_{rt} + y_{rsst}$ introduces $y's$ for each s, $r + 1 < s < t - 1$. For each pair r, t, one y can be eliminated by the following change of basis. Set

$$G_{rt} = \begin{cases} F_{rt} & \text{if } t - r < 4 \\ F_{rt} + y_{r,t-2,t-2,t} = [F_{r,t-2}, F_{t-2,t}] & \text{otherwise} \end{cases}$$

As stated in [2], "Thus $\{G(r,t)\}$ and $\{F(r,t)\}$ are complete sets of images of matrix units[.] Since the $y's$ are central, $G(r,t)$ and $F(r,t)$ induce the same multiplication in C." We freely use this fact when computing; that is, multiplying by $F's$ or $G's$ gives the same result and we use which ever is most convenient. We proceed to find all dependencies among the $y's$, using the Jacobi identity as our tool. After doing this, we have a basis for the multiplier and it remains to count the number of elements in this basis. We divide our investigation into two sections: elements produced by $[G_{rs}, G_{st}]$ and elements produced by $[G_{st}, G_{ab}]$, where $t \neq a$. Let $J(x, y, z) = 0$ denote the Jacobi identity.

3. $[G_{rs}, G_{st}]$

Theorem 3.1. *$[G_{rs}, G_{st}] = G_{rt}$, except in two cases: $s = r + 2, t = r + 5$ and $s = r + 3, t = r + 6$.*

Proof. Clearly $[G_{rs}, G_{st}] = F_{rt} + y_{rsst} = G_{rt}$ whenever $y_{rsst} = y_{r,t-2,t-2,t}$. Suppose first that $t \geq r+7$ and $s \neq t-2$. The Jacobi identity $J(F_{rs}, F_{s,t-2}, F_{t-2,t}) = 0$ immediately gives $y_{rsst} = y_{r,t-2,t-2,t}$ when $s = r+2, r+3$. However if $s > r+3$ we can use the identities $J(F_{rc}, F_{cs}, F_{st}) = 0$ and $J(F_{rc}, F_{c,t-2}, F_{t-2,t}) = 0$ together to achieve $y_{rsst} = y_{r,t-2,t-2,t}$, where $c = r + 2$.

Suppose $t = r + 6$. If $s = r + 4$ then $y_{rsst} = y_{r,t-2,t-2,t}$ is trivial. The Jacobi identity $J(F_{rs}, F_{s,t-2}, F_{t-2}, t) = 0$ gives $y_{rsst} = y_{r,t-2,t-2,t}$ when $s = r+2$. However if $s = r + 3$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-2,t-2,t}$. Therefore $[G_{r,r+3}, G_{r+3,r+6}] = F_{r,r+6} + y_{r,r+3,r+3,r+6}$ is not necessarily G_{rt}.

Suppose $t = r + 5$. If $s = r + 3$ then $y_{rsst} = y_{r,t-2,t-2,t}$ is trivial. If $s = r + 2$, there are no Jacobi identities available to equate y_{rsst} and $y_{r,t-2,t-2,t}$. Therefore $[G_{r,r+2}, G_{r+2,r+5}] = F_{r,r+5} + y_{r,r+2,r+2,r+5}$ is not necessarily G_{rt}.

Suppose $t = r+4$. The only choice for s is $t-2$. Therefore $[G_{rs}, G_{st}] = G_{rt}$. $\square$

Therefore the only non-zero multiplier element possibilities produced from $[G_{rs}, G_{st}]$ are of the form $y_{r,r+2,r+2,r+5}$ and $y_{r,r+3,r+3,r+6}$. This is an interesting contrast to the $\dim M(L)$ case where no multiplier elements result from $[G_{rs}, G_{st}]$. Please note that we consider $F_{rt} + y_{rsst} \neq G_{rt}$ whenever possible to get M of maximal dimension. We can also make a change in the choice of $y's$ to convert all $F's$ to $G's$ and hence describe C, but $\dim M(L^2)$ is our primary concern.

4. $[G_{st}, G_{ab}]$, $t \neq a$

Consider the second case where $t \neq a$ gives $[G_{st}, G_{ab}] = y_{stab}$. The assumption $s < a$ or $s = a$ and $t < b$ eliminates the $s = b$ possibility. Since no F is produced by the bracket notice that $[G_{st}, G_{ab}] = y_{stab} \in M(L^2)$. We will work with the $F's$ rather than the $G's$ as both produce the same elements in $M(L^2)$. We wish to find all the relationships between the values of the subscripts where $y_{stab} = 0$, otherwise we assume $y_{stab} \neq 0$ to get $M(L^2)$ of maximal dimension. We begin by showing an upper bound on the distance between subscripts for which y_{stab} may be non-zero.

Theorem 4.1. *If $b \geq a + 5$ or $t \geq s + 5$ then $y_{stab} = 0$.*

Proof. Suppose $b \geq a + 5$. If $t = a + 2$ then let $c = a + 3$, otherwise choose $c = a + 2$. By construction $c \neq s, t, a, b$. ($c \neq s$ because $s \leq a$.) Therefore $J(F_{st}, F_{ac}, F_{cb}) = 0 \;\Rightarrow\; y_{stab} = 0$.

Similarly suppose $t \geq s+5$. If $a = s+2$ then let $c = s+3$, otherwise choose $c = s+2$. By construction $c \neq s,t,a$, but we need $c \neq b$ also. Notice $s = a \Rightarrow t < b \Rightarrow c = s+2 < s+5 \leq t < b$. On the other hand if $s < a$ then $s+2 < a+2 \leq b$ (*i.e.* $b \geq s+3$). Therefore either $b > s+3 \geq c$ or $b = s+3 \Rightarrow a = s+1 \Rightarrow c = s+2 \neq b$. Therefore in all cases $c \neq b$. This gives $c \neq s,t,a,b$ and therefore $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. □

Theorem 4.2. *If $b = a+4$ then $y_{stab} = 0 \Leftrightarrow t \neq a+2$ or $s < a$ ($s \neq a$). Similarly if $t = s+4$ then $y_{stab} = 0 \Leftrightarrow a \neq s+2$ or $b \neq t$.*

Proof. ($\Leftarrow$) Suppose $b = a+4$. If $t \neq a+2$ let $c = a+2$, so $c \neq t,a,b$ and furthermore $s \leq a < c \Rightarrow c \neq s$. Therefore $c \neq s,t,a,b$ and $J(F_{st}, F_{ac}, F_{cb}) = 0 \Rightarrow y_{stab} = 0$. On the other hand if $t = a+2$ and $s < a$ then let $c = t-1$. In this case $J(F_{st}, F_{tb}, F_{at}) = 0 \Rightarrow y_{stab} = -y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ which together give $y_{stab} = 0$.

($\Rightarrow$) Suppose $b = a+4$, $t = a+2$, and $s = a$. There is no value of c such that F_{ac} and F_{cb} are both defined while $c \neq t$. As such, there are no Jacobi identities available to zero out y_{stab}.

($\Leftarrow$) Suppose $t = s+4$. If $a \neq s+2$ let $c = s+2$, so $c \neq s,t,a$. Notice when $s = a$ then $t < b$ so $s < c < t < b \Rightarrow c \neq b$. If $s < a$ then $c = s+2 < a+2 \leq b \Rightarrow c \neq b$. Therefore $c \neq s,t,a,b$ and $J(F_{sc}, F_{ct}, F_{ab}) = 0 \Rightarrow y_{stab} = 0$. On the other hand if $a = s+2$ and $b > t$ then let $c = a+1$. In this case $J(F_{sa}, F_{at}, F_{ab}) = 0 \Rightarrow y_{stab} = y_{sbat}$ and $J(F_{sc}, F_{cb}, F_{at}) = 0 \Rightarrow y_{sbat} = 0$ which together give $y_{stab} = 0$. If $a = s+2$ and $b < t$ then $s < s+2 = a < a+2 \leq b < t \Rightarrow s+4 < t$ and hence by Theorem 4.1, $y_{stab} = 0$.

($\Rightarrow$) Suppose $t = s+4$, $a = s+2$, and $b = t$. There is no value of c such that F_{sc} and F_{ct} are both defined while $c \neq a$. As such, there are no Jacobi identities available to zero out y_{stab}. □

Now that we have an upper bound on the distance for which non-zero y_{stab} values may be produced, we continue our search by separating the variable relationships into three cases. Either $s = a$, $a > t$, or $s < a < t$.

Case 1: $s = a$

For a fixed value of s, suppose $s = a$. Theorems 4.1 and 4.2 discuss $b \geq s+4$, so consider $b < s+4$. Let $t_{min} = s+2$. Since $s = a \Rightarrow t < b$ this gives $t_{min} < b < s+4 = t_{min}+2$, so $b = t_{min}+1$.

Theorem 4.3. *When $b = t_{min}+1$, we get one new non-zero value for y_{stab}.*

Proof. Observe $b = t_{min} + 1 = s + 3 = a + 3$. Additionally $t < b \Rightarrow t < s + 3$. Therefore $\nexists c$ such that F_{ac} and F_{cb} are both defined, similarly $\nexists c$ such that F_{sc} and F_{ct} are both defined. Therefore $y_{stab} \neq 0$. Also $b = t_{min} + 1$, $t < b \Rightarrow t = t_{min}$, therefore t may take on only one value for this fixed b, hence we get 1 distinct new non-zero value for y_{stab}. Putting this together we have $y(s, s+2, s, s+3) \neq 0$. □

Case 2: $a > t$

Theorem 4.4. *If $a > t$ then $y_{stab} \neq 0$ for all t and b such that both $t < s + 4$ and $b < a + 4$. Otherwise $y_{stab} = 0$ when $a > t$.*

Proof. If $t \geq s + 5$ or $b \geq a + 5$ then Theorem 4.1 $\Rightarrow$ $y_{stab} = 0$. If $t = s + 4$ or $b = a+4$ then Theorem 4.2 $\Rightarrow$ $y_{stab} = 0$ since $a > t \Rightarrow t \neq a+2$ and $a \neq s+2$. If $t < s + 4$ and $b < a + 4$ then there is no value of c, such that F_{sc} and F_{ct} are both defined for $s < c < t$. Similarly there is no value of c, such that F_{ac} and F_{cb} are both defined for $a < c < b$. Therefore the identities $J(F_{sc}, F_{ct}, F_{ab}) = 0$ and $J(F_{st}, F_{ac}, F_{cb}) = 0$ are not available. Also, placing a c such that $s < t < c < a < b$ will not provide any helpful Jacobi identities, no matter how large the gap between t and a. Thus y_{stab} will always be non-zero in this case. □

Case 3: $s < a < t$

Theorems 4.1 and 4.2 discuss $b \geq a + 4$ and $t \geq s + 4$.

Theorem 4.5. *If $s < a < t$, $y_{stab} \neq 0$ in the event that $t < s + 4$ and $b < a + 4$.*

Proof. There is not enough space available between s, t, a, b to define a suitable c to use any of the previous Jacobi identities, hence we cannot zero out y_{stab} when $b \in \{a + 2, a + 3\}$ and $t \in \{s + 2, s + 3\}$. Therefore $y_{stab} \neq 0$ in this case. □

Collecting all this information, Table 1 lists all non-zero y_{stab} possibilities.

TABLE 1. $y_{stab} \neq 0$ possibilities

Theorem 4.2	$b = a + 4$, $t = a + 2$, and $s = a$
Theorem 4.2	$t = s + 4$, $a = s + 2$ and $b = t$
Case 1/Theorem 4.3	$s = a$, $b = t_{min} + 1 = s + 3$, and $t = t_{min} = s + 2$
Case 2/Theorem 4.4	$a > t$, $t < s + 4$, and $b < a + 4$
Case 3/Theorem 4.5	$s < a < t$, $t = s + 2, s + 3$, and $b = a + 2, a + 3$

5. Counting the multiplier elements

To count all the cases when $y_{stab} \neq 0$, notice there are two types of elements: $y(s, s+x_1, s+x_2, s+x_3)$ where x_1, x_2, x_3 are all fixed and $y(s, s+x_1, a, a+x_2)$ where $a > s+x_1$ and x_1, x_2 are both fixed. Since $s+x_i$, $i = 1, 2, 3$ and $a+x_2$ were subscripts of the $G's$, and initially subscripts of the standard matrix units, these numbers may not exceed n.

Type 1: $y(s, s+x_1, s+x_2, s+x_3)$

Since $s \geq 1$ and $s + x_i \leq n$ for $i = 1, 2, 3$ this implies $s \in \{1, 2, \ldots, n-w\}$ where $w = \max\{x_1, x_2, x_3\}$. Therefore $y(s, s+x_1, s+x_2, s+x_3)$ may take on $n-w$ possible values. If $n \leq w$ there is no contribution to the multiplier, $M(L^2)$, since w represents the distance between matrix positions and n the total matrix positions.

Notice also that the elements y_{rsst} from the $[G_{rs}, G_{st}]$ case also fall into this category since all subscripts can be described as distances from r.

Type 2: $y(s, s+x_1, a, a+x_2)$ where $a > s+x_1$

Since no matrix positions may exceed n, this implies $a \leq n - x_2$. If $s = 1$ then $a \geq 2 + x_1$. Therefore $a \in \{2 + x_1 \ldots n - x_2\}$, giving $(n - x_2) - (2 + x_1) + 1 = n - x_2 - x_1 - 1$ different values for a. If $s = 2$, the minimum value of a increases by 1, yielding one fewer possible value of a. Every further increment of s in turn decrements the allowable values of a until a can assume only one value. Thus as s increases, the number of possible values of a go from $n - x_2 - x_1 - 1$ down to 1, giving $\displaystyle\sum_{i=1}^{n-x_2-x_1-1} i = \frac{(n-(x_2+x_1+1))(n-(x_2+x_1))}{2}$ possible values of $y(s, s+x_1, a, a+x_2)$.

As in Type 1, this type demands a distance of $x_2 + x_1 + 1$ between matrix positions, hence if $n \leq x_2 + x_1 + 1$ there is no contribution to the multiplier, $M(L^2)$, since the original matrix units are not available to work with.

Using these two counting techniques, Table 2 lists all non-trivial y_{rsst} and y_{stab} possibilities and the number of times they occur. Adding all of these together gives $\dim M(L^2) = (n-6) + 2(n-5) + 5(n-4) + 3(n-3) + (n-5)(n-6) + \frac{(n-6)(n-7)+(n-4)(n-5)}{2}$. Also, as stated above if any $n - w$ term is not positive, omit it from the formula because the matrices will not be large enough to produce the corresponding matrix units and the resulting multiplier elements they would have produced. Despite the appearance of a few zeros in the previous unsimplified formula when $n = 6$ and 7, modifying this formula is not absolutely necessary unless $n \leq 5$ in which case some $n - w$ terms will be negative. This gives $\dim M(L^2) = 2n^2 - 11n + 16$ for $n \geq 6$ and with the appropriate modifications $\dim M(L^2) = 11, 3, 0$ for $n = 5, 4, 3$ respectively.

TABLE 2. Counting multiplier elements for L^2

y_{rsst} or y_{stab} non-trivial	Number of occurences
$y_{r,r+2,r+2,r+5}$	$n-5$
$y_{r,r+3,r+3,r+6}$	$n-6$
$y_{s,s+2,s,s+4}$	$n-4$
$y_{s,s+4,s+2,s+4}$	$n-4$
$y_{s,s+2,s,s+3}$	$n-3$
$y_{s,s+2,a,a+2}$	$\frac{(n-4)(n-5)}{2}$
$y_{s,s+2,a,a+3}$	$\frac{(n-5)(n-6)}{2}$
$y_{s,s+3,a,a+2}$	$\frac{(n-5)(n-6)}{2}$
$y_{s,s+3,a,a+3}$	$\frac{(n-6)(n-7)}{2}$
$y_{s,s+2,s+1,s+3}$	$n-3$
$y_{s,s+2,s+1,s+4}$	$n-4$
$y_{s,s+3,s+1,s+3}$	$n-3$
$y_{s,s+3,s+1,s+4}$	$n-4$
$y_{s,s+3,s+2,s+4}$	$n-4$
$y_{s,s+3,s+2,s+5}$	$n-5$

6. Examples

Example 6.1. *Suppose $n = 4$, notice L^2 is abelian. The formula gives* $\dim M(L^2)$ $= 3$. *The Lie algebra corresponds to F_{13}, F_{14}, F_{24} in C. Considering all possible brackets gives*
$[F_{13}, F_{14}] = y_{1314}$
$[F_{13}, F_{24}] = y_{1324}$
$[F_{14}, F_{24}] = y_{1424}$

Each bracket produces an element in the multiplier, but no F's. Therefore any Jacobi identity will trivially give zero and no additional information about the y's. Counting the y's also $\dim M(L^2) = 3$.

Example 6.2. *Suppose* $n = 5$. *The formula for* $\dim M(L^2)$ *reduces to* $5(n-4)+3(n-3) = 11$. C *contains* $F_{13}, F_{14}, F_{15}, F_{24}, F_{25}, F_{35}$. *The possible brackets are*

$$
\begin{array}{lll}
[F_{13}, F_{14}] = y_{1314} & [F_{13}, F_{15}] = y_{1315} & [F_{13}, F_{24}] = y_{1324} \\
[F_{13}, F_{25}] = y_{1325} & [F_{13}, F_{35}] = F_{15} + y_{1335} = G_{15} & [F_{14}, F_{15}] = y_{1415} = 0 \\
[F_{14}, F_{24}] = y_{1424} & [F_{14}, F_{25}] = y_{1425} & [F_{14}, F_{35}] = y_{1435} \\
[F_{15}, F_{24}] = y_{1524} = 0 & [F_{15}, F_{25}] = y_{1525} = 0 & [F_{15}, F_{35}] = y_{1535} \\
[F_{24}, F_{25}] = y_{2425} & [F_{24}, F_{35}] = y_{2435} & [F_{25}, F_{35}] = y_{2535}
\end{array}
$$

We get y_{1415}, y_{1524}, $y_{1525} = 0$ *from* $J(F_{13}, F_{35}, F_{14}) = 0$, $J(F_{13}, F_{35}, F_{24}) = 0$, *and* $J(F_{13}, F_{35}, F_{25}) = 0$ *respectively. Counting the* y*'s also shows* $\dim M(L^2) = 11$.

Example 6.3. *Suppose* $n = 6$. *Then* $\dim M(L^2) = 2n^2 - 11n + 16 = 22$. *Computing all possible bracket operations produces Table 3. We place a* $*$ *wherever we would have* $[x, x]$ *or violate* $s < a$ *or* $s = a, t < b$. *All the zeros come from some Jacobi identity.*

TABLE 3. Bracket Operation $[F_{st}, F_{ab}]$, when $n = 6$

	F_{13}	F_{14}	F_{15}	F_{16}	F_{24}	F_{25}	F_{26}	F_{35}	F_{36}	F_{46}
F_{13}	$*$	y_{1314}	y_{1315}	0	y_{1324}	y_{1325}	0	G_{15}	$G_{16} + y_{1336}$	y_{1346}
F_{14}	$*$	$*$	0	0	y_{1424}	y_{1425}	0	y_{1435}	y_{1436}	G_{16}
F_{15}	$*$	$*$	$*$	0	0	0	0	y_{1535}	0	0
F_{16}	$*$	$*$	$*$	$*$	0	0	0	0	0	0
F_{24}	$*$	$*$	$*$	$*$	$*$	y_{2425}	y_{2426}	y_{2435}	y_{2436}	G_{26}
F_{25}	$*$	$*$	$*$	$*$	$*$	$*$	0	y_{2535}	y_{2536}	y_{2546}
F_{26}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0	y_{2646}
F_{35}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y_{3536}	y_{3546}
F_{36}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	y_{3646}
F_{46}	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$

Notice that counting the y*'s also gives* $\dim M(L^2) = 22$.

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Louis A. Levy
Department of Mathematics
Millersville University
P.O. Box 1002
Millersville, PA 17551-0302
e-mail: louisalevy@yahoo.com