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QUADRATIC DESCENT OF GENERALIZED QUADRATIC FORMS

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Abstract. Quadratic descent of generalized quadratic forms over a division algebra with involution of the first kind in characteristic two is investigated. Using the notion of transfer, it is shown that a system of quadratic forms, associated to such a generalized quadratic form, can be used to characterize its descent properties.

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1. Introduction

Over fields of characteristic two, the theory of quadratic forms divides into two distinct theories, that of quadratic forms and symmetric bilinear forms (which are otherwise equivalent). Involutions on central simple algebras appear as twisted forms of symmetric (or alternating) bilinear forms, up to a scalar factor [5]. Again, when the base field has characteristic two, one has the distinct notion of quadratic pairs on central simple algebras. This notion may be regarded as twisted analogues of quadratic forms, up to a scalar factor (see [5, §5.B]). Using the natural correspondence between central simple algebras with involution and hermitian forms over division algebras with involution, quadratic pairs correspond to generalized quadratic forms, introduced first in [11].

Let (D, θ) be a central division algebra with involution of the first kind over a field F and let (V, h) be a hermitian space over (D, θ). Then for every $\phi \in \mathrm{Hom}_F(D, F)$, the map $\phi \circ h : V \to F$ is a quadratic form over F. Fixing an F-basis $\mathcal{B}$ of D and letting $\mathcal{B}^* \subseteq \mathrm{Hom}_F(D, F)$ denote its dual basis, a system of quadratic forms $q_h := \{\phi \circ h \mid \phi \in \mathcal{B}^*\}$ was associated to h in [6] and used to study the isometry class and the isotropy behaviour of hermitian forms. Similarly, for a generalized quadratic space (V, ρ) over (D, θ), a system of quadratic forms q_ρ was defined in [9].

It was shown that this system determines ρ, up to isomorphism, as well as reflects its isotropy behaviour.

Let K/F be a quadratic field extension such that $D_K = D \otimes_F K$ is a division algebra. Hermitian forms can be extended from F to K in a straightforward manner. Let (V, h) be a generalized quadratic space over (D_K, θ_K). We say that (V, h) has a descent to (D, θ) if it is extended from a hermitian space over (D, θ). In [7], it was shown that (V, h) has a descent to (D, θ) if and only if the system of quadratic forms q_h over K is extended from a system of quadratic forms over F. In this work, we study a similar descent problem for generalized quadratic forms. Our main result is Theorem 4.5, which shows that a generalized quadratic spaces over (D_K, θ_K) has a descent to (D, θ) if and only if the system q_ρ can be descended to F.

2. Systems of quadratic forms

Throughout this work, all fields are implicitly supposed to be of characteristic two.

Let V be a finite dimensional vector space over a field F. A *quadratic form* on V is a map $q : V \to F$ satisfying

(i) $q(av) = a^2 q(v)$ for all $a \in F$ and $v \in V$;
(ii) the map $\mathfrak{b}_q : V \times V \to F$ defined by $\mathfrak{b}_q(u, v) = q(u + v) - q(u) - q(v)$ is a bilinear form.

The bilinear form $\mathfrak{b}_q$ is called *the polar form* of q. Set

$$\mathrm{rad}(\mathfrak{b}_q) = \{v \in V \mid \mathfrak{b}_q(v, w) = 0 \quad \text{for all } w \in V\}.$$

The form q is called *nonsingular* if $\mathrm{rad}(\mathfrak{b}_q) = \{0\}$ and *totally singular* if $\mathfrak{b}_q$ is trivial. Also, q is called *regular* if $q(v) \neq 0$ for every nonzero vector $v \in \mathrm{rad}(\mathfrak{b}_q)$.

By an (m-fold) *system of quadratic forms* on V we mean an m-tuple $Q = (q_1, \cdots, q_m)$, where every $q_i : V \to F$ is a quadratic form. Note that we may identify Q with a quadratic map $Q : V \to F^m$, which induces in turn a bilinear map $\mathfrak{b}_Q : V \times V \to F^m$ given by $\mathfrak{b}_Q(u, v) = Q(u+v) - Q(u) - Q(v)$. As in the case of a single quadratic form, one can define the set

$$\mathrm{rad}(\mathfrak{b}_Q) = \{v \in V \mid \mathfrak{b}_Q(v, w) = 0 \quad \text{for all } w \in V\}.$$

We say that Q is *nonsingular* if $\mathrm{rad}(\mathfrak{b}_Q) = \{0\}$ and *totally singular* if $\mathfrak{b}_Q$ is trivial. Also, as in [9], we say that Q is *strongly nonsingular* if q_i is nonsingular for some i, and *totally nonsingular* if every q_i is nonsingular.

A system of quadratic forms Q on V is called *metabolic* if there exists a subspace L of V such that $\dim_F L \geqslant \frac{1}{2}\dim_F V$ and $Q|_L = 0$. The orthogonal sum of two systems of quadratic forms Q and Q' is denoted by $Q \perp Q'$. We also write $Q \simeq Q'$ to say that Q and Q' are isometric (see [10, Ch. 9] for more details). Let Q be a system of quadratic forms on V and let W be a complement of $\mathrm{rad}(\mathfrak{b}_Q)$ in V. Then $Q \simeq Q|_{\mathrm{rad}(\mathfrak{b})} \perp Q|_W$. Hence, every system of quadratic forms decomposes as an orthogonal sum of a totally singular and a nonsingular system of quadratic forms. It is easy to see that the totally singular part in this decomposition is unique.

Lemma 2.1. *Let (V, Q) be a system of quadratic forms over F. If $Q \simeq Q_{\mathrm{ts}} \perp Q_{\mathrm{ns}}$, where Q_{ts} is totally singular and Q_{ns} is nonsingular, then $Q_{\mathrm{ts}} \simeq Q|_{\mathrm{rad}(\mathfrak{b}_Q)}$.*

Proof. Let W and W' be underlying vector spaces of Q_{ts} and Q_{ns}, respectively. We may identify W and W' with subspaces of V so that $V = W + W'$, $Q_{\mathrm{ts}} = Q|_W$ and $Q_{\mathrm{ns}} = Q|_{W'}$. It is enough to prove that $W = \mathrm{rad}(\mathfrak{b}_Q)$. Clearly, $W \subseteq \mathrm{rad}(\mathfrak{b}_Q)$. To prove the converse inclusion, let $v \in \mathrm{rad}(\mathfrak{b}_Q)$ and write $v = w + w'$ for some $w \in W$ and $w' \in W'$. Let $w'' \in W'$. Then

$$\mathfrak{b}_{Q_{\mathrm{ns}}}(w', w'') = 0 + \mathfrak{b}_{Q_{\mathrm{ns}}}(w', w'') = \mathfrak{b}_Q(w, w'') + \mathfrak{b}_{Q_{\mathrm{ns}}}(w', w'') = \mathfrak{b}_Q(v, w'') = 0.$$

Since Q_{ns} in nonsingular, one concludes that $w' = 0$, hence $v \in W$. $\square$

Corollary 2.2. *Let $Q_{\mathrm{ts}} \perp Q_{\mathrm{ns}} \simeq Q'_{\mathrm{ts}} \perp Q'_{\mathrm{ns}}$ be an isometry of systems of quadratic forms, where Q_{ts} and Q'_{ts} are totally singular and Q_{ns} and Q'_{ns} are nonsingular. Then $Q_{\mathrm{ts}} \simeq Q'_{\mathrm{ts}}$. Also, if Q_{ns} is strongly nonsingular (resp. totally nonsingular) then so is Q'_{ns}.*

Proof. The first statement follows from Lemma 2.1. The second one follows from the fact that if q is a single quadratic form on a vector space V and $V \simeq \mathrm{rad}(\mathfrak{b}_q) \oplus W$ for some subspace W of V, then $q|_W$ is nonsingular. $\square$

Let L/F be a field extension and let (V, Q) be a system of quadratic forms over F. Then there exists a system of quadratic forms (V_L, Q_L) over L, where $V_L = V \otimes_F L$ and $Q_L(v \otimes \alpha) = \alpha^2 Q(v)$ for every $v \in V$ and $\alpha \in L$. Note that if $Q = (q_1, \cdots, q_m)$, then $Q_L = ((q_1)_L, \cdots, (q_m)_L)$, where every $(q_i)_L : V_L \to L$ is the scalar extension of the quadratic form q_i to L. Let (V, Q) be a system of quadratic forms over L. We say that (V, Q) (or simply Q) has a *descent* to F if there exists a system of quadratic forms (V', Q') over F such that $(V, Q) \simeq (V', Q')_L$.

Let L/F be a finite field extension and let $s : L \to F$ be a nonzero F-linear functional. The *transfer* $s_*(q)$ of a quadratic form $q : V \to L$ is the quadratic form $s_*(q) : V \to F$ defined by $(s_*(q))(v) = s(q(v))$ for $v \in V$. It is easily seen that $\mathfrak{b}_{s_*(q)} = s_*(\mathfrak{b}_q)$, where $s_*(\mathfrak{b}_q) : V \times V \to F$ is the bilinear form given by $s_*(\mathfrak{b}_q)(u,v) = s(\mathfrak{b}_q(u,v))$ for all $u, v \in V$. Also, as in [8], we define the *transfer* of a system of quadratic forms $Q = (q_1, \cdots, q_n)$ over L as $s_*(Q) := (s_*(q_1), \cdots, s_*(q_n))$. Note that $s_*(Q)$ is a system of quadratic forms over F. It is readily seen that if Q is totally singular (resp. totally nonsingular, strongly nonsingular), then so is $s_*(Q)$.

3. Generalized quadratic forms

Let D be a finite dimensional central division algebra over a field F and let θ be an involution of the first kind on D, i.e., an antiautomorphism of D of period two which restricts to the identity on F. Set

$$\operatorname{Symd}(D,\theta) = \{x + \theta(x) \mid x \in A\}.$$

Let V be a finite dimensional right vector space over D. A *generalized quadratic form* over (D,θ) is a map $\rho : V \to D/\operatorname{Symd}(D,\theta)$ satisfying

(i) $\rho(v\alpha) = \theta(\alpha)\rho(v)\alpha$ for every $v \in V$ and $\alpha \in D$;
(ii) $\rho(u+v) - \rho(u) - \rho(v) = h_\rho(u,v) + \operatorname{Symd}(D,\theta)$ for every $u, v \in V$, where $h_\rho : V \times V \to D$ is a hermitian form.

The form h_ρ is called *the polar form* of ρ. The pair (V,ρ) is also called a *generalized quadratic space* over (D,θ).

Let (V,ρ) be a generalized quadratic space over (D,θ). Set

$$\operatorname{rad}(h_\rho) = \{v \in V \mid h_\rho(v,w) = 0 \quad \text{for all } w \in V\}.$$

We say that ρ is *nonsingular* if $\operatorname{rad}(h_\rho) = \{0\}$ and *regular* if $\rho(v) \neq 0 \in D/\operatorname{Symd}(D,\theta)$ for every nonzero vector $v \in \operatorname{rad}(h_\rho)$. The form ρ is also called *totally singular* if h_ρ is trivial.

An *isometry* between two generalized quadratic spaces (V,ρ) and (V',ρ') over (D,θ) is an isomorphism of right vector spaces $f : V \to V'$ satisfying $\rho'(f(v)) = \rho(v)$ for all $v \in V$. A generalized quadratic space (V,ρ) (or the form ρ itself) is called *isotropic* if there exists a nonzero vector $v \in V$ such that $\rho(v) = 0$ and *anisotropic* otherwise. A generalized quadratic space (V,ρ) is called *hyperbolic* if it is nonsingular and there exists a D-subspace L of V with $\dim_D L = \frac{1}{2}\dim_D V$ such that $\rho|_L = 0$. Such a subspace L is called a *lagrangian* of (V,ρ). It is easy to see that every hyperbolic generalized quadratic form of dimension $2n$ is isometric to $n\mathbb{H}_{(D,\theta)}$, where $\mathbb{H}_{(D,\theta)}$ is the 2-dimensional generalized quadratic form over (D,θ) given by $(x,y) \mapsto \theta(x)y$ (see [4, Ch. I, (5.6.1)]).

Let ρ be a generalized quadratic forms on V and let W be a complement of $\mathrm{rad}(h_\rho)$ in V. Then $\rho \simeq \rho|_{\mathrm{rad}(h_\rho)} \perp \rho|_W$. Hence, every generalized quadratic form decomposes as an orthogonal sum of a totally singular and a nonsingular generalized quadratic forms.

We now fix (D,θ) as a finite dimensional division algebra with involution of the first kind over a field F. As in [9], for $d \in D$ we denote the element $d + \mathrm{Symd}(D,\theta)$ in the quotient $D/\mathrm{Symd}(D,\theta)$ by $\bar{d}$. We also denote $D/\mathrm{Symd}(D,\theta)$ by $\overline{D}$. Fix L/F as a finite field extension such that D_L is a division ring.

Let $s : L \to F$ be a nonzero F-linear map and extend it to a D-linear map $s_D : D_L \to D$. Let (V,h) be a hermitian space over $(D,\theta)_L$. The hermitian form $s_*(h) : V \times V \to D$ defined by

$$s_*(h)(u,v) = s_D(h(u,v)) \quad \text{for all } u,v \in V,$$

is called the *transfer* of h. Note that if h is nonsingular, then so is $s_*(h)$ (see [1, p. 362]).

We now proceed to define the transfer of a generalized quadratic form over $(D,\theta)_L$. Observe first that $\mathrm{Symd}((D,\theta)_L)) = \mathrm{Symd}(D,\theta) \otimes L$. Also, identifying $D \subseteq D_L$, one has $s_D|_D = \mathrm{id}$, so

$$s_D(\mathrm{Symd}((D,\theta)_L)) \subseteq \mathrm{Symd}(D,\theta).$$

Hence, the map s_D induces a well-defined map $\bar{s} : \overline{D_L} \to \overline{D}$ given by

$$\bar{s}(\alpha + \mathrm{Symd}((D,\theta)_L)) = \overline{s_D(\alpha)},$$

where $\overline{D_L} = D_L/\mathrm{Symd}((D,\theta)_L)$. By abuse of notation, for $\alpha \in D_L$, we denote the element $\alpha + \mathrm{Symd}((D,\theta)_L)) \in D_L/\mathrm{Symd}((D,\theta)_L$ by $\bar{\alpha}$. Hence,

$$\bar{s}(\bar{\alpha}) = \overline{s_D(\alpha)} \quad \text{for } \bar{\alpha} \in \overline{D_L}.$$

Let $\rho : V \to \overline{D}_L$ be a generalized quadratic space over $(D,\theta)_L$. Define the map $s_*(\rho) : V \to \overline{D}$ via

$$s_*(\rho)(v) = \bar{s}(\rho(v)) \quad \text{for all } v \in V.$$

We call $s_*(\rho)$ the *transfer* of ρ.

Lemma 3.1. *The map $s_*(\rho)$ is a generalized quadratic form over (D,θ) with the polar form $h_{s_*(\rho)} = s_*(h_\rho)$. Moreover, if ρ is nonsingular (resp. totally singular), then so is $s_*(\rho)$.*

Proof. Let $v \in V$. Then for every $\alpha \in D$, we have

$$s_*(\rho)(v\alpha) = \bar{s}(\rho(v\alpha)) = \bar{s}(\theta(\alpha)\cdot\rho(v)\cdot\alpha) = \theta(\alpha)\cdot\bar{s}(\rho(v))\cdot\alpha = \theta(\alpha)\cdot s_*(\rho)(v)\cdot\alpha.$$

Also, for every $u, v \in V$, we have

$$\begin{aligned}\overline{h_{s_*(\rho)}(u,v)} &= s_*(\rho)(u+v) - s_*(\rho)(u) - s_*(\rho)(v) = \bar{s}(\rho(u+v) - \rho(u) - \rho(v)) \\ &= \bar{s}(\overline{h_\rho(u,v)}) = \overline{s_D(h_\rho(u,v))} = \overline{s_*(h_\rho(u,v))}.\end{aligned}$$

By [3, (1.1)], the polar form of a generalized quadratic form is unique. Hence, $h_{s_*(\rho)} = s_*(h_\rho)$. The rest statements of the result are now evident. □

Remark 3.2. If the form ρ in Lemma 3.1 is regular, then $s_*(\rho)$ is not necessarily regular. For example, let ρ be a regular generalized quadratic form with $\operatorname{rad}(h_\rho) \neq 0$. Suppose that there exists $v \in \operatorname{rad}(h_\rho)$ with $\rho(v) = \alpha$, where α is a nonzero element of D. Then for every linear map $s : L \to F$ with $s(1) = 0$, we have $s_*(\rho)(v) = 0$, hence $s_*(\rho)$ is not regular.

Corollary 3.3. *Let (V, ρ) be a generalized quadratic space over $(D, \theta)_L$. If ρ is hyperbolic, then so is $s_*(\rho)$.*

Proof. If ρ is hyperbolic, then it is nonsingular and there exists a D_L-subspace L of V with $\dim_{D_L} L = \frac{1}{2}\dim_{D_L} V$ such that $\rho|_L = 0$. By Lemma 3.1, $s_*(\rho)$ is nonsingular. We also have $\dim_D L = \frac{1}{2}\dim_D V$ and $s_*(\rho)|_L = 0$, hence $s_*(\rho)$ is hyperbolic. □

4. The main result

Throughout this section, (D, θ) is a finite dimensional division algebra with involution of the first kind over a field F. We recall some constructions forms [9]. Let (V, ρ) be a generalized quadratic space over (D, θ). Let $\mathcal{B} = \{u_1, \cdots, u_m\}$ be a basis of $\overline{D}$ over F and denote by $\{\pi_1, \cdots, \pi_m\}$ its dual basis of $\operatorname{Hom}(\overline{D}, F)$. For $i = 1, \cdots, m$, define the map $q^{u_i}_{\rho,\mathcal{B}} : V \to F$ via $q^{u_i}_{\rho,\mathcal{B}}(v) = \pi_i(\rho(v))$. Let

$$Q_{\rho,\mathcal{B}} = (q^{u_1}_{\rho,\mathcal{B}}, \cdots, q^{u_m}_{\rho,\mathcal{B}}).$$

As observed in [9, p. 380], $Q_{\rho,\mathcal{B}}$ is a system of quadratic forms. We now fix K/F as a separable quadratic extension such that D_K is a division ring. For $i = 1, \cdots, m$, write $u_i = \overline{u'_i}$, where $u'_i \in D$. Then the set

$$\{u'_1 \otimes 1 + \operatorname{Symd}((D,\theta)_K), \cdots, u'_m \otimes 1 + \operatorname{Symd}((D,\theta)_K)\},$$

is a basis of $\overline{D_K} = D_K / \operatorname{Symd}((D,\theta)_K)$. By abuse of notation, we denote $u'_i \otimes 1 + \operatorname{Symd}((D,\theta)_K)$ by $u_i \otimes 1$, $i = 1, \cdots, m$. Set $\mathcal{B}_K = \{u_1 \otimes 1, \cdots, u_m \otimes 1\}$. Clearly, $\mathcal{B}_K$ is a basis of $\overline{D_K}$.

Let (V,ρ) be a generalized quadratic space over (D,θ). Then there exists a unique generalized quadratic form $\rho_K : V_K \to \overline{D_K}$ satisfying

$$\rho_K(v\otimes\alpha)=\rho(v)\otimes\alpha^2 \quad \text{for all } v\in V \text{ and } \alpha\in K.$$

Now, let (V,ρ) be a generalized quadratic space over $(D,\theta)_K$. We say that (V,ρ) (or the form ρ itself) has a *descent* to (D,θ) if there exists a generalized quadratic space (V',ρ') over (D,θ) such that $(V,\rho)\simeq(V'_K,\rho'_K)$.

Lemma 4.1. *Let (V,ρ) be a totally singular generalized quadratic space over $(D,\theta)_K$. If there exists a basis $\mathcal{B}$ of $\overline{D}$ for which $Q_{\rho,\mathcal{B}_K}$ has a descent to F, then ρ has a descent to (D,θ).*

Proof. Write $\mathcal{B}=\{u_1,\cdots,u_m\}$ for some $u_1,\cdots,u_m\in\overline{D}$ and let $q_i=q^{u_i\otimes1}_{\rho,\mathcal{B}_K}$, $i=1,\cdots,m$. Let (V',Q') be a descent of $(V,Q_{\rho,\mathcal{B}_K})$. Write $Q'=(q'_1,\cdots,q'_m)$, so that every q'_i is a descent of q_i. Let $\{v_1,\cdots,v_n\}$ be a basis of V' over F. Since the set $\{v_1\otimes1,\cdots,v_n\otimes1\}$ generates V as a vector space over D_K, it contains a basis of V over D_K. By re-indexing we may assume that $\{v_1\otimes1,\cdots,v_r\otimes1\}$ is a D_K-basis of V. Set $W=v_1D+\cdots+v_rD$ and let $\rho':W\to\overline{D}$ be the totally singular generalized quadratic form over (D,θ) induced by $\rho'(v_j)=\sum_{i=1}^{r}q'_i(v_j)u_i$, $j=1,\cdots,r$. Then $\rho\simeq\rho'_K$, proving the claim. □

The following result is analogous to [8, (2.1)] with essentially the same proof.

Lemma 4.2. *Let (V,ρ) be a generalized quadratic space over $(D,\theta)_K$ and let $\mathcal{B}$ be a basis of $\overline{D}$. If $s:K\to F$ is a nonzero F-linear map, then $s_*(Q_{\rho,\mathcal{B}_K})=Q_{s_*(\rho),\mathcal{B}}$.*

Proof. Let $\mathcal{B}=\{u_1,\cdots,u_m\}$, where $u_1,\cdots,u_m\in\overline{D}$. Fix an index $1\leqslant i\leqslant m$ and a vector $v\in V$. It is enough to prove that $s_*(q^{u_i\otimes1}_{\rho,\mathcal{B}_K})(v)=q^{u_i}_{s_*(\rho),\mathcal{B}}(v)$. Write $K=F(\eta)$ for some $\eta\in K$ and $\rho(v)=\overline{\alpha+\eta\beta}\in\overline{D_K}$ for some $\alpha,\beta\in D$. Choose $a_j,b_j\in F$, $j=1,\cdots,m$, such that

$$\overline{\alpha}=\textstyle\sum_{j=1}^{m}a_ju_j \quad \text{and} \quad \overline{\beta}=\textstyle\sum_{j=1}^{m}b_ju_j.$$

Since $\mathrm{Symd}((D,\theta)_K)=\mathrm{Symd}(D,\theta)\otimes K$, one can extend $\pi_i:\overline{D}\to F$ to a K-linear map $(\pi_i)_K:\overline{D_K}\to K$. Then

$$\begin{aligned} s_*(q^{u_i\otimes 1}_{\rho,\mathcal{B}_K})(v) &= s((\pi_i)_K(\rho(v)))\\ &= s((\pi_i)_K(\textstyle\sum_{j=1}^m (a_ju_j+b_ju_j\eta)))\\ &= s(a_i+b_i\eta)=a_is(1)+b_is(\eta)\\ &= \pi_i(\textstyle\sum_{j=1}^m a_ju_js(1)+\sum_{j=1}^m b_ju_js(\eta))\\ &= \pi_i(\overline{\alpha}s(1)+\overline{\beta}s(\eta))=\pi_i(\overline{s}(\overline{\alpha+\beta\eta}))\\ &= \pi_i(\overline{s}(\rho(v)))=\pi_i(s_*(\rho)(v))=q^{u_i}_{s_*(\rho),\mathcal{B}}(v). \end{aligned}$$ □

Lemma 4.3. *Let (V,ρ) be a generalized quadratic form over $(D,\theta)_K$ and $s\colon K\to F$ be a nonzero F-linear map with $s(1)=0$. If v is an isotropic vector of $s_*(\rho)$, then $\rho(v)\in\bar{D}$.*

Proof. Write $\rho(v)=\overline{\alpha+\beta\eta}$ for some $\alpha,\beta\in D$. Since $s(1)=0$, we have

$$0=s_*(\rho)(v)=\overline{\beta s(\eta)}=\overline{\beta}s(\eta).$$

The assumption $s\neq 0$ implies that $\overline{\beta}=0$, i.e., $\rho(v)=\bar{\alpha}\in\overline{D}$. □

The following result is analogous to [8, (2.2)].

Proposition 4.4. *Let (V,ρ) be a nonsingular generalized quadratic form over $(D,\theta)_K$ and let $s\colon K\to F$ be a nonzero F-linear map with $s(1)=0$. Then ρ has a descent to (D,θ) if and only if $s_*(\rho)$ is hyperbolic.*

Proof. Suppose first that $(V,\rho)\simeq(V',\rho')_K$ for some generalized quadratic space (V',ρ') over (D,θ). Identifying $D\subseteq D_K$, the assumption $s(1)=0$ implies that $s_D|_D=0$, hence $\bar{s}|_{\overline{D}}=0$. Considering $V'\subseteq V$, this means that $s_*(\rho)|_{V'}=0$. Note that $\dim_D V'=\frac{1}{2}\dim_D V$, so $s_*(\rho)$ is hyperbolic.

To prove the converse, write $\rho\simeq\rho_{\mathrm{an}}\perp\rho_{\mathrm{hyp}}$, where ρ_{an} is anisotropic and ρ_{hyp} is hyperbolic (see [4, Ch. I, (6.5.1)]). Observe first that ρ_{hyp} has a descent to (D,θ), because

$$\rho_{\mathrm{hyp}}\simeq(m\mathbb{H}_{(D,\theta)})_K,$$

where $m=\frac{1}{2}\dim_{D_K}\rho_{\mathrm{hyp}}$. Hence, it is enough to prove that ρ_{an} has a descent to (D,θ). By Corollary 3.3, $s_*(\rho_{\mathrm{hyp}})$ is hyperbolic. Since $s_*(\rho)\simeq s_*(\rho_{\mathrm{an}})\perp s_*(\rho_{\mathrm{hyp}})$, the form $s_*(\rho_{\mathrm{an}})$ is also hyperbolic. Let W be an underlying vector space of ρ_{an} and set $n=\dim_{D_K}W=\dim_{D_K}\rho_{\mathrm{an}}$. Then $\dim_D s_*(\rho_{\mathrm{an}})=2n$. Let $L\subseteq W$ be a lagrangian of $s_*(\rho_{\rm an})$ with a D-basis $\{v_1, \cdots, v_n\}$. We claim that $\{v_1, \cdots, v_n\}$ is linearly independent over D_K. Suppose that

$$\sum_{i=1}^{n} v_i(a_i + b_i\eta) = 0, \quad \text{for } a_i, b_i \in D,\ i = 1, \cdots, n.$$

Then $\sum_{i=1}^{n} v_i a_i = \sum_{i=1}^{n} v_i b_i \eta$, which implies that

$$\rho_{\rm an}(\sum_{i=1}^{n} v_i a_i) = \rho_{\rm an}(\sum_{i=1}^{n} v_i b_i \eta) = \eta^2 \rho_{\rm an}(\sum_{i=1}^{n} v_i b_i). \tag{1}$$

Since K/F is separable, one has $\eta^2 = \lambda\eta + \mu$ for some $\lambda, \mu \in F$ with $\lambda \neq 0$. Using (1), one concludes that

$$\rho_{\rm an}(\sum_{i=1}^{n} v_i a_i) + \mu\rho_{\rm an}(\sum_{i=1}^{n} v_i b_i) = \lambda\eta\rho_{\rm an}(\sum_{i=1}^{n} v_i b_i).$$

In view of Lemma 4.3, $\rho_{\rm an}(v) \in \bar{D}$ for every $v \in L$. Hence, the left side of the above relation belongs to $\bar{D}$. Since $\lambda \neq 0$, this implies that $\rho_{\rm an}(\sum_{i=1}^{n} v_i b_i) = 0$. However, $\rho_{\rm an}$ is anisotropic, hence $\sum_{i=1}^{n} v_i b_i = 0$, which implies that $\sum_{i=1}^{n} v_i a_i = 0$, thanks to (1). It follows that $a_i = b_i = 0$ for $i = 1, \cdots, n$, because $\{v_1, \cdots, v_n\}$ is linearly independent over D. This proves the claim.

Now, let $\rho' : L \to \bar{D}$ be the generalized quadratic form over (D, θ) defined by $\rho'(v) = \rho_{\rm an}(v) \in \bar{D}$. Note that the above claim implies that $L_K = W$. Hence, $\rho_{\rm an} = \rho'_K$, i.e., $\rho_{\rm an}$ has a descent to (D, θ). This completes the proof. □

The following result is analogous to [7, (4.2)].

Theorem 4.5. *Let ρ be a generalized quadratic form over $(D, \theta)_K$. Then ρ has a descent to (D, θ) if and only if there exists a basis $\mathcal{B}$ of $\overline{D}$ for which $Q_{\rho,\mathcal{B}_K}$ has a descent to F.*

Proof. Suppose first that ρ has a descent ρ' to (D, θ). Then for every basis $\mathcal{B}$ of $\overline{D}$, we have $Q_{\rho,\mathcal{B}_K} \simeq (Q_{\rho',\mathcal{B}})_K$, thanks to [9, (4.2)].

Conversely, suppose that there exists a basis $\mathcal{B}$ of $\overline{D}$ for which $Q_{\rho,\mathcal{B}_K}$ has a descent to F. Write $\rho \simeq \rho_{\rm ts} \perp \rho_{\rm ns}$, where $\rho_{\rm ts}$ is totally singular and $\rho_{\rm ns}$ is nonsingular. Then $Q_{\rho,\mathcal{B}_K} \simeq Q_{\rho_{\rm ts},\mathcal{B}_K} \perp Q_{\rho_{\rm ns},\mathcal{B}_K}$. By [9, (4.7)], the system $Q_{\rho_{\rm ts},\mathcal{B}_K}$ is totally singular and $Q_{\rho_{\rm ns},\mathcal{B}_K}$ is strongly nonsingular. Let Q' be a descent of $Q_{\rho,\mathcal{B}_K}$ and write $Q' \simeq Q'_{\rm ts} \perp Q'_{\rm ns}$, where $Q'_{\rm ts}$ is totally singular and $Q'_{\rm ns}$ is nonsingular. Then

$$Q_{\rho_{\rm ts},\mathcal{B}_K} \perp Q_{\rho_{\rm ns},\mathcal{B}_K} \simeq (Q'_{\rm ts})_K \perp (Q'_{\rm ns})_K. \tag{2}$$

In view of Lemma 2.1, the isometry (2) implies that $Q_{\rho_{\rm ts},\mathcal{B}_K} \simeq (Q'_{\rm ts})_K$, i.e., $Q_{\rho_{\rm ts},\mathcal{B}_K}$ has a descent to F. Hence, the form $\rho_{\rm ts}$ has a descent ρ' to (D, θ) by Lemma 4.1.

Let $s : K \to F$ be a nonzero F-linear map with $s(1) = 0$. Then the isometry (2) implies that

$$s_*(Q_{\rho_{\mathrm{ts}},\mathcal{B}_K}) \perp s_*(Q_{\rho_{\mathrm{ns}},\mathcal{B}_K}) \simeq s_*((Q'_{\mathrm{ts}})_K) \perp s_*((Q'_{\mathrm{ns}})_K). \tag{3}$$

On the other hand, the system $Q_{\rho_{\mathrm{ns}},\mathcal{B}_K}$ in (2) is strongly nonsingular, hence the system $(Q'_{\mathrm{ns}})_K$ is also strongly nonsingular by Corollary 2.2. It follows from [2, (20.4)] that $s_*((Q'_{\mathrm{ns}})_K)$ is strongly nonsingular. Also, the system $s_*((Q'_{\mathrm{ns}})_K)$ is metabolic by [8, (2.3)]. Hence, in view of [9, (3.1)], the isometry (3) implies that $s_*(Q_{\rho_{\mathrm{ns}},\mathcal{B}_K})$ is metabolic. It follows from Lemma 4.2 that $Q_{s_*(\rho_{\mathrm{ns}}),\mathcal{B}}$ is metabolic. Hence, $s_*(\rho_{\mathrm{ns}})$ is hyperbolic, thanks to [9, (4.8)]. By Proposition 4.4, this means that ρ_{ns} has a descent to (D, θ). Hence, $\rho \simeq \rho_{\mathrm{ts}} \perp \rho_{\mathrm{ns}}$ has a descent to (D, θ), completing the proof. □

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References

[1] E. Bayer-Fluckiger and H. W. Lenstra, Jr., *Forms in odd degree extensions and self-dual normal bases*, Amer. J. Math., 112(3) (1990), 359-373.

[2] R. Elman, N. Karpenko and A. Merkurjev, The Algebraic and Geometric Theory of Quadratic Forms, American Mathematical Society Colloquium Publications, 56, American Mathematical Society, Providence, RI, 2008.

[3] M. A. Elomary and J.-P. Tignol, *Classification of quadratic forms over skew fields of characteristic* 2, J. Algebra, 240(1) (2001), 366-392.

[4] M.-A. Knus, Quadratic and Hermitian Forms Over Rings, Grundlehren der Mathematischen Wissenschaften, 294, Springer-Verlag, Berlin, 1991.

[5] M.-A. Knus, A. Merkurjev, M. Rost and J.-P. Tignol, The Book of Involutions, American Mathematical Society Colloquium Publications, 44, American Mathematical Society, Providence, RI, 1998.

[6] A. H. Nokhodkar, *Hermitian forms and systems of quadratic forms*, Doc. Mat., 23 (2018), 747-758.

[7] A. H. Nokhodkar, *Quadratic descent of hermitian forms*, Math. Nachr., 292(10) (2019), 2294-2299.

[8] A. H. Nokhodkar, *Erratum to the paper "Quadratic descent of hermitian forms"*, Math. Nachr., 293(9) (2020), 1836-1838.

[9] A. H. Nokhodkar, *Applications of systems of quadratic forms to generalised quadratic forms*, Bull. Aust. Math. Soc., 102(3) (2020), 374-386.

[10] A. Pfister, Quadratic forms with Applications to Algebraic Geometry and Topology, London Mathematical Society Lecture Note Series, 217, Cambridge University Press, Cambridge, 1995.

[11] J. Tits, *Formes quadratiques, groupes orthogonaux et algèbres de Clifford*, Invent. Math., 5 (1968), 19-41.

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