International Electronic Journal of Algebra
Published Online: December 25, 2024
DOI: 10.24330/ieja.1607238

ON A VARIETY OF LIE-ADMISSIBLE ALGEBRAS

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Received: 16 July 2024; Accepted: 17 December 2024
Communicated by Meltem Altun Özarslan

Dedicated to the memory of our dear friend Tariq Rizvi

Abstract. The aim of this paper is to propose the study of a class of Lie-admissible algebras. It is the class (variety) of all the (not-necessarily associative) algebras M over a commutative ring k with identity 1_k for which $(x, y, z) = (y, x, z) + (z, y, x)$ for every $x, y, z \in M$. Here (x, y, z) denotes the associator of M. We call such algebras *algebras of type* $\mathcal{V}_2$. Very little is known about these algebras.

Mathematics Subject Classification (2020): 17A15, 17A30, 17D25
Keywords: Lie-admissible algebra, pre-Lie algebra

1. Introduction

This paper is devoted to the study of *algebras of type* $\mathcal{V}_2$, that is, the algebras M for which

$$(x, y, z) = (y, x, z) + (z, y, x) \tag{1}$$

for every $x, y, z \in M$. Here (x, y, z) denotes the *associator*, that is, the k-trilinear mapping $(-, -, -)\colon M \times M \times M \to M$ defined by $(x, y, z) = (xy)z - x(yz)$ for every $x, y, z \in M$.

Let us recall the basic notions and fix the notation. In this paper, k will always denote a commutative ring with identity 1_k. By a *k-algebra* $(M, \cdot)$, we mean a k-module M with a further operation $\cdot\colon M \times M \to M$, $(x, y) \mapsto x \cdot y = xy$, which is assumed to be k-bilinear. Equivalently, M is a k-module endowed with a k-module morphism $M \otimes_k M \to M$. Clearly, k-algebras form a category Alg_k, whose morphisms are the k-module morphisms that also respect algebra multiplication.

There is an endofunctor U of the category of k-algebras Alg_k that associates with any k-algebra $(M, \cdot)$ the k-algebra $(M, [-, -])$, where $[x, y] = xy - yx$ for every $x, y \in M$. It associates with any morphism $f\colon (M, \cdot) \to (N, \cdot)$ in Alg_k, the same mapping $U(f) = f\colon (M, [-, -]) \to (N, [-, -])$.

Dually, there is an endofunctor D of the category Alg_k that associates with any k-algebra $(M,\cdot)$ the k-algebra $(M,\circ)$, where $x\circ y=xy+yx$ for every $x,y\in M$. It also associates with any morphism $f\colon (M,\cdot)\to(N,\cdot)$ in Alg_k, the same mapping $D(f)=f\colon (M,\circ)\to(N,\circ)$.

As we have implicitly already mentioned above, if M is any k-module, the set of all k-bilinear mappings $M\times M\to M$ is a k-module isomorphic to the k-module $\mathrm{Hom}_k(M\otimes_k M,M)$. If C is the k-submodule of $M\otimes_k M$ generated by the set $\{\,x\otimes y-y\otimes x\mid x,y\in M\,\}$, then the set of all commutative k-bilinear operations $M\times M\to M$ is a sub-k-module of $\mathrm{Hom}_k(M\otimes_k M,M)$ isomorphic to the k-module $\mathrm{Hom}_k(M\otimes_k M/C,M)$. If A is the k-submodule of $M\otimes_k M$ generated by the set $\{\,x\otimes y+y\otimes x\mid x,y\in M\,\}$, then the set of all anticommutative k-bilinear operations $M\times M\to M$ is a sub-k-module of $\mathrm{Hom}_k(M\otimes_k M,M)$ isomorphic to $\mathrm{Hom}_k(M\otimes_k M/A,M)$. Let ${}_kM$ be any k-module and let Comm and AntiComm be the k-submodules of $\mathrm{Hom}_k(M\otimes_k M,M)$ consisting of all k-bilinear commutative and anticommutative operations on ${}_kM$, respectively. If 2 is invertible in k, then $\mathrm{Hom}_k(M\otimes_k M,M)=\mathrm{Comm}\oplus\mathrm{AntiComm}$ (see for instance [3, Theorem 4.1]).

For every k-algebra M, the k-algebra $U(M)$ is always anticommutative (i.e., $[x,y]=-[y,x]$) and the k-algebra $D(M):=(M,\circ)$ is always a commutative algebra. By definition, a k-algebra $(M,\cdot)$ is *Lie-admissible* if the anticommutative k-algebra $U(M):=(M,[-,-])$ is a Lie algebra, that is, if the Jacobi identity

$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]] \;=\; 0$$

holds for every $x,y,z\in M$, and is a *pre-Lie algebra* if $(xy)z-x(yz)=(yx)z-y(xz)$ for every $x,y,z\in M$. If the *associator* of a k-algebra M is defined by $(x,y,z)=(xy)z-x(yz)$ for all x,y,z in M, then M is associative if and only if $(x,y,z)=0$ for all $x,y,z\in M$; the algebra M is a pre-Lie algebra if and only if $(x,y,z)=(y,x,z)$ for all $x,y,z\in M$; and M is Lie-admissible if and only if

$$(x,y,z)+(y,z,x)+(z,x,y)=(y,x,z)+(x,z,y)+(z,y,x) \qquad (2)$$

for all $x,y,z\in M$. Therefore associative algebras are pre-Lie, and pre-Lie algebras are Lie-admissible.

Lemma 1.1. *Algebras of type $\mathcal{V}_2$ are Lie-admissible.*

Proof. If M is an algebra of type $\mathcal{V}_2$, we have that $(x,y,z)=(y,x,z)+(z,y,x)$. Swapping y and z in this equation, we get that $(z,x,y)+(y,z,x)=(x,z,y)$. Summing up these two equalities, we get equality (2). □

Equality (2) can be written explicitly as

$$\begin{aligned}(xy)z+(yz)x+(zx)y+y(xz)+x(zy)+z(yx)-\\ -x(yz)-y(zx)-z(xy)-(yx)z-(xz)y-(zy)x=0\end{aligned} \tag{3}$$

This is a sum of 12 terms, corresponding to the six permutations of $\{x, y, z\}$ and the two possibilities $(ab)c$ and $a(bc)$ of writing the parentheses in a product of three terms a, b, c. The sign of each of these twelve terms depends on the way $(ab)c$ or $a(bc)$ in which the parentheses are written and the sign of the permutation of $\{x, y, z\}$ (cf. [4, pp. 131–132]).

In this paper, we will study the first properties of the variety of algebras of type $\mathcal{V}_2$. The variety $\mathcal{V}_2$ is properly contained between the variety of associative algebras and the variety of all algebras M for which $(x, y, z) + (y, z, x) + (z, x, y) = 0$ for all $x, y, z \in M$ (Theorem 3.6 and Examples 3.7 and 3.8).

We are able to show that every 2-torsion-free k-algebra of type $\mathcal{V}_2$ is right alternative, hence power-associative (Proposition 3.2 and Corollary 3.4). Also, we show that a 2-torsion-free algebra is of type $\mathcal{V}_2$ if and only if $(z, x, y) + (z, y, x) = 0$ and $(x, y, z) + (y, z, x) + (z, x, y) = 0$ for every $x, y, z \in M$ (Theorem 3.6). The notion of algebras M of type $\mathcal{V}_2$ seems to be related to the notion of module over the commutative algebra $D(M)$ and the notion of module over pre-Lie algebras (Section 4).

I am grateful to Professor Carmelo Antonio Finocchiaro for several discussions on this topic.

2. Varieties of Lie-admissible algebras

When k is a field and $F := k\langle x_1, x_2, x_3, \dots\rangle$ is the non-associative countably generated free k-algebra, call T-ideal any *totally invariant* ideal of F, that is, any ideal invariant under all endomorphisms of the k-algebra F. There is a one-to one correspondence between the set of all T-ideals of F and the class of all varieties of k-algebras. The T-ideal corresponding to an arbitrary variety of non-associative algebras over a field k corresponds to the set of all polynomial identities of the variety. For instance, our variety of algebras of type $\mathcal{V}_2$ corresponds to the principal T-ideal of F generated by the non-associative polynomial

$$(x_1, x_2, x_3) - (x_2, x_1, x_3) - (x_3, x_2, x_1).$$

This is a homogeneous polynomial of degree three.

Remark 2.1. It is important to notice that our algebras are not required to have an identity in general. Thus, for instance, the k-algebra F introduced in the previous paragraph, is an $\mathbb{N}$-graded algebra, whose component of degree zero is zero, whose component of degree one is the vector space over k with basis all monomials x_i, the component of degree two is the vector space over k with basis all monomials x_ix_j, and the component of degree three is the vector space over k with basis all monomials $(x_ix_j)x_k$ and all monomials $x_i(x_jx_k)$. Of course, it would be also possible to consider the category of all k-algebras with an identity 1_M, or the category of all k-algebras M with an identity 1_M and an augmentation $M \to k$, that is, a morphism of k-algebras with identity that composed with the embedding $k \to M$, $\lambda \in k \mapsto \lambda \cdot 1_M$, gives the identity automorphism of k. Clearly, the category Alg_k of our k-algebras is equivalent to the category of all k-algebras M with an identity 1_M and an augmentation.

There is a "hierarchy" of varieties of Lie-admissible algebras corresponding to the lattice of all T-ideals of the free k-algebra F containing the pricipal T-ideal generated by the non-commutative non-associative polynomial

$$(x_1, x_2, x_3) + (x_2, x_3, x_1) + (x_3, x_1, x_2) - (x_2, x_1, x_3) - (x_1, x_3, x_2) - (x_3, x_2, x_1).$$

Let us examine some of these varieties (cf. [4, pp. 131–132]). Of course, our list cannot be exaustive, because it has been proved in [5] that the variety of right-symmetric algebras (see (14) below) over an arbitrary field does not have the Specht property, that is, it has a subvariety that has not a finite basis of identities.

Now, there is an involutive category automorphism ${}^{\mathrm{op}}\colon \mathsf{Alg}_k \to \mathsf{Alg}_k$. It associates with any k-algebra $(M, \cdot)$ its *opposite algebra* M^{op}, which is the algebra $(M, *)$, where $*$ is defined by $x * y = y \cdot x$ for every $x, y \in M$. In our list of variety, we will denote by $\mathcal{W}_i$ the varieties fixed by the automorphism ${}^{\mathrm{op}}$, and by $\mathcal{V}_j^*$ the variety that is the image of the variety $\mathcal{V}_j$ via the automorphism ${}^{\mathrm{op}}$.

(1) The smallest subvariety of Alg_k is trivially the variety $\mathcal{W}_1$ of all k-algebras of cardinality 1, corresponding to the improper ideal of the free k-algebra F, which is the principal T-ideal generated by x_1.

(2) Then we have the variety $\mathcal{W}_2$ of all *abelian* k-algebras, that is, the k-algebras M for which $xy = 0$ for every $x, y \in M$. Clearly, the full subcategory of the category Alg_k whose objects are all abelian k-algebras is equivalent to the category k-Mod of all modules over k. The variety $\mathcal{W}_2$ of all abelian k-algebras corresponds to the principal T-ideal of F generated by the monomial x_1x_2 of degree 2. This T-ideal is the direct sum of all the homogeneous components of degree ≥ 2 of the $\mathbb{N}$-graded k-algebra F.

(3) Then we have the variety $\mathcal{W}_3$ of all k-algebras M for which $(xy)z = 0$ and $x(yz) = 0$ for all $x, y, z \in M$. These are the algebras for which all the 12 terms in Identity (3) are zero. This shows, trivially, that the algebras in the variety $\mathcal{W}_3$ are Lie-admissible. The algebras in $\mathcal{W}_3$ can also be described as the k-algebras M for which both $M^2 \cdot M = 0$ and $M \cdot M^2 = 0$, that is, equivalently, $M^2 \subseteq \operatorname{r.ann}(M) \cap \operatorname{l.ann}(M)$, where $\operatorname{r.ann}(M)$ and $\operatorname{l.ann}(M)$ denote the right annihilator and the left annihilator of M, respectively. The variety $\mathcal{W}_3$ corresponds to the T-ideal of F generated by the two monomials $(x_1x_2)x_3$ and $x_1(x_2x_3)$ of degree 3. As a k-vector space, this T-ideal is the direct sum of all the homogeneous components of degree ≥ 3 of the graded algebra F.

(4) The variety $\mathcal{W}_4$ of all associative k-algebras, that is, the k-algebras M for which $(x, y, z) = 0$ for all $x, y, z \in M$.

(5) The variety $\mathcal{W}_5$ of all k-algebras M for which $(xy)z = (zy)x$ and $x(yz) = z(yx)$ for all $x, y, z \in M$.

(6) Then we have a number of varieties in which the 12 terms in Identity (3) annihilates in pair. The first example of such a variety is the variety $\mathcal{W}_6$ of all commutative (not-necessarily associative) k-algebras. This variety corresponds to the principal T-ideal of F generated by the homogeneous polynomial $x_1x_2 - x_2x_1$ of degree two. Clearly, every commutative algebra is Lie-admissible (they are exactly the algebras for which the sub-adjacent Lie algebra is abelian).

(7) The variety $\mathcal{W}_7$ of all k-algebras M for which $(x, y, z) + (y, z, x) + (z, x, y) = 0$ for all $x, y, z \in M$. Notice that an algebra that satisfies $x(yz) + y(zx) + z(xy) = 0$ is not necessarily Lie-admissible [6, pp. 287–288]. Clearly, every Lie algebra belongs to $\mathcal{W}_7$.

(8) The variety $\mathcal{W}_8$ of all Lie-admissible k-algebras, that is, the k-algebras in which Identity (2) holds.

(9) The variety $\mathcal{V}_1$ of all k-algebras M for which $(xy)z = (xz)y$ and $x(yz) = z(yx)$ for all $x, y, z \in M$.

(10) The variety $\mathcal{V}_1^*$ of all k-algebras M for which $x(yz) = y(xz)$ and $(xy)z = (zy)x$ for all $x, y, z \in M$.

(11) The variety $\mathcal{V}_2$ of all k-algebras M for which $(x, y, z) = (y, x, z) + (z, y, x)$ for all $x, y, z \in M$. These algebras are our main object of study in this paper.

Let us prove that:

Lemma 2.2. *Let M be a commutative k-algebra and assume that the abelian group M is 3-torsion-free (that is, that $x \in M$ and $3x = 0$ imply $x = 0$). Then the k-algebra M is of type $\mathcal{V}_2$ if and only if M is associative.*

Proof. Trivially, associative algebras are of type $\mathcal{V}_2$. Conversely assume M 3-torsion-free and of type $\mathcal{V}_2$. Then $(x,y,z)=(y,x,z)+(z,y,x)$ for all $x,y,z\in M$, that is, $(xy)z-x(yz)=(yx)z-y(xz)+(zy)x-z(yx)$. From commutativity, it follows that

$$2(yz)x-(xz)y-(xy)z=0. \tag{4}$$

Exchanging x and y in this identity, we get that

$$2(xz)y-(yz)x-(yx)z=0 \tag{5}$$

for every $x,y,z\in M$. Subtracting these two identities, we get $3(yz)x-3(xz)y=0$. But M is 3-torsion-free, so $(yz)x=(xz)y$. Because of commutativity, this identity can be written $(yz)x=y(zx)$. This proves that M must be associative. □

We have thus shown that $\mathcal{V}_2\cap\mathcal{W}_6=\mathcal{W}_4\cap\mathcal{W}_6$. We will show in Theorem 3.6 and Example 3.7 that the class $\mathcal{V}_2$ is properly contained in the class $\mathcal{W}_7$.

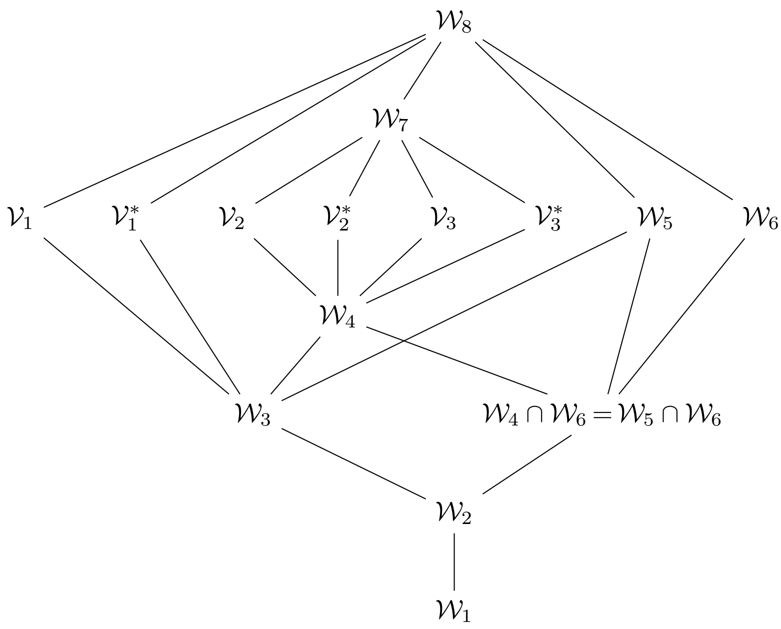
(12) The variety $\mathcal{V}_2^*$ of all k-algebras M for which $(z,y,x)=(z,x,y)+(x,y,z)$ for all $x,y,z\in M$.

(13) The variety $\mathcal{V}_3$ of all left-symmetric (or pre-Lie) algebras, that is, the k-algebras M for which $(x,y,z)=(y,x,z)$ for all $x,y,z\in M$.

Lemma 2.3. *The intersection of the class of all algebras of type $\mathcal{V}_2$ and the class $\mathcal{V}_3$ of all pre-Lie algebras is the class $\mathcal{W}_4$ of all associative algebras.*

Proof. It is clear that $\mathcal{W}_4\subseteq\mathcal{V}_2\cap\mathcal{V}_3$, because an algebra M is associative if and only if $(x,y,z)=0$ for every $x,y,z\in M$. Conversely, if M is of type $\mathcal{V}_2$, then $(x,y,z)=(y,x,z)+(z,y,x)$; and if M is pre-Lie, we know that $(x,y,z)=(y,x,z)$. Subtracting these two equalities, we get that (z,y,x) for every $x,y,z\in M$, as desired. □

(14) The variety $\mathcal{V}_3^*$ of all right-symmetric algebras, that is, the k-algebras M for which $(x,y,z)=(x,z,y)$ for all $x,y,z\in M$.



3. Algebras of type $\mathcal{V}_2$

We have already seen some elementary properties of algebras of type $\mathcal{V}_2$ in Lemmas 1.1, 2.2 and 2.3. In this section we will give further properties of these algebras. We begin with the following proposition.

Proposition 3.1. *Let k be a field. Every k-algebra of dimension ≤ 2 and of type $\mathcal{V}_2$ is associative.*

Proof. The proof is rather long, but it only consists in elementary calculations. Here, we just give a quick sketch of it. Let M be a k-algebra of dimension ≤ 2 and of type $\mathcal{V}_2$. Let $U(M)$ be its sub-adjacent Lie algebra. If $U(M)$ is abelian, we conclude by Lemma 2.2. Hence we can suppose $U(M)$ non-abelian, and there is only one such algebra up to isomorphism. It is the Lie algebra of dimension 2 in which a basis can be chosen to be of the type $\{v, w\}$ with $[v, w] = v$. As a consequence, in M we have the k-basis $\{v, w\}$ subject to the relation $vw - wu = v$. Elementary but rather long calculations show that there are exactly two algebras M satisfying this relation and Identity (1). They are the two algebras whose multiplication tables are given by

$$\begin{cases} vv = 0 \\ vw = 0 \\ wv = -v \\ ww = -w \end{cases} \quad \text{and} \quad \begin{cases} vv = 0 \\ vw = v \\ wv = 0 \\ ww = w. \end{cases}$$

Thus there are at most two non-commutative algebras of type $\mathcal{V}_2$ up to isomorphism. Another easy calculation shows that both of them are associative. □

Proposition 3.2. *Let M be an algebra of type $\mathcal{V}_2$ over a commutative ring k with identity.*

(a) *If M is 2-torsion-free, then M is right alternative, that is, $(xy)y = x(yy)$ for every $x, y \in M$.*

(b)
$$(z, x, y) + (z, y, x) = 0 \tag{6}$$
for every $x, y, z \in M$.

Proof. Let $(M, \cdot)$ be an algebra of type $\mathcal{V}_2$, so that Identity (1) holds. Exchanging the two variables x and y in Identity (1), we get that

$$(y, x, z) = (x, y, z) + (z, x, y), \tag{7}$$

and summing up the two identities (1) and (7) we get that $(z, x, y) + (z, y, x) = 0$. This proves (b). In particular, for $x = y$, we get that $2(z, x, x) = 0$. If M is 2-torsion-free, it follows that $(z, x, x) = 0$, that is, $(zx)x = z(xx)$. This proves (a). □

Remark 3.3. In the proof of Proposition 3.2 we have shown that if M is any 2-torsion-free k-algebra, then Identity (6) implies that M is right alternative. But it is easy to see that the converse of this implication is also true, that is, Identity $(z, x, y) + (z, y, x) = 0$ holds in every right alternative k-algebra M. To see this, let M be any right alternative k-algebra. Then $0 = (x, y + z, y + z) = (x, y, y) + (x, y, z) + (x, z, y) + (x, z, z) = (x, y, z) + (x, z, y)$, and Identity (6) holds.

Corollary 3.4. *Every 2-torsion-free k-algebra M of type $\mathcal{V}_2$ is power-associative, that is, $x^n \cdot x^m = x^{n+m}$ for every $x \in M$ and every pair of positive integers n, m. Here x^n is defined by induction on $n \geq 1$ setting $x^1 = x$ and $x^{n+1} = x^n \cdot x$. Moreover, $(x^n)^m = x^{nm}$.*

The corollary follows immediately from [8, p. 343, Theorem 1]. Recall that an algebra is power-associative if and only if all its cyclic subalgebras are associative. In particular we have:

Theorem 3.5. *Let k be a field of characteristic $\neq 2$. The free k-algebra of type $\mathcal{V}_2$ on one object x is the k-algebra*

$$k[x] = kx \oplus kx^2 \oplus kx^3 \oplus kx^4 \oplus \dots$$

In order to illustrate the previous theorem, let us give an explicit computation of the first four homogeneous component of the algebra. The free non-associative k-algebra on one object x is an $\mathbb{N}$-graded k-algebra F whose homogeneous component of degree 0 is 0, the homogeneous component of degree 1 has dimension one and basis $\{x\}$, the homogeneous component of degree 2 has dimension one and basis $\{x^2\}$, the homogeneous component of degree 3 has dimension two and basis $\{x \cdot x^2, x^2 \cdot x\}$, and so on. The free k-algebra of type $\mathcal{V}_2$ on one object x is a quotient Q of this k-algebra F. In Q one has $x \cdot x^2 = x^2 \cdot x$, because of Proposition 3.2(a). We denote this element by x^3. Then the homogeneous component of Q of degree $1, 2, 3$ have all dimension one and basis $\{x\}$, $\{x^2\}$ and $\{x^3\}$, respectively. The homogeneous component of Q of degree 4 is generated by $\{x \cdot x^3, x^2 \cdot x^2, x^3 \cdot x\}$. From Proposition 3.2(a) we find that

$$x^3 \cdot x = (x^2 \cdot x) \cdot x = x^2(x \cdot x) = x^2 \cdot x^2. \tag{8}$$

Finally, from Identity (1) we get that $(x^2, x, x) = (x, x^2, x) + (x, x, x^2)$. In this equation, the term on the left is $(x^2, x, x) = x^3 \cdot x - x^2 \cdot x^2$, and this is zero because of (8). Therefore $(x, x^2, x) + (x, x, x^2) = 0$. This equality can be written as $x^3 \cdot x - x \cdot x^3 + x^2 \cdot x^2 - x \cdot x^3$. From (8), $2(x^3 \cdot x - x \cdot x^3) = 0$, so $x^3 \cdot x = x \cdot x^3$. Therefore $x \cdot x^3 = x^2 \cdot x^2 = x^3 \cdot x$, and the homogeneous component of Q of degree four is also one-dimensional.

The next theorem gives another, maybe more natural, presentation of the class of algebras of type $\mathcal{V}_2$.

Theorem 3.6. *Let M be a 2-torsion-free algebra over a commutative ring k with identity. Then M is of type $\mathcal{V}_2$ if and only if $(z, x, y) + (z, y, x) = 0$ and $(x, y, z) + (y, z, x) + (z, x, y) = 0$ for every $x, y, z \in M$.*

Proof. Let M be a 2-torsion-free algebra over a commutative ring k with identity. Assume M of type $\mathcal{V}_2$. Then $(z, x, y) + (z, y, x) = 0$ for every $x, y, z \in M$ by Proposition 3.2(b). Then $(x, y, z)+(x, z, x) = 0$ and $(y, z, x)+(y, x, z) = 0$ for every $x, y, z \in M$. Since algebras of type $\mathcal{V}_2$ are Lie-admissible, identity (2) holds, so that $(x, y, z) + (y, z, x) + (z, x, y) = -(y, z, x) - (x, y, z) - (z, x, y)$ for every $x, y, z \in M$. But M is 2-torsion-free, hence $(x, y, z) + (y, z, x) + (z, x, y) = 0$. This proves one of the two mutually inverse implications. Conversely, $(z, x, y) + (z, y, x) = 0$ and

$(x,y,z)+(y,z,x)+(z,x,y)=0$ imply $(x,y,z)-(y,x,z)-(z,y,x)=0$. Thus M is of type $\mathcal{V}_2$. □

Example 3.7. Consider the cross product $\times$ of vectors in $\mathbb{R}^3$. Let $\{i,j,k\}$ be the standard basis of $\mathbb{R}^3$. This algebra $(\mathbb{R}^3,\times)$ is a three-dimensional Lie real algebra, hence it belongs to $\mathcal{W}_7$. But $\mathbb{R}^3$ does not satisfy the identity $(x,y,z)=(y,x,z)+(z,y,x)$, as can be seen taking $x=y=i$ and $z=k$, in which case the identity becomes $(i,i,j)=(i,i,j)+(j,i,i)$, equivalently $(j,i,i)=0$, while $(j,i,i)=(j\times i)\times i-j\times(i\times i)=-k\times i=-j$. This proves that the algebra $\mathbb{R}^3$ does not belong to $\mathcal{V}_2$.

This example can be immediately adapted to any commutative ring k with identity $1\neq 0$, getting the free k-module k^3 with free set of generators $\{i,j,k\}$ and the same multiplication table as $(\mathbb{R}^3,\times)$, showing that for any such ring k the class of k-algebras $\mathcal{V}_2$ is not contained in $\mathcal{W}_7$.

Theorem 3.6 and Example 3.7 show that the class $\mathcal{W}_7$ properly contains the class $\mathcal{V}_2$.

We conclude this section with an example of an algebra of type $\mathcal{V}_2$ that is not associative, i.e., that the class $\mathcal{W}_4$ is properly contained in $\mathcal{V}_2$.

Example 3.8. Here is an example of a non-associative algebra of type $\mathcal{V}_2$. The example is given in https://math.stackexchange.com/a/4505089. It is an example of a right alternative algebra that is not left alternative. The example was obtained with MAGMA by Thomas Preu. Let k be any commutative ring with identity and M be a free k-module of rank 3 with free set of generators $\{x,y,z\}$. This is a k-algebra with respect to the multiplication defined, for every $\alpha,\beta,\gamma,\alpha',\beta',\gamma'\in k$, by

$$(\alpha x+\beta y+\gamma z)(\alpha' x+\beta' y+\gamma' z)=\alpha\alpha' x+\beta\alpha' y+\alpha\beta' z.$$

Since M is right alternative but not left alternative, the k-algebra M is not associative. If

$$A:=\alpha x+\beta y+\gamma z$$
$$A':=\alpha' x+\beta' y+\gamma' z$$
$$A'':=\alpha'' x+\beta'' y+\gamma'' z$$

are three arbitrary elements of M, then

$$(A,A',A'')=(\alpha\alpha'\beta''-\alpha\beta'\alpha'')z,$$

which also shows that M is not associative. Then Identity (1), that is

$$(A,A',A'')=(A',A,A'')+(A'',A',A),$$

becomes $(\alpha\alpha'\beta'' - \alpha\beta'\alpha'')z = (\alpha'\alpha\beta'' - \alpha'\beta\alpha'')z + (\alpha''\alpha'\beta - \alpha''\beta'\alpha)z$, which is trivially true. Hence M is of type $\mathcal{V}_2$.

4. Modules

This section is devoted to the study of modules over our algebras. Our motivation is the following. As we saw in Proposition 3.2(b), our algebras of type $\mathcal{V}_2$ satisfy the identity

$$(z, x, y) + (z, y, x) = 0. \tag{9}$$

This identity is very similar, except for the sign, to the identity $(z, x, y) = (z, y, x)$ that defines right-symmetric algebras, the right/left dual of pre-Lie algebras. Identity (9) can be written explicitly as $(zx)y - z(xy) + (zy)x - z(yx) = 0$. Now the concept of pre-Lie algebras is strictly connected to the study of modules of pre-Lie algebras, and similarly for anti-pre-Lie algebras and Jordan algebras. Let us briefly review the concept of modules over these algebras.

If M is an associative algebra over a commutative ring k, its left modules are the pairs (N, λ), where N is a k-module and $\lambda\colon M \to \mathrm{End}(N_k)$ is a k-algebra morphism. If M is a Lie k-algebra, its left modules are the pairs (N, λ), where N is a k-module and $\lambda\colon M \to U(\mathrm{End}(N_k))$ is a k-algebra morphism. The morphism λ is usually called the *adjoint*. If M is a pre-Lie algebra, its left modules are the pairs (N, λ), where N is a k-module and $\lambda\colon U(M) \to U(\mathrm{End}(N_k))$ is a k-algebra morphism [2, Section 4.1]. Notice that there is not a natural concept of left module over an arbitrary non-associative k-algebra M. Cf. [7], where the following notion of module over a Jordan k-algebra M is also developed. For a Jordan algebra M, let M' be M with unity adjoined and let E be the subalgebra of $\mathrm{End}_k(M')$ generated by all right multiplications r_x for x in M. A k-module N is an M-module if N is an E-module and there is a k-module morphism $\rho\colon M \to \mathrm{End}(N_k)$ such that $n\rho(x) = nr_x$ for all $n \in N$ and $x \in M$.

Making use of the mapping $\rho\colon M \to \mathrm{End}_{k\text{-}\mathsf{Mod}}(M)$, $\rho\colon x \mapsto \rho_x$ (right multiplication by x), Identity (9) can be equivalently written as

$$\rho_x\rho_y - \rho_{xy} + \rho_y\rho_x - \rho_{yx} = 0,$$

that is, $\rho_x\rho_y + \rho_y\rho_x = \rho_{xy} + \rho_{yx}$ for every $x, y \in M$. Via the Jordan product $\circ$, this can be written as $\rho_x \circ \rho_y = \rho_{x\circ y}$. That is, right multiplication $\rho\colon M \to \mathrm{End}_{k\text{-}\mathsf{Mod}}(M_k)$ in $(M, \cdot)$ is a *Jordan homomorphism*, that is, a k-algebra morphism $\rho\colon (M, \circ) \to (\mathrm{End}_{k\text{-}\mathsf{Mod}}(M_k), \circ)$.

Also recall that an algebra M is Lie-admissible if and only if

$$U(\rho-\lambda)\colon (U(M),[-,-]) \to (U(\mathrm{End}(_kM)),[-,-]) \tag{10}$$

is a k-algebra morphism [1, p. 573]. (Notice that here, in Equation (10), what we write in not completely correct, because we write $U(\rho-\lambda)$, while $\rho-\lambda\colon M \to \mathrm{End}(_kM)$ is not a morphism in the category Alg_k, but only in the category of k-modules. Nevertheless we simply mean that the mapping $\rho-\lambda\colon M \to \mathrm{End}(_kM)$ is a k-algebra morphism $(U(M),[-,-]) \to (U(\mathrm{End}(_kM)),[-,-])$.)

Identity (1), which defines algebras M of type $\mathcal{V}_2$, can be also expressed in terms of left mutiplication $\lambda\colon M_k \to \mathrm{End}(M_k)$ and right multiplication $\rho\colon M_k \to \mathrm{End}(M_k)$. In fact, exchanging x and y in (1), one sees that M is of type $\mathcal{V}_2$ if and only if $(x,z,y)=(z,x,y)+(y,z,x)$, that is, if and only if $(xy)z+y(xz)-z(yx)=x(xz)+(yx)z+(zy)x$. This can be re-written as

$$\lambda_{xy}-\lambda_y\circ\lambda_x-\rho_{yx}=\lambda_x\circ\lambda_y+\lambda_{yx}+\rho_y\circ\rho_x. \tag{11}$$

Here $\circ$ denotes composition of mappings (written on the left, as usual). Finally, (11) is equivalent to

$$\lambda_{[x,y]}-[\lambda_x,\lambda_y]=\rho_y\circ\rho_x+\rho_{yx}.$$

Notice the similarity between this formula and the formula in [2, Theorem 16(b)].

Disclosure statement. The author reports that there are no competing interests to declare.

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