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SOME RESULTS OF WEAK (σ, δ)-SKEW MCCOY RINGS

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Abstract. Rege and Chhawchharia, as well as Nielsen, introduced the concept of a right McCoy ring, based on McCoy's theorem in 1942 for the annihilators in polynomial rings over commutative rings. Alhevaz et al. in 2012 investigated a natural generalization of a right McCoy ring that is called a *weak* (σ, δ)*-skew McCoy ring*, where σ is an endomorphism and δ is a σ-derivation of a given ring. In this paper, we continue to investigate the structural properties of these rings and examine their behavior under various extensions. Specifically, we prove that while the full matrix ring over a weak (σ, δ)-skew McCoy ring does not generally preserve this property, the upper triangular matrix ring does under suitable conditions. Furthermore, we explore the stability of this property in polynomial extensions, direct products, classical right quotient rings, and skew power series rings.

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1. Introduction

Throughout this paper R denotes an associative ring with identity unless otherwise stated. Let $N(R)$ be the set of all nilpotent elements in R. We use $R[x]$ to denote the polynomial ring with an indeterminate x over R. Let $C_{f(x)}$ denote the set of all coefficients of $f(x) \in R[x]$. Denote the n by n full matrix ring over R by $Mat_n(R)$ and the n by n upper triangular matrix ring over R by $U_n(R)$. Use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. By $\mathbb{Z}_n$ we mean the ring of integers modulo n.

In 1942, McCoy [18] showed that if two polynomials annihilate each other over a commutative ring, then each polynomial has a nonzero annihilator in the base ring. Weiner [22] showed this fact fails in noncommutative rings. Based on this result, Nielsen [20] and Rege and Chhawchharia [21] called a noncommutative ring R *right McCoy* (resp., *left McCoy*) if whenever any nonzero polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $f(x)c = 0$ (resp., $cg(x) = 0$) for some nonzero $c \in R$, and a ring R is called *McCoy* if it is both left and right McCoy. The concept of a McCoy ring is generalized in [7] to a *weak McCoy* ring. A ring R is called *right weak McCoy* if for any nonzero polynomials $f(x)$ and $g(x)$ in $R[x]$, $f(x)g(x) = 0$ implies $f(x)c \in N(R)[x]$ for some $0 \neq c \in R$, equivalently, there exists $0 \neq c \in R$ such that $ac \in N(R)$ for any $a \in C_{f(x)}$. Left weak McCoy rings are defined analogously, and a ring R is called *weak McCoy* if it is both left and right weak McCoy. To have the terminology be more expressive, Camillo et al. [4] called a weak McCoy ring a *nilpotent coefficient McCoy* ring, or an *NC-McCoy* ring for short. Furthermore, the structure and several kinds of extensions of right weak McCoy rings are investigated, and the structure of minimal right weak McCoy rings is also examined in [4]. Recall that a ring R is called *reduced* if $N(R) = 0$. There exist several generalizations of a reduced ring. Cohn [5] called a ring R *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$. Due to Narbonne [19], a ring R is called *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. A ring is called *Abelian* if every idempotent is central. Semicommutative rings are easily shown to be Abelian, but not conversely in general. Nielsen developed and extended the concept of a McCoy ring. In particular, he showed that any reversible ring is McCoy [20, Theorem 2] and gave an example that is a semicommutative ring but not McCoy [20, Section 3]. Let $N_*(R)$ and $N^*(R)$ denote the prime radical and the upper nilradical (i.e., the sum of all nil two-sided ideals) of a ring R, respectively. Note $N_*(R) \subseteq N^*(R) \subseteq N(R)$. A generalization of a semicommutative ring is the 2-primal condition. A ring R (possibly without identity) is called *2-primal* [3] if $N_*(R) = N(R)$. Note that a ring R is 2-primal if and only if $R/N_*(R)$ is reduced. In [17], a ring R (possibly without identity) is called *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms a two-sided ideal if and only if $R/N^*(R)$ is reduced. It is well-known that 2-primal rings are NI, but the converse need not hold. But if R is an NI ring of bounded index of nilpotency, then R is 2-primal by [14, Proposition 1.4]. Recall that for an endomorphism σ of a ring R, the additive map $\delta : R \to R$ is called a *σ-derivation* if $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for any $a, b \in R$. For a ring R with an endomorphism σ of R and a σ-derivation δ, the *Ore extension* $R[x;\, \sigma,\, \delta]$ of R is the ring obtained by giving the polynomial ring over R a new skew-multiplication $xr = \sigma(r)x + \delta(r)$ for all $r \in R$ (while $R[[x; \sigma, \delta]]$ is called a *skew power series* ring). If $\delta = 0$, then we write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and it is called a *skew polynomial ring* (or, an *Ore extension of endomorphism type*), and for an identity endomorphism 1_R of R, we write $R[x; \delta]$ for $R[x; 1_R, \delta]$ and it is called an *Ore extension of derivation type.* In particular, $R[x; 1_R, 0]$ is the polynomial ring $R[x]$ over R. In 2009, Başer et al. [2] introduced the notion of a skew McCoy ring R with respect to an endomorphism σ

of R. Let σ be an endomorphism of a ring R. The ring R is called a *skew McCoy ring with respect to* σ (simply, σ*-skew McCoy ring*) if for any nonzero polynomials $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma]$, $p(x)q(x) = 0$ implies $p(x)c = 0$ for some nonzero $c \in R$. Habibi et al. [10] called a ring R (σ, δ)*-skew McCoy* if for any nonzero polynomials $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma, \delta]$, $p(x)q(x) = 0$ implies $a_i x^i c = 0$ for some nonzero $c \in R$ for each i, where δ is the σ-derivation. Each McCoy ring is (σ, δ)-skew McCoy, where $\sigma = 1_R$ and $\delta = 0$, but not conversely in general. In the special case, the concept of a $(\sigma, 0)$-skew McCoy ring coincides with what Başer et al. [2] called σ-skew McCoy ring.

From now on, we let σ denote an endomorphism with $\sigma(1) = 1$ (where 1 is the identity of R) of a given ring R; we do not assume that σ is nonzero or nonidentity, so that the special case $\sigma = 1_R$ discussed later in the paper is included, and the additive map δ denote a σ-derivation, unless specified otherwise. Alhevaz et al. [1] generalized the concept of a (σ, δ)-skew McCoy ring as follows and they just studied the class of rings which strictly contains such rings.

2. (σ, δ)-skew McCoy rings

Definition 2.1. [1, Definition 3.1] A ring R is called *weak* (σ, δ)*-skew McCoy* if for any nonzero polynomials $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j \in R[x; \sigma, \delta]$ with $p(x)q(x) = 0$, there exists nonzero $c \in R$ such that $a_i x^i c \in N(R)[x; \sigma, \delta]$ for all i.

In this paper, we continue the study of weak (σ, δ)-skew McCoy rings, such as their structure, characterization and extensions. Basic properties of weak (σ, δ)-skew McCoy rings are observed and some of the known results on right weak McCoy rings are obtained as corollaries.

As basic examples of weak (σ, δ)-skew McCoy rings, we have the following Proposition 2.2 and Proposition 3.3 which extend the results of [4, Proposition 3(1, 2)].

Recall that an ideal (a subset) I of a ring R is called a σ*-stable ideal (subset)* (resp., δ*-stable ideal (subset)*) of R if $\sigma(I) \subseteq I$ (resp., $\delta(I) \subseteq I$), and if I is both a σ-stable ideal (subset) and a δ-stable ideal (subset), then we call I a (σ, δ)*-stable ideal (subset)*.

Proposition 2.2. *If a ring R contains a nonzero nil (σ, δ)-stable ideal, then R is a weak (σ, δ)-skew McCoy ring.*

Proof. Let I be a nonzero nil (σ, δ)-stable ideal of R, and choose $0 \neq c \in I$. Since I is stable under both σ and δ, every coefficient occurring in the expansion of $x^i c$

belongs to I for each $i \geq 0$. Hence, for every $r \in R$, all coefficients of $rx^i c$ belong to $RI \subseteq I$, and therefore are nilpotent. Now let

$$p(x) = \sum_{i=0}^{m} a_i x^i, \qquad q(x) \in R[x;\sigma,\delta]$$

be nonzero polynomials such that $p(x)q(x) = 0$. The above argument shows that, for each i, the polynomial $a_i x^i c$ has all its coefficients in $N(R)$. Thus R is weak (σ,δ)-skew McCoy. □

Corollary 2.3. (1) *If $0 \neq N^*(R)$ is a (σ,δ)-stable ideal of a ring R, then R is a weak (σ,δ)-skew McCoy ring.*

(2) [7, Proposition 2.7] *If R is an NI ring, then R is weak McCoy.*

Proof. (1) follows from Proposition 2.2. □

3. Matrix rings

For a ring R with an endomorphism σ and σ-derivation δ, the corresponding $(a_{ij}) \mapsto (\sigma(a_{ij}))$ induces endomorphism of $Mat_n(R)$. We denote each of them by $\bar{\sigma}$. Similarly, the σ-derivation δ is also extended to $\bar{\delta}$ defined by $\bar{\delta}((a_{ij})) = (\delta(a_{ij}))$.

One may conjecture that the $n \times n$ full matrix ring over any ring is weak $(\bar{\sigma},\bar{\delta})$-skew McCoy for $n \geq 2$, but the possibility is erased by the next example, in general.

Example 3.1. We adopt the argument in the proof of [4, Theorem 5]. Let R be a reduced ring with a monomorphism σ. Consider the ring $Mat_n(R)$ for $n \geq 2$. Note that $Mat_n(R)[x;\bar{\sigma}] \cong Mat_n(R[x;\sigma])$ for $n \geq 2$. Consider nonzero polynomials

$$p(x) = \begin{pmatrix} 1 & x & \cdots & x^{n-1} \\ x^n & x^{n+1} & \cdots & x^{2n-1} \\ \vdots & \vdots & \cdots & \vdots \\ x^{n(n-1)} & x^{n(n-1)+1} & \cdots & x^{n^2-1} \end{pmatrix} \text{ and } q(x) = \begin{pmatrix} x & x & \cdots & x \\ -1 & -1 & \cdots & -1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

in $Mat_n(R)[x;\bar{\sigma}]$ with $p(x)q(x) = 0$. Assume to the contrary that $Mat_n(R)$ is weak $\bar{\sigma}$-skew McCoy. Then there exists $0 \neq C = (c_{ij}) \in Mat_n(R)$ such that $p(x)C \in N(Mat_n(R))[x;\bar{\sigma}]$. For

$$\begin{aligned} p(x) = {} & E_{11} + E_{12}x + \cdots + E_{1n}x^{n-1} + E_{21}x^n + E_{22}x^{n+1} + \cdots + E_{2n}x^{2n-1} + \cdots \\ & + E_{n1}x^{n(n-1)} + E_{n2}x^{n(n-1)+1} + \cdots + E_{nn}x^{n^2-1}, \end{aligned}$$

we have

$$E_{11}C, E_{12}\bar{\sigma}(C), \ldots, E_{1n}\bar{\sigma}^{n-1}(C), E_{21}\bar{\sigma}^{n}(C), E_{22}\bar{\sigma}^{n+1}(C), \ldots, E_{2n}\bar{\sigma}^{2n-1}(C), \ldots,$$
$$E_{n1}\bar{\sigma}^{n(n-1)}(C), E_{n2}\bar{\sigma}^{n(n-1)+1}(C), \ldots, E_{nn}\bar{\sigma}^{n^2-1}(C) \in N(Mat_n(R)).$$

Then there exists k_{ij} for any i and j such that

$$[E_{11}((c_{ij}))]^{k_{11}} = 0, [E_{12}(\sigma(c_{ij}))]^{k_{12}} = 0, \ldots, [E_{1n}(\sigma^{n-1}(c_{ij}))]^{k_{1n}} = 0,$$
$$[E_{21}(\sigma^{n}(c_{ij}))]^{k_{21}} = 0, [E_{22}(\sigma^{n+1}(c_{ij}))]^{k_{22}} = 0, \ldots, [E_{2n}(\sigma^{2n-1}(c_{ij}))]^{k_{2n}} = 0, \ldots,$$
$$[E_{n1}(\sigma^{n(n-1)}(c_{ij}))]^{k_{n1}} = 0, [E_{n2}(\sigma^{n(n-1)+1}(c_{ij}))]^{k_{n2}} = 0, \ldots, [E_{nn}(\sigma^{n^2-1}(c_{ij}))]^{k_{nn}} = 0.$$

To see why this forces $C = 0$, fix $1 \le p, q \le n$ and $k \ge 0$, and write $\bar{\sigma}^k(C) = (\sigma^k(c_{ij}))$. Since E_{pq} has 1 in the (p,q) position and 0 elsewhere, matrix multiplication gives

$$(E_{pq}\bar{\sigma}^k(C))_{p,l} = \sigma^k(c_{ql}) \text{ for all } l, \qquad (E_{pq}\bar{\sigma}^k(C))_{s,l} = 0 \text{ for } s \ne p.$$

Thus $E_{pq}\bar{\sigma}^k(C)$ has all its nonzero entries confined to row p, and an easy induction on m gives

$$\left(E_{pq}\bar{\sigma}^k(C)\right)^m_{p,l} = \left(\sigma^k(c_{qp})\right)^{m-1}\sigma^k(c_{ql}) \quad (m \ge 1), \text{ and } \left(E_{pq}\bar{\sigma}^k(C)\right)^m_{s,l} = 0 \ (s \ne p).$$

Hence if $E_{pq}\bar{\sigma}^k(C)$ is nilpotent, say its k_{pq}-th power is 0, then taking $l = p$ gives $\left(\sigma^k(c_{qp})\right)^{k_{pq}} = 0$, so $\sigma^k(c_{qp}) \in N(R)$. Since R is reduced, $N(R) = 0$, and so $\sigma^k(c_{qp}) = 0$; as σ is a monomorphism, so is σ^k, and therefore $c_{qp} = 0$. Applying this to each of the n^2 terms listed above - where, following the indexing of $p(x)$, the pair (i,j) ranges over all $1 \le i, j \le n$ with exponent $k = n(i-1) + (j-1)$ - yields $c_{ji} = 0$ for every $i, j \in \{1, \ldots, n\}$. Thus $C = 0$; which is a contradiction. Thus $Mat_n(R)$ is not weak $(\bar{\sigma}, \bar{0})$-skew McCoy.

Note that the class of weak (σ,δ)-skew McCoy rings is not closed under homomorphic images by the following example. For a ring R with an endomorphism σ and the σ-derivation δ, let I be a (σ,δ)-ideal of R. Then $\bar{\sigma} : R/I \to R/I$ defined by $\bar{\sigma}(r+I) = \sigma(r) + I$ is an endomorphism of the factor ring R/I and $\bar{\delta} : R/I \to R/I$ defined by $\bar{\delta}(r+I) = \delta(r) + I$ is a $\bar{\sigma}$-derivation of R/I.

Example 3.2. It is a well-known fact that if D is any domain with a monomorphism σ and a σ-derivation δ, then $D[x;\sigma,\delta]$ is also a domain. Hence D is (σ,δ)-skew McCoy, and therefore weak (σ,δ)-skew McCoy. Now, let R be the ring of quaternions with integer coefficients with the identity endomorphism $\sigma = 1_R$ and $\delta = 0$ (so that the derivation induced on the factor ring below is also 0, matching the hypothesis of Example 3.1), then R is weak (σ,δ)-skew McCoy, since R is a domain.

But for any odd prime integer q, we have $R/qR \cong Mat_2(\mathbb{Z}_q)$ by the argument in [8, Exercise 2A]. Since q is prime, $\mathbb{Z}_q$ is a field and in particular reduced, and σ induces a monomorphism $\bar{\sigma}$ on R/qR; since $\delta = 0$, Example 3.1 applies directly to $R/qR \cong Mat_2(\mathbb{Z}_q)$. Then the factor ring R/qR is not weak $(\bar{\sigma}, \bar{0})$-skew McCoy by Example 3.1.

Alhevaz et al. [1] introduced a *skew triangular matrix ring* as a set of all upper triangular matrices with addition point-wise and new multiplication defined by $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i(i+1)}\alpha(b_{(i+1)j}) + \cdots + a_{ij}\alpha^{j-i}(b_{jj})$ for each $i \leq j$, where $(a_{ij}), (b_{ij}) \in U_n(R)$ $(n \geq 2)$ over a ring R with an endomorphism α, and denoted by $U_n(R, \alpha)$. Note that $U_n(R, 1_R) = U_n(R)$, where 1_R is the identity endomorphism of R. Let $D_n(R)$ be the ring of all matrices in $U_n(R)$ whose diagonal entries are all equal, and $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Then $D_n(R, \alpha)$ and $V_n(R, \alpha)$ are subrings of $U_n(R, \alpha)$, and $V_n(R, \alpha)$ is a subring of $D_n(R, \alpha)$. There is a ring isomorphism $\phi : R[x; \alpha]/(x^n) \to V_n(R, \alpha)$, given by $\phi(a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + (x^n)) = (a_0, a_1, \ldots, a_{n-1})$, with $a_i \in R$, where (x^n) is the ideal generated by x^n. So $V_n(R, \alpha) \cong R[x; \alpha]/(x^n)$.

Let σ and α be endomorphisms and δ a σ-derivation of R such that $\sigma\alpha = \alpha\sigma$, $\delta\alpha = \alpha\delta$ and $\sigma\delta = \delta\sigma$. Then the corresponding $(a_{ij}) \mapsto (\sigma(a_{ij}))$ induces endomorphisms of $U_n(R, \alpha)$, $D_n(R, \alpha)$ and $V_n(R, \alpha)$, respectively. We denote each of them by $\bar{\sigma}$. Similarly, the σ-derivation δ is also extended to $\bar{\delta}$ defined by $\bar{\delta}((a_{ij})) = (\delta(a_{ij}))$.

The following generalizes the results of [4, Proposition 3(4, 5, 6)] and [7, Proposition 2.2].

Proposition 3.3. *Let σ and α be endomorphisms and δ a σ-derivation of R such that $\sigma\alpha = \alpha\sigma$, $\delta\alpha = \alpha\delta$ and $\sigma\delta = \delta\sigma$. Then we have the following.*

(1) *$U_n(R, \alpha)$ is a weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy ring for $n \geq 2$.*
(2) *$D_n(R, \alpha)$ is a weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy ring for $n \geq 2$.*
(3) *$V_n(R, \alpha)$ is a weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy ring for $n \geq 2$.*

Proof. (1) Put $c = E_{1n}$. Then $0 \neq c \in U_n(R, \alpha)$. For every $B = (b_{ij}) \in U_n(R, \alpha)$, the definition of the multiplication in $U_n(R, \alpha)$ gives $BE_{1n} = b_{11}E_{1n}$. Consequently, $(BE_{1n})^2 = 0$ since $n \geq 2$, and hence $BE_{1n} \in N(U_n(R, \alpha))$.

Since $\sigma(1) = 1$ and δ is a σ-derivation, $\delta(1) = 0$. Thus $\bar{\sigma}(E_{1n}) = E_{1n}$ and $\bar{\delta}(E_{1n}) = 0$. Therefore, in $U_n(R, \alpha)[x; \bar{\sigma}, \bar{\delta}]$, $xE_{1n} = E_{1n}x$, and hence $x^iE_{1n} = E_{1n}x^i$ for every $i \geq 0$.

Now let $p(x) = \sum_{i=0}^{m} A_i x^i$ be a nonzero polynomial. For every i, $A_i x^i c = A_i E_{1n} x^i \in N(U_n(R,\alpha))[x;\bar{\sigma},\bar{\delta}]$. Thus the same nonzero element $c = E_{1n}$ satisfies the condition in Definition 2.1, and $U_n(R,\alpha)$ is weak $(\bar{\sigma},\bar{\delta})$-skew McCoy.

(2) and (3) Since $E_{1n} \in V_n(R,\alpha) \subseteq D_n(R,\alpha)$, and both subrings are stable under $\bar{\sigma}$ and $\bar{\delta}$, the same argument applies verbatim. Hence $D_n(R,\alpha)$ and $V_n(R,\alpha)$ are weak $(\bar{\sigma},\bar{\delta})$-skew McCoy. □

The next example shows that the class of weak (σ,δ)-skew McCoy rings strictly contains (σ,δ)-skew McCoy rings, in general.

Example 3.4. For any ring R with a monomorphism σ, $\alpha = 1_R$ and $\delta = 0$, let $A = U_2(R)$. Then A is weak $\bar{\sigma}$-skew McCoy by Proposition 3.3(1). For

$$p(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x, \; q(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x \in A[x;\bar{\sigma}],$$

we have $p(x)q(x) = 0$. To see that there cannot exist a nonzero $C \in A$ with $p(x)C = 0$, write $C = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in A$. Since $p(x) = E_{11} - E_{12}x$ and $\delta = 0$, we have $xC = \bar{\sigma}(C)x$, we compute

$$p(x)C = E_{11}C - E_{12}\bar{\sigma}(C)x = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma(d) \\ 0 & 0 \end{pmatrix} x.$$

Thus $p(x)C = 0$ forces $a = 0$, $b = 0$, and $\sigma(d) = 0$; since σ is a monomorphism, $\sigma(d) = 0$ implies $d = 0$, so $C = 0$. Hence there cannot exist nonzero $C \in A$ such that $p(x)C = 0$, and thus A is not $\bar{\sigma}$-skew McCoy.

A ring R is called *Dedekind finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Every Abelian ring is Dedekind finite but not conversely. Moreover there exists a weak (σ,δ)-skew McCoy ring which is not Dedekind finite by almost the same argument as in [4, Example 7].

4. Polynomial rings

For an endomorphism σ and a σ-derivation δ of a ring R, the map $\bar{\sigma} : R[y] \to R[y]$ defined by $\bar{\sigma}(\sum_{i=0}^{m} a_i y^i) = \sum_{i=0}^{m} \sigma(a_i) y^i$ is an endomorphism of the polynomial ring $R[y]$, and clearly this map extends σ, and the σ-derivation δ of R is also extended to $\bar{\delta} : R[y] \to R[y]$ defined by $\bar{\delta}(\sum_{i=0}^{m} a_i y^i) = \sum_{i=0}^{m} \delta(a_i) y^i$. It can be easily checked that $\bar{\delta}$ is a $\bar{\sigma}$-derivation of the ring $R[y]$. Throughout this section, we reserve x for the indeterminate of an Ore extension (such as $R[x;\sigma,\delta]$ or $(R[y])[x;\bar{\sigma},\bar{\delta}]$) and y for the indeterminate of the ordinary polynomial ring $R[y]$, so as to avoid any confusion between the two.

Theorem 4.1. *For a ring R, let σ be an endomorphism of R and δ a σ-derivation with $\delta\sigma = \sigma\delta$. If $R[y]$ is weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy, then R is weak (σ, δ)-skew McCoy.*

Proof. Suppose that $R[y]$ is weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy, and let $p(x) = \sum_{i=0}^{m} a_i x^i$ and $q(x) = \sum_{j=0}^{n} b_j x^j$ be nonzero elements of $R[x; \sigma, \delta]$ such that $p(x)q(x) = 0$.

Via the natural embedding $\iota : R[x; \sigma, \delta] \longrightarrow (R[y])[x; \bar{\sigma}, \bar{\delta}]$, we may regard $p(x)$ and $q(x)$ as nonzero polynomials in $(R[y])[x; \bar{\sigma}, \bar{\delta}]$, still satisfying $p(x)q(x) = 0$. Hence there exists $0 \neq c(y) = c_0 + c_1 y + \cdots + c_r y^r \in R[y]$ such that $a_i x^i c(y) \in N(R[y])[x; \bar{\sigma}, \bar{\delta}]$ for every $0 \leq i \leq m$.

Since $\sigma\delta = \delta\sigma$, we have $x^i c(y) = \sum_{k=0}^{i} \binom{i}{k} \bar{\sigma}^{\,i-k} \bar{\delta}^{\,k}(c(y)) x^{i-k}$. Therefore $a_i x^i c(y) = \sum_{k=0}^{i} \binom{i}{k} a_i \bar{\sigma}^{\,i-k} \bar{\delta}^{\,k}(c(y)) x^{i-k}$. Thus, for every i and every $0 \leq k \leq i$, $h_{ik}(y) := \binom{i}{k} a_i \bar{\sigma}^{\,i-k} \bar{\delta}^{\,k}(c(y)) \in N(R[y])$.

Let l be the least integer such that $c_l \neq 0$. Since $c_j = 0$ for $j < l$, the polynomial $h_{ik}(y)$ has no terms of degree less than l, and the coefficient of y^l is $\binom{i}{k} a_i \sigma^{\,i-k} \delta^k(c_l)$.

Since $h_{ik}(y)$ is nilpotent, say $h_{ik}(y)^t = 0$, the coefficient of y^{lt} in $h_{ik}(y)^t$ is $\left(\binom{i}{k} a_i \sigma^{\,i-k} \delta^k(c_l)\right)^t$. Hence $\binom{i}{k} a_i \sigma^{\,i-k} \delta^k(c_l) \in N(R)$ for every i and k.

Consequently, every coefficient of $a_i x^i c_l = \sum_{k=0}^{i} \binom{i}{k} a_i \sigma^{\,i-k} \delta^k(c_l) x^{i-k}$ is nilpotent. Therefore $a_i x^i c_l \in N(R)[x; \sigma, \delta]$ for every i. Since $c_l \neq 0$, Definition 2.1 shows that R is weak (σ, δ)-skew McCoy. $\square$

Remark 4.2. For a ring R, let σ be an endomorphism and δ a σ-derivation of R. If R is a ring with the nonzero nilpotent (σ, δ)-stable ideal $N^*(R)$, then $N^*(R)$ and $N^*(R)[y]$ are nil (σ, δ)-ideals of R and $R[y]$, respectively. Thus R is weak (σ, δ)-skew McCoy and $R[y]$ is weak $(\bar{\sigma}, \bar{\delta})$-skew McCoy by Proposition 2.2.

According to Krempa [16], an endomorphism σ of a ring R is called *rigid* if $a\sigma(a) = 0$ implies $a = 0$ for $a \in R$. Hong et al. [12] called a ring R a *σ-rigid ring* if there exists a rigid endomorphism σ of R. Note that any rigid endomorphism of a ring is a monomorphism and σ-rigid rings are reduced rings by [12, Proposition 5]. Following [11], a ring R is called *σ-compatible* if for each $a, b \in R$, $ab = 0 \Leftrightarrow a\sigma(b) = 0$, and R is called *δ-compatible* if for each $a, b \in R$, $ab = 0 \Rightarrow a\delta(b) = 0$. If R is both σ-compatible and δ-compatible, then R is called (σ, δ)-compatible, in this case, the endomorphism σ is clearly a monomorphism. In [1, Corollary 3.7], every (σ, δ)-compatible and semicommutative ring is weak (σ, δ)-skew McCoy. On the other hand, for a σ-ideal I of a ring R, I is called a *σ-rigid ideal* of R [13] if $a\sigma(a) \in I$ for $a \in R$ implies $a \in I$. Obviously, R is a σ-rigid ring if and only if the zero ideal of R is a σ-rigid ideal. If R is a σ-rigid ring, then $N^*(R) = 0$ is clearly a σ-rigid ideal, but not conversely by [15, Example 2.4].

Proposition 4.3. *For a ring R, let σ be an endomorphism and δ a σ-derivation of R. If $N^*(R)$ is a σ-rigid and δ-ideal of R, then R is weak (σ,δ)-skew McCoy.*

Proof. Suppose that $N^*(R)$ is a σ-rigid and δ-ideal and let $p(x)q(x) = 0$ for nonzero polynomials $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_nx^n$ in $R[x;\sigma,\delta]$. Since $p(x)q(x) = 0$, in particular $p(x)q(x) \in N(R)[x;\sigma,\delta]$ (the zero polynomial trivially has all coefficients nilpotent). As $N^*(R)$ is a σ-rigid δ-ideal of R, [15, Theorem 2.12] applies with $n = 2$ to $p(x), q(x) \in R[x;\sigma,\delta]$ and gives

$$a_ib_j \in N(R) \quad \text{for every } a_i \in C_{p(x)} = \{a_0,\dots,a_m\} \text{ and } b_j \in C_{q(x)} = \{b_0,\dots,b_n\}.$$

Fix any nonzero coefficient b_{j_0} of $q(x)$ (one exists since $q(x) \neq 0$). By [15, Proposition 2.5(1)] (again using that $N^*(R)$ is a σ-rigid δ-ideal), $a_ib_{j_0} \in N(R)$ for each i implies $a_if^i_k(b_{j_0}) \in N(R) \quad$ for all $0 \le k \le i$, where f^i_k is the sum of all words in σ,δ occurring in the generalized Leibniz expansion $x^ir = \sum_{k=0}^{i} f^i_k(r)x^k$ of $R[x;\sigma,\delta]$. Hence every coefficient of

$$a_ix^ib_{j_0} = a_i\Big(\sum_{k=0}^{i} f^i_k(b_{j_0})x^k\Big) = \sum_{k=0}^{i} a_if^i_k(b_{j_0})x^k$$

lies in $N(R)$, that is, $a_ix^ib_{j_0} \in N(R)[x;\sigma,\delta]$ for every i, showing that R is weak (σ,δ)-skew McCoy. □

Corollary 4.4. *Every (σ,δ)-compatible NI ring is weak (σ,δ)-skew McCoy.*

Proof. By [15, Proposition 2.3], if R is a (σ,δ)-compatible NI ring, then $N^*(R)$ is a σ-rigid δ-ideal of R. It follows from Proposition 4.3 that R is weak (σ,δ)-skew McCoy. □

Note that we don't know whether or not the weak (σ,δ)-skew McCoy property of R can go up to the weak $(\bar{\sigma},\bar{\delta})$-skew McCoy property of $R[x]$. But the following provides us some condition for a weak σ-skew McCoy ring R such that $R[x]$ becomes a weak $\bar{\sigma}$-skew McCoy ring.

Proposition 4.5. *For a ring R, let σ be an endomorphism of R such that $\sigma^t = 1_R$ for some positive integer t where 1_R denotes the identity endomorphism of R. If R is a weak σ-skew McCoy ring with $N(R)[x] \subseteq N(R[x])$, then $R[x]$ is weak $\bar{\sigma}$-skew McCoy.*

Proof. Let R be a weak σ-skew McCoy ring and $N(R)[x] \subseteq N(R[x])$. Suppose that $p(y) = f_0(x) + \cdots + f_m(x)y^m$ and $q(y) = g_0(x) + \cdots + g_n(x)y^n$ are nonzero polynomials in $(R[x])[y;\bar{\sigma}]$ with $p(y)q(y) = 0$. We also let $f_i(x) = a_{i_0} + \cdots + a_{i_{w_i}}x^{i_{w_i}}$, $g_j(x) = b_{j_0} + \cdots + b_{j_{v_j}}x^{j_{v_j}}$ for each $0 \le i \le m$ and $0 \le j \le n$, where $a_{i_0},\ldots,a_{i_{w_i}},b_{j_0},\ldots,b_{j_{v_j}} \in R$. Take a positive integer k such that $k = 1+\sum_{i=0}^{m}\deg(f_i)+\sum_{j=0}^{n}\deg(g_j)$ where the degree of the zero polynomial is taken to be 0. Now set

$$P = f_0(x^t) + f_1(x^t)x^{tk+1} + \cdots + f_m(x^t)x^{(tk+1)m},$$

and

$$Q = g_0(x^t) + g_1(x^t)x^{tk+1} + \cdots + g_n(x^t)x^{(tk+1)n}$$

in $R[x;\sigma]$. Then $P \neq 0$ and $Q \neq 0$ from $p(y) \neq 0$ and $q(y) \neq 0$. Thus $p(y)q(y) = 0$ in $(R[x])[y;\bar{\sigma}]$ implies $PQ = 0$ in $R[x;\sigma]$, since the set of coefficients of the f_i's (resp., g_j's) equals the set of coefficients of P (resp., Q) and x^t commutes with all elements of R in $R[x;\sigma]$ from $\sigma^t = 1_R$ by hypothesis. Since R is weak σ-skew McCoy, there exists nonzero $c \in R$ such that $Pc \in N(R)[x;\sigma]$. Then $f_i(x^t)\sigma^i(c) \in N(R)[x;\sigma,0]$ and so $a_{i_j}\sigma^i(c) \in N(R)$ for each $i = 0,\ldots,m$ and $j = 0,\ldots,w_i$. Hence $(a_{i_0} + \cdots + a_{i_{w_i}}x^{iw_i})\sigma^i(c) \in N(R)[x] \subseteq N(R[x])$ for all i and thus $f_i(x)\bar{\sigma}^i(c)y^i \in N(R[x])[y;\bar{\sigma},\bar{0}]$, entailing that $p(y)c \in N(R[x])[y;\bar{\sigma}]$ for $0 \neq c \in R[x]$. Therefore $R[x]$ is a weak $\bar{\sigma}$-skew McCoy ring. □

In Proposition 4.5, if $N(R[x])$ is a subring of $R[x]$, then the condition "$N(R)[x] \subseteq N(R[x])$" can be obtained. In fact, for any $a \in N(R)$ and nonnegative integer t, ax^t is nilpotent. Thus $ax^t \in N(R[x])$, and so $N(R)[x] \subseteq N(R[x])$ as the latter is closed under addition.

5. Other relations

Let R_γ be a ring with an endomorphism σ_γ and a σ_γ-derivation δ_γ for each $\gamma \in \Gamma$. Let $\prod_{\gamma\in\Gamma} R_\gamma$ and $\oplus_{\gamma\in\Gamma}R_\gamma$ be the direct product and direct sum of R_γ, respectively. If we take $R = \prod_{\gamma\in\Gamma} R_\gamma$ or $R = \oplus_{\gamma\in\Gamma}R_\gamma$, then the map $\sigma : R \to R$ defined by $\sigma((a_\gamma)) = (\sigma_\gamma(a_\gamma))$ is an endomorphism and the map $\delta : R \to R$ defined by $\delta((a_\gamma)) = (\delta_\gamma(a_\gamma))$ is a σ-derivation.

Recall that a ring R is said to be *of bounded index (of nilpotency)* if there exists a positive integer n such that $a^n = 0$ for all $a \in N(R)$.

For the direct-sum assertion below, rings are allowed to be without identity when Γ is infinite.

Proposition 5.1. *Let R_γ be a ring with an endomorphism σ_γ and a σ_γ-derivation δ_γ for each $\gamma \in \Gamma$, and let σ be an endomorphism and δ a σ-derivation of $R = \prod_{\gamma\in\Gamma} R_\gamma$ as defined above.*

(1) *Suppose that the direct product $R=\prod_{\gamma\in\Gamma}R_\gamma$ is of bounded index of nilpotency. If R_γ is weak $(\sigma_\gamma,\delta_\gamma)$-skew McCoy for each $\gamma\in\Gamma$, then the direct product of R_γ is weak (σ,δ)-skew McCoy.*

(2) *If R_γ is weak $(\sigma_\gamma,\delta_\gamma)$-skew McCoy for each $\gamma\in\Gamma$, then the direct sum of R_γ is weak (σ,δ)-skew McCoy.*

Proof. (1) Let $R=\prod_{\gamma\in\Gamma}R_\gamma$ and suppose that R has bounded index of nilpotency. Then $N(R)=\prod_{\gamma\in\Gamma}N(R_\gamma)$. Indeed, one inclusion is immediate. Conversely, the common bound on the nilpotency indices in R also bounds the nilpotency indices in every R_γ, and hence any element of $\prod_{\gamma\in\Gamma}N(R_\gamma)$ is nilpotent in R.

Let $p(x)=\sum_{i=0}^m a_ix^i$ and $q(x)=\sum_{j=0}^n b_jx^j$ be nonzero elements of $R[x;\sigma,\delta]$ with $p(x)q(x)=0$. Write $p(x)=(p^{(\gamma)}(x))_{\gamma\in\Gamma}$ and $q(x)=(q^{(\gamma)}(x))_{\gamma\in\Gamma}$.

For each $\gamma\in\Gamma$, choose $c_\gamma\in R_\gamma$ as follows. If $q^{(\gamma)}(x)=0$, put $c_\gamma=0$. If $q^{(\gamma)}(x)\neq 0$ and $p^{(\gamma)}(x)=0$, choose any $0\neq c_\gamma\in R_\gamma$. Finally, if both $p^{(\gamma)}(x)$ and $q^{(\gamma)}(x)$ are nonzero, the weak $(\sigma_\gamma,\delta_\gamma)$-skew McCoy property of R_γ provides $0\neq c_\gamma\in R_\gamma$ such that, for every i, $a_i^{(\gamma)}x^ic_\gamma\in N(R_\gamma)[x;\sigma_\gamma,\delta_\gamma]$.

Put $c=(c_\gamma)_{\gamma\in\Gamma}$. For every i, each coefficient of a_ix^ic has its γ-coordinate in $N(R_\gamma)$. Hence, by $N(R)=\prod_{\gamma\in\Gamma}N(R_\gamma)$, we obtain $a_ix^ic\in N(R)[x;\sigma,\delta]$ for every i. Since $q(x)\neq 0$, there exists γ_0 such that $q^{(\gamma_0)}(x)\neq 0$. By the construction above, $c_{\gamma_0}\neq 0$, and hence $c\neq 0$. Therefore R is weak (σ,δ)-skew McCoy.

(2) Let $R=\bigoplus_{\gamma\in\Gamma}R_\gamma$. Use exactly the same choice of c_γ as above. Since $c_\gamma=0$ whenever $q^{(\gamma)}(x)=0$, we have $\operatorname{supp}(c)\subseteq\operatorname{supp}(q)$, and the latter is finite. Thus $c\in\bigoplus_{\gamma\in\Gamma}R_\gamma$.

Moreover, $N\big(\bigoplus_{\gamma\in\Gamma}R_\gamma\big)=\bigoplus_{\gamma\in\Gamma}N(R_\gamma)$, because every element of the direct sum has finite support. The same coordinatewise argument therefore gives $a_ix^ic\in N(R)[x;\sigma,\delta]$ for every i, with $c\neq 0$. Hence the direct sum is weak (σ,δ)-skew McCoy. □

Recall that a subset S of a ring R is strictly multiplicative if $0\notin S$, $1\in S$, and $ab\in S$ for any $a,b\in S$. The set S is a right Ore set if for any $a\in R$ and $s\in S$, $aS\cap sR\neq\emptyset$. If S is the set of all regular elements of R, and S is a right Ore set, then R has a classical right quotient ring $Q(R)=\{as^{-1}\mid a\in R,s\in S\}$.

To extend σ and δ to $Q(R)$, we assume throughout this part of the section that

$$(*)\qquad \sigma(s)=s\quad\text{and}\quad\delta(s)=0\qquad\text{for every } s\in S.$$

This is the natural compatibility needed to make the extension well defined, and, as shown below, is exactly what allows elements of S to be moved freely past x when clearing denominators. Under $(*)$, the extensions $\overline{\sigma},\overline{\delta}:Q(R)\to Q(R)$ defined by

$\overline{\sigma}(as^{-1}) = \sigma(a)\sigma(s)^{-1} = \sigma(a)s^{-1}$, $\overline{\delta}(as^{-1}) = \delta(a)\sigma(s)^{-1} - \sigma(a)\sigma(s)^{-1}\delta(s)\sigma(s)^{-1} = \delta(a)s^{-1}$ make $\overline{\sigma}$ an endomorphism of $Q(R)$ extending σ and $\overline{\delta}$ a $\overline{\sigma}$-derivation of $Q(R)$ extending δ. In particular, $(*)$ gives, for every $s \in S$,

$$xs = \overline{\sigma}(s)x + \overline{\delta}(s) = sx,$$

so every element of S commutes with x in $Q(R)[x; \overline{\sigma}, \overline{\delta}]$, and hence $x^i s = sx^i$ for all $i \geq 0$.

Remark 5.2. For any nonzero $F(x) = \sum_{i=0}^{k} \gamma_i x^i \in Q(R)[x; \overline{\sigma}, \overline{\delta}]$, there exist $v \in S$ and $c_0, \dots, c_k \in R$ with $\gamma_i v = c_i$ for every i (a common right denominator for the finitely many coefficients γ_i), obtained by the standard iterated application of the right Ore condition (see e.g. [8, Ch. 9]). By $(*)$,

$$F(x)v = \sum_i \gamma_i x^i v = \sum_i \gamma_i v x^i = \sum_i c_i x^i =: f(x) \in R[x; \sigma, \delta],$$

and $f(x) \neq 0$ since v is invertible in $Q(R)$. Note that the correspondence $c_i = \gamma_i v$ is degree-preserving: distinct powers of x are never mixed.

Theorem 5.3. *Let R be a ring with a classical right quotient ring $Q(R)$, where S satisfies $(*)$, and let $\overline{\sigma}, \overline{\delta}$ be the extensions of σ, δ to $Q(R)$ described above. If R is weak (σ, δ)-skew McCoy, then $Q(R)$ is weak $(\overline{\sigma}, \overline{\delta})$-skew McCoy.*

Proof. Let $P(x)Q(x) = 0$ for nonzero $P(x) = \sum_i \alpha_i x^i$, $Q(x) = \sum_j \beta_j x^j \in Q(R)[x; \overline{\sigma}, \overline{\delta}]$. By Remark 5.2 applied to $P(x)$, there is $v \in S$ with $p(x) := P(x)v = \sum_i c_i x^i \in R[x; \sigma, \delta]$ nonzero, where $c_i = \alpha_i v$. Since $P(x) = p(x)v^{-1}$, $0 = P(x)Q(x) = p(x)\big(v^{-1}Q(x)\big)$, and $F(x) := v^{-1}Q(x)$ is nonzero, as v^{-1} is a unit of $Q(R)$ and $Q(x) \neq 0$. Applying Remark 5.2 again, this time to $F(x)$, gives $u \in S$ with $q(x) := F(x)u \in R[x; \sigma, \delta]$ nonzero. Right multiplying the equation $p(x)F(x) = 0$ by u yields $p(x)q(x) = p(x)F(x)u = 0$. Thus $p(x), q(x)$ are nonzero elements of $R[x; \sigma, \delta]$ with $p(x)q(x) = 0$.

Since R is weak (σ, δ)-skew McCoy, there exists $0 \neq \gamma \in R$ such that $c_i x^i \gamma \in N(R)[x; \sigma, \delta]$ for every i. Put $C := v\gamma \in R \subseteq Q(R)$; since v is a unit of $Q(R)$ and $\gamma \neq 0$, $C \neq 0$. Using associativity and then $(*)$ (so that v may be moved past each power of x via $x^i v = vx^i$), $\alpha_i x^i C = \alpha_i x^i (v\gamma) = (\alpha_i x^i v)\gamma = (\alpha_i v x^i)\gamma = (\alpha_i v)x^i\gamma = c_i x^i \gamma$ for every i. Hence $\alpha_i x^i C = c_i x^i \gamma \in N(R)[x; \sigma, \delta] \subseteq N(Q(R))[x; \overline{\sigma}, \overline{\delta}]$ for every i, where the last inclusion holds because every nilpotent element of R remains nilpotent in the overring $Q(R)$. Since $C \neq 0$, this shows that $Q(R)$ is weak $(\overline{\sigma}, \overline{\delta})$-skew McCoy. $\square$

In the Ore extension the twisted multiplication is governed by

$$(\dagger) \qquad x^n c = \sum_{t=0}^{n} \pi_t^n(c)\, x^t,$$

where each π_t^n is a finite sum of length-n words in σ and δ (in particular $\pi_n^n = \sigma^n$); every π_t^n maps a (σ, δ)-stable subset of R into itself. From now on, we assume that $R[[x; \sigma, \delta]]$ is well defined; a standard sufficient condition is that δ be *locally nilpotent* (for each $b \in R$, $\delta^m(b) = 0$ for some m), as shown by [9].

Lemma 5.4. *Suppose $N(R)$ is a nilpotent (σ, δ)-stable ideal, say $N(R)^k = 0$. Then $N(R)[[x; \sigma, \delta]]$ is a two-sided ideal of $R[[x; \sigma, \delta]]$ with $(N(R)[[x; \sigma, \delta]])^k = 0$; consequently, $N(R)[x; \sigma, \delta] \subseteq N(R)[[x; \sigma, \delta]] \subseteq N(R[[x; \sigma, \delta]])$.*

Proof. Put $I = N(R)$. By hypothesis, $I^k = 0$. As I is a (σ, δ)-stable ideal, we verify directly that $N(R)[[x; \sigma, \delta]] = I[[x; \sigma, \delta]]$ is a two-sided ideal of $R[[x; \sigma, \delta]]$: for $c \in I$, closure under right multiplication follows from $(cx^i)r = \sum_{t=0}^{i} c\, \pi_t^i(r)\, x^t \in I[[x; \sigma, \delta]]$ since I is a right ideal, while closure under left multiplication follows from $r(cx^i) = (rc)x^i$ and $x(cx^i) = \sigma(c)x^{i+1} + \delta(c)x^i$, using $\sigma(c), \delta(c) \in I$. Furthermore, by repeated use of the multiplication rule (†), each coefficient of a product of k elements of $I[[x; \sigma, \delta]]$ is obtained as a finite sum. Expanding each map π_t^m into its constituent words and applying distributivity, every such coefficient becomes a finite sum of terms of the form $c_1\, w_1(c_2)\, w_2(c_3) \cdots w_{k-1}(c_k)$, where $c_1, \ldots, c_k \in I$ and each w_i is a word in σ and δ. Since I is (σ, δ)-stable, both σ and δ map I into itself; hence every composition of σ and δ also maps I into itself, so $w_i(c_j) \in I$ and every summand belongs to I^k. Therefore every coefficient lies in $I^k = 0$, and $\big(N(R)[[x; \sigma, \delta]]\big)^k = 0$. Since $N(R)[[x; \sigma, \delta]]$ is a nilpotent ideal of $R[[x; \sigma, \delta]]$, every element of it is nilpotent; hence $N(R)[[x; \sigma, \delta]] \subseteq N(R[[x; \sigma, \delta]])$. Also $N(R)[x; \sigma, \delta] \subseteq N(R)[[x; \sigma, \delta]]$ trivially. □

Theorem 5.5. *Let R be a ring with an automorphism σ and a σ-derivation δ with $\delta\sigma = \sigma\delta$. Assume that*

1. *every nonzero right annihilator in $R[[x; \sigma, \delta]]$ contains a nonzero polynomial in $R[x; \sigma, \delta]$;*
2. *R is a weak (σ, δ)-skew McCoy ring; and*
3. *$N(R)$ is a nilpotent (σ, δ)-stable ideal.*

If $0 \neq f(x) \in R[x; \sigma, \delta]$ and $0 \neq g(x) \in R[[x; \sigma, \delta]]$ satisfy $f(x)g(x) = 0$, then there exists $0 \neq c \in R$ such that $f(x)\, c \in N(R)[x; \sigma, \delta] \subseteq N\big(R[[x; \sigma, \delta]]\big)$.

Proof. Let $I = r_{R[[x;\sigma,\delta]]}(f(x))$ be the right annihilator of $f(x)$ in $R[[x; \sigma, \delta]]$. Since $f(x)g(x) = 0$ with $g(x) \neq 0$, we have $g(x) \in I$, so I is a nonzero right annihilator.

By (1), I contains a nonzero polynomial $h(x)$. Hence $f(x)$ and $h(x)$ are nonzero elements of $R[x;\sigma,\delta]$ satisfying $f(x)h(x)=0$. By (2) and Definition 2.1 applied to the pair (f,h), there is $0\neq c\in R$ with $a_ix^ic\in N(R)[x;\sigma,\delta]$ for all i, where $f(x)=\sum_{i=0}^n a_ix^i$. Since $N(R)[x;\sigma,\delta]$ is closed under finite sums (being an additive subgroup, indeed an ideal), $f(x)\,c=\sum_{i=0}^n a_ix^ic\in N(R)[x;\sigma,\delta]$. Finally, by Lemma 5.4, $N(R)[x;\sigma,\delta]\ \subseteq\ N(R)[[x;\sigma,\delta]]\ \subseteq\ N\big(R[[x;\sigma,\delta]]\big)$. □

The following examples show that hypothesis (1) is not vacuous: it holds, together with hypotheses (2) and (3), on rings carrying genuine zero divisors, so Theorem 5.5 has content. In both examples $\sigma=1_R$ and $\delta=0$ (a permitted special case), and we verify the stronger statement that every nonzero annihilator contains a nonzero constant.

Example 5.6. (The reduced case: $R=F\times F$) Let F be a field and $R=F\times F$. Then $N(R)=0$, so (3) holds with the zero ideal, and, since R is commutative, R is McCoy by [18], hence weak McCoy, i.e., (2). Here $T:=R[[x]]=F[[x]]\times F[[x]]$ with idempotents $e_1=(1,0),\ e_2=(0,1)$, and since $F[[x]]$ is a domain,

$$\mathrm{ann}_T\,\big((\phi,\psi)\big)=\mathrm{ann}_{F[[x]]}(\phi)\times\mathrm{ann}_{F[[x]]}(\psi)\in\{\,0,\ e_1T,\ e_2T,\ T\,\}.$$

The annihilator of any subset is an intersection of such sets; as $e_1T\cap e_2T=0$, it again lies in $\{0,e_1T,e_2T,T\}$, and each nonzero member contains a nonzero constant (e_1, e_2, or 1). Therefore hypothesis (1) holds. The theorem has content: for the nonzero polynomial $f=e_1$ and the nonzero non-polynomial series $g=\big(0,\frac{1}{1-x}\big)$, we have $fg=0$, and the proof yields $c=e_2$ with $f(x)c=e_1e_2=0\in N(R)[x;\sigma,\delta]$.

Example 5.7. (The nilpotent case: $R=F[t]/(t^2)$) Let $R=F[t]/(t^2)$, where F is a field. Then $N(R)=(t)$ is a nonzero nil ideal with $N(R)^2=0$; as $\sigma=1_R$ and $\delta=0$, it is trivially (σ,δ)-stable, hence a nonzero nil (σ,δ)-stable ideal and thus (3) holds and, by Proposition 2.2, so does (2). Writing an element of $T:=R[[x]]$ as $u+tv$ with $u,v\in F[[x]]$ (and $t^2=0$), one has $(u_1+tv_1)(u_2+tv_2)=u_1u_2+t(u_1v_2+v_1u_2)$, so for $g=u+tv$, $h=p+tq$, we have $gh=0\iff up=0$ and $uq+vp=0$. As $F[[x]]$ is a domain: if $u\neq0$, then $\mathrm{ann}_T(g)=0$; if $u=0,\ v\neq0$, then $\mathrm{ann}_T(tv)=t\,F[[x]]$. Hence every nonzero annihilator of an element or a subset is $t\,F[[x]]$ or T, each containing a nonzero constant (t or 1). Therefore hypothesis (1) holds; a zero divisor is $t\cdot t=0$, e.g. $f=t,\ g=t\,\frac{1}{1-x}$ give $fg=0$ with $g\notin R[x]$.

Remark 5.8. The preceding computations instantiate Fields' theorem [6]: over a commutative Noetherian ring, every zero divisor of $R[[x]]$ is annihilated by a nonzero constant of R; thus any commutative Noetherian ring furnishes a setting in which nonzero power-series annihilators meet R nontrivially, which is the mechanism behind hypothesis (1). Examples 5.6 and 5.7 are the simplest reduced and nonreduced examples.

Corollary 5.9. *Let R be an NI ring with an automorphism σ and a σ-derivation δ satisfying $\delta\sigma = \sigma\delta$. Suppose that $N(R)$ is nilpotent, $\delta(N(R)) \subseteq N(R)$, and every nonzero right annihilator in $R[[x;\sigma,\delta]]$ contains a nonzero polynomial. If R is weak (σ,δ)-skew McCoy, then, for any $0 \neq f(x) \in R[x;\sigma,\delta]$ and $0 \neq g(x) \in R[[x;\sigma,\delta]]$ with $f(x)g(x) = 0$, there exists $0 \neq c \in R$ such that $f(x)c \in N(R)[x;\sigma,\delta] \subseteq N(R[[x;\sigma,\delta]])$.*

Proof. Since R is NI, $N(R)$ is a two-sided ideal, and it is nilpotent by hypothesis. Moreover, every ring endomorphism preserves nilpotency, so $\sigma(N(R)) \subseteq N(R)$, while $\delta(N(R)) \subseteq N(R)$ holds by assumption. Hence $N(R)$ is a nilpotent (σ,δ)-stable ideal. Thus all the hypotheses of Theorem 5.5 are satisfied, and the conclusion follows. $\square$

Remark 5.10. In Theorem 5.5, the nilpotency of $N(R)$ (hypothesis (3)) is what places the conclusion in the nilradical of the power series ring: by Lemma 5.4, it makes $N(R)[[x;\sigma,\delta]]$ a nilpotent ideal of $R[[x;\sigma,\delta]]$, so that

$$N(R)[x;\sigma,\delta] \;\subseteq\; N\big(R[[x;\sigma,\delta]]\big).$$

The well-definedness of $R[[x;\sigma,\delta]]$ is a separate matter, assumed here; local nilpotence of δ is a standard sufficient condition.

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