

ONE-SIDED CORE-EP ORTHOGONAL ELEMENTS IN A RING WITH INVOLUTION

Olivera Stanimirović and Dijana Mosić

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ABSTRACT. Our motivation is to explore the concepts of the left and right core-EP orthogonality for elements of a ring with involution. In this manner, we generalize notions of one-sided core-EP orthogonality for operators and core-EP orthogonality for elements of a ring with involution. Many properties and characterizations of left and right core-EP orthogonal elements are given. As consequences, we characterize left and right core orthogonal elements. We also study equivalent conditions for additive property of the core-EP inverse and the Drazin inverse.

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1. Introduction

In this paper, an associative ring \mathcal{R} with the unit 1 is observed. Let us mark with \mathcal{R}^{-1} the set of all invertible elements of \mathcal{R} and assign with \mathcal{R}^{nil} the set of all nilpotent elements of \mathcal{R} . Define $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ for $a \in \mathcal{R}$, and $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$ as the image ideals. Now, we denote the kernel ideals with: $a^\circ = \{x \in \mathcal{R} : ax = 0\}$ and ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$.

An element $x = a^D \in \mathcal{R}$ is the Drazin inverse of $a \in \mathcal{R}$ if the following equations hold:

$$x = xax, \quad ax = xa, \quad a^k = a^{k+1}x,$$

for some positive integer k . If a^D exists, it is unique. The smallest integer k which satisfies this definition is called the index of a , marked as $\text{ind}(a) = k$. If $\text{ind}(a) \leq 1$,

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$a^D = a^\#$ is the group inverse of a . The symbol $\mathcal{R}^\#$ represents the set of all group invertible elements of \mathcal{R} .

If equalities $(a^*)^* = a$, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$ are satisfied for $a, b \in \mathcal{R}$, we say that $a \mapsto a^*$ is an involution in a ring \mathcal{R} . The unique element $x = a^\dagger \in \mathcal{R}$ (if it exists) is the Moore–Penrose inverse of $a \in \mathcal{R}$ which satisfies the next four equations: $axa = a$, $xax = x$, $(ax)^* = ax$ and $(xa)^* = xa$ [12]. The set of all Moore–Penrose invertible elements of \mathcal{R} is assigned with \mathcal{R}^\dagger . Recent results about the Moore–Penrose inverse and the Drazin inverse were given in [18,19].

The core–EP inverse was introduced for elements in a ring with involution in [5], extending the same concept for square matrices presented in [14]. The unique $x = a^\mathfrak{D} \in \mathcal{R}$ is the core–EP inverse of the element $a \in \mathcal{R}$ if it satisfies the following equations:

$$ax^2 = x, \quad (ax)^* = ax, \quad xa^{k+1} = a^k,$$

for some positive integer k . Recall that this k is equal to $\text{ind}(a)$ and $a^\mathfrak{D} = a^D a^k (a^k)^\dagger$ [5]. Also, $a^\mathfrak{D} = a^\oplus$ is the core inverse of a [1] if $\text{ind}(a) \leq 1$. Denote with $\mathcal{R}^\mathfrak{D}$ the set of all core–EP elements of \mathcal{R} and \mathcal{R}^\oplus is the set of the core invertible elements of \mathcal{R} .

For $a, b \in \mathcal{R}^\mathfrak{D}$, the core–EP pre-order [6] is defined as

$$a \leq^\mathfrak{D} b \quad \text{if} \quad aa^\mathfrak{D} = ba^\mathfrak{D} \quad \text{and} \quad a^\mathfrak{D}a = a^\mathfrak{D}b.$$

The equality $1 = e_1 + e_2 + \cdots + e_n$ with idempotents $e_1, e_2, \dots, e_n \in \mathcal{R}$ satisfying $e_i e_j = 0$ for $i \neq j$, presents a decomposition of the identity. For two decompositions of the identity $1 = e_1 + e_2 + \cdots + e_n$ and $1 = f_1 + f_2 + \cdots + f_n$, arbitrary $a \in \mathcal{R}$ has the unique matrix form:

$$a = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}_{e \times f}, \quad (1)$$

where $a_{ij} = e_i a f_j \in e_i \mathcal{R} f_j$. If $a = (a_{ij})_{e \times f}$ and $b = (b_{ij})_{e \times f}$, then $a + b = (a_{ij} + b_{ij})_{e \times f}$. When $1 = g_1 + \cdots + g_n$ is a third decomposition of the identity and $c = (c_{ij})_{f \times g}$, $ac = \left(\sum_{k=1}^n a_{ik} c_{kj} \right)_{e \times g}$. So, the usual algebraic operations in \mathcal{R} can be presented as simple operations between corresponding $n \times n$ matrices over \mathcal{R} . Also,

$$a^* = \begin{bmatrix} a_{11}^* & \cdots & a_{n1}^* \\ \vdots & \ddots & \vdots \\ a_{1n}^* & \cdots & a_{nn}^* \end{bmatrix}_{f^* \times e^*},$$

where this matrix representation of a^* is given relative to the decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_n^*$. When $n = 2$ and $p \in \mathcal{R}$ is an idempotent, we note

$$a = pap + pa(1-p) + (1-p)ap + (1-p)a(1-p) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{p \times p},$$

where $a_{11} = pap$, $a_{12} = pa(1-p)$, $a_{21} = (1-p)ap$ and $a_{22} = (1-p)a(1-p)$.

Using one-sided orthogonality conditions, many kinds of orthogonality are established for matrices and operators. Here, we are extending many of these results to the elements of a ring with involution. For one-sided orthogonality concepts, note that $a \in \mathcal{R}$ is left orthogonal to $b \in \mathcal{R}$ (denoted by $a \perp_l b$) if $ab = 0$ and $a \in \mathcal{R}$ is right orthogonal to $b \in \mathcal{R}$ (denoted by $a \perp_r b$) if $ba = 0$.

Let us recall that $a, b \in \mathcal{R}$ are

- (i) orthogonal (assigned with $a \perp b$) if $ab = 0$ and $ba = 0$;
- (ii) $*$ -orthogonal (assigned with $a \perp_* b$) if $a^*b = 0$ and $ba^* = 0$ [7].

Various concepts of orthogonality are related to generalized inverses. Clearly, if $a \in \mathcal{R}^\#$, then $a \perp b$ is equivalent to $a^\#b = 0$ and $ba^\# = 0$. In the case that $a \in \mathcal{R}^\dagger$, $a \perp_* b$ if and only if $a^\dagger b = 0$ and $ba^\dagger = 0$.

Using the core inverse, the notion of core orthogonal matrices was introduced in [4] and further investigated in [8,9,10,20]. Based on the core-EP inverse, the core-EP orthogonality was defined in [13] for Drazin invertible operators as an extension of the core orthogonality. The core-EP orthogonality was presented in [17] for elements of a ring with involution. For $a \in \mathcal{R}^\mathfrak{D}$ and $b \in \mathcal{R}$, a is core-EP orthogonal to b (assigned with $a \perp_{\mathfrak{D}} b$) if

$$a^\mathfrak{D}b = 0 \quad \text{and} \quad ba^\mathfrak{D} = 0.$$

If $\text{ind}(a) \leq 1$, the core-EP orthogonality reduces to the core orthogonality. A new extension of the core-EP orthogonality was given in [15].

As weaker kinds of the core orthogonality and the $*$ -orthogonality, their one-sided versions were considered in [3] for matrices. The notions of one-sided core orthogonality and core-EP orthogonality were generalized in [16] to one-sided types of the core-EP orthogonality for operators. Applying all versions of core and core-EP orthogonality, significant additive results for core and core-EP inverses were developed [9,16,20].

Different types of orthogonality and their applications motivated us to continue studying this topic. Our goal is to further extend concepts of core-EP orthogonality for elements of a ring [17] and one-sided core-EP orthogonality for operators [16].

More precisely, we define one-sided types of core-EP orthogonality for elements of a ring with involution. Some properties and several characterizations of one-sided types of core-EP orthogonality are established. We develop matrix forms of one-sided core-EP orthogonal elements. Consequently, we get results about one-sided types of core orthogonality in a ring with involution. Under certain conditions, we present additive results for the core-EP inverse and the Drazin inverse.

This paper is organized in three sections. Section 2 contains definitions and properties of one-sided kinds of core-EP orthogonality for elements of a ring with involution. We give conditions for the core-EP inverse (or the Drazin inverse) of the sum of two elements to be equal to the sum of their core-EP inverses (or Drazin inverses) in Section 3.

2. One-sided core-EP orthogonality

In this section, we introduce one-sided kinds of core-EP orthogonality for elements of a ring with involution in order to extend the notions of the core-EP orthogonality for elements of a ring with involution [17] and the one-sided core-EP orthogonality for operators [16].

Definition 2.1. For $a \in \mathcal{R}^{\mathfrak{D}}$ and $b \in \mathcal{R}$, we say that:

- (i) a is left core-EP orthogonal to b (assigned with $a \perp_{\mathfrak{D},l} b$) if $a^{\mathfrak{D}}b = 0$;
- (ii) a is right core-EP orthogonal to b (assigned with $a \perp_{\mathfrak{D},r} b$) if $ba^{\mathfrak{D}} = 0$.

Obviously, if $a \in \mathcal{R}^{\mathfrak{D}}$ and $b \in \mathcal{R}$, $a \perp_{\mathfrak{D}} b$ if and only if $a \perp_{\mathfrak{D},l} b$ and $a \perp_{\mathfrak{D},r} b$. Thus, the core-EP orthogonality implies both left and right core-EP orthogonality, but in general, by [16, Example 2.1], we see that the left (or right) core-EP orthogonality does not give the core-EP orthogonality.

For $a \in \mathcal{R}^{\oplus}$, the left (or right) core orthogonality for elements of a ring can be defined as the special case of the left (or right) core-EP orthogonality.

Definition 2.2. For $a \in \mathcal{R}^{\oplus}$ and $b \in \mathcal{R}$, we say that:

- (i) a is left core orthogonal to b (assigned with $a \perp_{\oplus,l} b$) if $a^{\oplus}b = 0$;
- (ii) a is right core orthogonal to b (assigned with $a \perp_{\oplus,r} b$) if $ba^{\oplus} = 0$.

Notice that one-sided core orthogonality and one-sided *-orthogonality were stated for matrices and operators in [3,16], and here for elements of a ring.

Definition 2.3. For $a, b \in \mathcal{R}$, we say that:

- (i) a is left *-orthogonal to b (assigned with $a \perp_{*,l} b$) if $a^*b = 0$;
- (ii) a is right *-orthogonal to b (assigned with $a \perp_{*,r} b$) if $ba^* = 0$.

In the case when $a \in \mathcal{R}^\dagger$, it follows that $a \perp_{*,l} b$ is equivalent to $a^\dagger b = 0$, and $a \perp_{*,r} b$ if and only if $ba^\dagger = 0$.

We can first characterize $a \perp_{\mathcal{D},l} b$ as [16, Lemma 2.1].

Lemma 2.4. *If $a \in \mathcal{R}^\mathcal{D}$, $b \in \mathcal{R}$ and $\text{ind}(a) = k$, the next assertions are equivalent:*

- (i) $a \perp_{\mathcal{D},l} b$;
- (ii) $(a^k)^\dagger b = 0$;
- (iii) $(a^k)^* b = 0$ (i.e., $a^k \perp_{*,l} b$);
- (iv) $b^* a^k = 0$ (i.e., $b \perp_{*,l} a^k$);
- (v) $b^* a^\mathcal{D} = 0$ (i.e., $b \perp_{*,l} a^\mathcal{D}$);
- (vi) $b\mathcal{R} \subseteq (a^\mathcal{D})^\circ$;
- (vii) $b\mathcal{R} \subseteq [(a^k)^*]^\circ$;
- (viii) $a^k \mathcal{R} \subseteq (b^*)^\circ$;
- (ix) $a^\mathcal{D} \mathcal{R} \subseteq (b^*)^\circ$.

When $\text{ind}(a) = 1$ in Lemma 2.4, we obtain characterizations for $a \perp_{\oplus,l} b$.

Corollary 2.5. *If $a \in \mathcal{R}^\oplus$ and $b \in \mathcal{R}$, the next assertions are equivalent:*

- (i) $a \perp_{\oplus,l} b$;
- (ii) $a^\dagger b = 0$;
- (iii) $a^* b = 0$ (i.e., $a \perp_{*,l} b$);
- (iv) $b^* a = 0$ (i.e., $b \perp_{*,l} a$);
- (v) $b^\dagger a = 0$;
- (vi) $b\mathcal{R} \subseteq (a^\oplus)^\circ$;
- (vii) $b\mathcal{R} \subseteq (a^*)^\circ$;
- (viii) $a\mathcal{R} \subseteq (b^*)^\circ$;
- (ix) $a^\oplus \mathcal{R} \subseteq (b^*)^\circ$.

Also, we can check the following result.

Lemma 2.6. *If $a, b \in \mathcal{R}^\mathcal{D}$, $k = \text{ind}(a)$ and $a \perp_{\mathcal{D},l} b$, then $b \perp_{\mathcal{D},l} a^k$, $aa^\mathcal{D} \perp_{\oplus} bb^\mathcal{D}$, $a \perp_{\mathcal{D}} b^\mathcal{D}$ and $b \perp_{\mathcal{D}} a^\mathcal{D}$.*

Similarly, we obtain necessary and sufficient conditions for $a \perp_{\mathcal{D},r} b$ and $a \perp_{\oplus,r} b$ to be satisfied in a ring.

Lemma 2.7. *If $a \in \mathcal{R}^\mathcal{D}$, $b \in \mathcal{R}$ and $\text{ind}(a) = k$, the next assertions are equivalent:*

- (i) $a \perp_{\mathcal{D},r} b$;
- (ii) $ba^\mathcal{D} = 0$;
- (iii) $ba^k = 0$;

- (iv) $(a^k)^*b^* = 0$ (i.e., $b \perp_{*,r}(a^k)^*$);
- (v) $a^{\mathfrak{D}}b^* = 0$ (i.e., $a \perp_{\mathfrak{D},l}b^*$);
- (vi) $a^{\mathfrak{D}}\mathcal{R} \subseteq b^\circ$;
- (vii) $a^k\mathcal{R} \subseteq b^\circ$;
- (viii) $b^*\mathcal{R} \subseteq (a^k)^\circ$;
- (ix) $b^*\mathcal{R} \subseteq (a^{\mathfrak{D}})^\circ$.

Corollary 2.8. *If $a \in \mathcal{R}^{\oplus}$ and $b \in \mathcal{R}$, the next assertions are equivalent:*

- (i) $a \perp_{\oplus,r}b$;
- (ii) $ba^\# = 0$;
- (iii) $ba = 0$;
- (iv) $a^*b^* = 0$ (i.e., $b \perp_{*,r}a^*$);
- (v) $a^{\oplus}b^* = 0$;
- (vi) $a^{\oplus}\mathcal{R} \subseteq b^\circ$;
- (vii) $a\mathcal{R} \subseteq b^\circ$;
- (viii) $b^*\mathcal{R} \subseteq a^\circ$;
- (ix) $b^*\mathcal{R} \subseteq (a^{\oplus})^\circ$.

We show the next useful property of core-EP invertible elements.

Lemma 2.9. *Let $b \in \mathcal{R}^{\mathfrak{D}}$ and $p = p^* = p^2 \in \mathcal{R}$. If*

$$b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times p},$$

then $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$.

Proof. Set

$$b^{\mathfrak{D}} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p \times p}.$$

From $b(b^{\mathfrak{D}})^2 = b^{\mathfrak{D}}$, we have $x_1 = 0$, $x_2 = 0$ and $b_4x_4^2 = x_4$. Since $bx = (bx)^*$ and

$$bx = \begin{bmatrix} 0 & 0 \\ b_4x_3 & b_4x_4 \end{bmatrix}_{p \times p},$$

we have that $b_4x_4 = (b_4x_4)^*$. The fact that $xb^{k+1} = b^k$, for some k , implies

$$\begin{bmatrix} 0 & 0 \\ x_4b_4^k b_3 & x_4b_4^{k+1} \end{bmatrix}_{p \times p} = \begin{bmatrix} 0 & 0 \\ b_4^{k-1}b_3 & b_4^k \end{bmatrix}_{p \times p}$$

and so $x_4b_4^{k+1} = b_4^k$. We conclude that $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$ and $b_4^{\mathfrak{D}} = x_4$. \square

Now, we introduce decompositions in matrix forms for one-sided orthogonal elements of a ring with involution.

Theorem 2.10. *If $a, b \in \mathcal{R}^\mathfrak{D}$, $\text{ind}(a) = k$ and $p = aa^\mathfrak{D}$, the next assertions are equivalent:*

- (i) $a \perp_{\mathfrak{D}, l} b$;
- (ii) *there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$, $a_3^k = 0$ and $b_4 \in ((1-p)\mathcal{R}(1-p))^\mathfrak{D}$;

- (iii) *there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ with $e_1 = p$ and $e_2 = e_2^*$ such that*

$$a = \begin{bmatrix} a_1 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \\ 0 & a_{33} & a_{34} \end{bmatrix}_{e \times e} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 \\ b_{31} & b_{41} & b_{42} \\ b_{32} & 0 & b_{43} \end{bmatrix}_{e \times e},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$, $(a_{31} + a_{32} + a_{33} + a_{34})^k = 0$, $b_{41} \in (e_2\mathcal{R}e_2)^{-1}$ and $b_{43}^j = 0$, for some j ;

- (iv) $b = (1 - aa^\mathfrak{D})w$, for arbitrary $w \in \mathcal{R}$;
- (v) $a^\mathfrak{D}b$ is idempotent and $a^\mathfrak{D}(a+b)a^\mathfrak{D} = a^\mathfrak{D}$;
- (vi) $a^\mathfrak{D}b$ is idempotent and $a^\mathfrak{D}ba^\mathfrak{D} = 0$;
- (vii) $a^\mathfrak{D}b$ is idempotent and $aa^\mathfrak{D}ba^\mathfrak{D}a = 0$;
- (viii) $a^\mathfrak{D}b$ is idempotent and $(a^k)^*ba^k = 0$.

Proof. (i) \Rightarrow (ii): Relative to $p = aa^\mathfrak{D}$, by [2] and [11, Lemma 2.3], note that a and $a^\mathfrak{D}$ have the next matrix forms:

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad a^\mathfrak{D} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $a_3^k = 0$. Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p}.$$

Then, from the assertion $0 = a^\mathfrak{D}b$, we have that

$$0 = \begin{bmatrix} a_1^{-1}b_1 & a_1^{-1}b_2 \\ 0 & 0 \end{bmatrix}_{p \times p}.$$

Hence, $b_1 = 0$ and $b_2 = 0$. Because

$$b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times p} \in \mathcal{R}^\mathfrak{D},$$

applying Lemma 2.9, it follows that $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$.

(ii) \Rightarrow (iii): Since $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$, for $q = b_4 b_4^{\mathfrak{D}}$, we observe that, by [2] and [11, Lemma 2.3],

$$b_4 = \begin{bmatrix} b_{41} & b_{42} \\ 0 & b_{43} \end{bmatrix}_{q \times q},$$

where $b_{41} \in (q\mathcal{R}q)^{-1}$, $\text{ind}(b_4) = j$ and $b_{43}^j = 0$. Here, $b_4^{\mathfrak{D}} = \begin{bmatrix} b_{41}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{q \times q}$.

We have $1 = p + q + (1-p-q)$, $p^* = p$ and $q^* = q$. Using $b_4 \in (1-p)\mathcal{R}$, $pq = pb_4 b_4^{\mathfrak{D}} = 0$. Also, $q = q^* = (b_4^{\mathfrak{D}})^* b_4^* \in \mathcal{R}(1-p)$ gives $qp = 0$. Set $e_1 = p$, $e_2 = q$ and $e_3 = 1-p-q$. Now, $1 = e_1 + e_2 + e_3$ and $e_i e_j = 0$, $i \neq j$, $i, j = \overline{1, 3}$. This part sure can be finished with the next results:

$$b = b_3 + b_4 = (1-p)bp + b_4 = qbp + (1-p-q)bp + b_4$$

with

$$\begin{aligned} b_4 &= b_{41} + b_{42} + b_{43} \\ &= q(1-p)b(1-p)q + q(1-p)b(1-p)(1-q) + (1-q)(1-p)b(1-p)(1-q) \\ &= qbq + qb(1-p-q) + (1-p-q)b(1-p-q). \end{aligned}$$

So, b has the next matrix form:

$$b = \begin{bmatrix} 0 & 0 & 0 \\ b_{31} & b_{41} & b_{42} \\ b_{32} & 0 & b_{43} \end{bmatrix}_{e \times e},$$

where we denote $b_{31} = qbp$, $b_{32} = (1-p-q)bp$, $b_{41} = qbq$, $b_{42} = qb(1-p-q)$ and $b_{43} = (1-p-q)b(1-p-q)$. Also,

$$\begin{aligned} a &= a_1 + a_2 + a_3 = pap + pa(1-p) + (1-p)a(1-p) \\ &= pap + paq + pa(1-p-q) + qa(1-p) + (1-p-q)a(1-p) \\ &= pap + paq + pa(1-p-q) + qaq + qa(1-p-q) \\ &+ (1-p-q)aq + (1-p-q)a(1-p-q) \end{aligned}$$

and thus

$$a = \begin{bmatrix} a_1 & a_{21} & a_{22} \\ 0 & a_{31} & a_{32} \\ 0 & a_{33} & a_{34} \end{bmatrix}_{e \times e},$$

where $a_1 = pap \in (p\mathcal{R}p)^{-1}$, $a_{21} = paq$, $a_{22} = pa(1-p-q)$, $a_{31} = qaq$, $a_{32} = qa(1-p-q)$, $a_{33} = (1-p-q)aq$ and $a_{34} = (1-p-q)a(1-p-q)$ and $(a_{31} + a_{32} + a_{33} + a_{34})^k = a_3^k = 0$.

(iii) \Rightarrow (i): From

$$a^{\mathfrak{D}} = \begin{bmatrix} a_1^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e},$$

we obtain $a^{\mathfrak{D}}b = 0$. Thus, $a \perp_{\mathfrak{D},l} b$.

The rest follows as in [16, Theorem 2.1]. \square

Using Theorem 2.10, we observe the next equivalent conditions for $a \perp_{\oplus,l} b$ to be satisfied.

Corollary 2.11. *If $a, b \in \mathcal{R}^{\oplus}$ and $p = aa^{\oplus}$, the next assertions are equivalent:*

- (i) $a \perp_{\oplus,l} b$;
- (ii) *there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 \\ b_3 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $b_4 \in ((1-p)\mathcal{R}(1-p))^{\oplus}$;

- (iii) *there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ with $e_1 = p$ and $e_2 = e_2^*$ such that*

$$a = \begin{bmatrix} a_1 & a_{21} & a_{22} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e} \quad \text{and} \quad b = \begin{bmatrix} 0 & 0 & 0 \\ b_{31} & b_{41} & b_{42} \\ b_{32} & 0 & 0 \end{bmatrix}_{e \times e},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $b_{41} \in (e_2\mathcal{R}e_2)^{-1}$;

- (iv) $b = (1 - aa^{\oplus})w$, for arbitrary $w \in \mathcal{R}$;
- (v) $a^{\oplus}b$ is idempotent and $a^{\oplus}(a+b)a^{\oplus} = a^{\oplus}$;
- (vi) $a^{\oplus}b$ is idempotent and $a^{\oplus}ba^{\oplus} = 0$;
- (vii) $a^{\oplus}b$ is idempotent and $aa^{\oplus}ba^{\oplus}a = 0$;
- (viii) $a^{\oplus}b$ is idempotent and $a^*ba = 0$.

We develop the matrix forms for right core-EP orthogonal elements of a ring with involution.

Theorem 2.12. *If $b \in \mathcal{R}$, $a \in \mathcal{R}^{\mathfrak{D}}$, $\text{ind}(a) = k$ and $p = aa^{\mathfrak{D}}$, then the next assertions are equivalent:*

- (i) $a \perp_{\mathfrak{D},r} b$;

(ii) *there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $a_3^k = 0$;

- (iii) $b = w(1 - a^\mathfrak{D}a)$, for arbitrary $w \in \mathcal{R}$;
- (iv) $ba^\mathfrak{D}$ is idempotent and $a^\mathfrak{D}(a + b)a^\mathfrak{D} = a^\mathfrak{D}$;
- (v) $ba^\mathfrak{D}$ is idempotent and $a^\mathfrak{D}ba^\mathfrak{D} = 0$;
- (vi) $ba^\mathfrak{D}$ is idempotent and $aa^\mathfrak{D}ba^\mathfrak{D}a = 0$;
- (vii) $ba^\mathfrak{D}$ is idempotent and $(a^k)^*ba^k = 0$.

Proof. (i) \Rightarrow (ii): According to [2] and [11, Lemma 2.3], we have, for $p = aa^\mathfrak{D}$, that is

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad a^\mathfrak{D} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $a_3^k = 0$. Suppose that

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p}.$$

Then

$$0 = ba^\mathfrak{D} = \begin{bmatrix} b_1a_1^{-1} & 0 \\ b_3a_1^{-1} & 0 \end{bmatrix}_{p \times p}$$

gives $b_1 = 0$ and $b_3 = 0$.

(ii) \Rightarrow (i): Since

$$a^\mathfrak{D} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

it follows $ba^\mathfrak{D} = 0$ and thus $a \perp_{\mathfrak{D}, r} b$.

(i) \Leftrightarrow (iii)–(viii): These equivalences follow by characteristics of the core-EP inverse. \square

Under an extra condition $(1 - aa^\mathfrak{D})b \in \mathcal{R}^\mathfrak{D}$, we obtain the new equivalent conditions for $a \perp_{\mathfrak{D}, r} b$.

Theorem 2.13. *If $b \in \mathcal{R}$, $a, (1 - aa^\mathfrak{D})b \in \mathcal{R}^\mathfrak{D}$, $\text{ind}(a) = k$, $p = aa^\mathfrak{D}$, then the next assertions are equivalent:*

- (i) $a \perp_{\mathfrak{D}, r} b$;

- (ii) *there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ with $e_1 = p$ and $e_2 = e_2^*$ such that*

$$a = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}_{e \times e} \quad \text{and} \quad b = \begin{bmatrix} 0 & b_{21} & b_{22} \\ 0 & b_{41} & b_{42} \\ 0 & 0 & b_{43} \end{bmatrix}_{e \times e},$$

where $a_{11} \in (p\mathcal{R}p)^{-1}$, $(a_{22} + a_{23} + a_{32} + a_{33})^k = 0$, $b_{41} \in (e_2\mathcal{R}e_2)^{-1}$ and $b_{43}^j = 0$, for some j .

Proof. (i) \Rightarrow (ii): We can represent a and b as in part (ii) of Theorem 2.12. Since

$$(1 - aa^{\mathfrak{D}})b = \begin{bmatrix} 0 & 0 \\ 0 & b_4 \end{bmatrix}_{p \times p} \in \mathcal{R}^{\mathfrak{D}},$$

we can conclude that $b_4 \in ((1 - p)\mathcal{R}(1 - p))^{\mathfrak{D}}$. Further, for $q = b_4 b_4^{\mathfrak{D}}$,

$$b_4 = \begin{bmatrix} b_{41} & b_{42} \\ 0 & b_{43} \end{bmatrix}_{q \times q},$$

where $b_{41} \in (q\mathcal{R}q)^{-1}$, $b_{43}^j = 0$ and $\text{ind}(b_4) = j$.

As in the proof of Theorem 2.10, this part is completed.

(ii) \Rightarrow (i): It is clear by

$$a^{\mathfrak{D}} = \begin{bmatrix} a_{11}^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e}.$$

□

We can characterize $a \perp_{\oplus, r} b$ by Theorem 2.12 and Theorem 2.13.

Corollary 2.14. *If $b \in \mathcal{R}$, $a \in \mathcal{R}^{\oplus}$ and $p = aa^{\oplus}$, the next assertions are equivalent:*

- (i) $a \perp_{\oplus, r} b$;
 (ii) *there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that*

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} 0 & b_2 \\ 0 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $b_4 \in ((1 - p)\mathcal{R}(1 - p))^{\oplus}$;

- (iii) $b = w(1 - a^{\oplus}a)$, for arbitrary $w \in \mathcal{R}$;
 (iv) ba^{\oplus} is idempotent and $a^{\oplus}(a + b)a^{\oplus} = a^{\oplus}$;
 (v) ba^{\oplus} is idempotent and $a^{\oplus}ba^{\oplus} = 0$;
 (vi) ba^{\oplus} is idempotent and $aa^{\oplus}ba^{\oplus}a = 0$;

(vii) ba^{\oplus} is idempotent and $a^*ba = 0$.

Corollary 2.15. *If $b \in \mathcal{R}$, $a, (1 - aa^{\mathfrak{D}})b \in \mathcal{R}^{\oplus}$ and $p = aa^{\oplus}$, the next assertions are equivalent:*

- (i) $a \perp_{\oplus, r} b$;
- (ii) *there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ with $e_1 = p$ and $e_2 = e_2^*$ such that*

$$a = \begin{bmatrix} a_1 & a_{21} & a_{22} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{e \times e} \quad \text{and} \quad b = \begin{bmatrix} 0 & b_{21} & b_{22} \\ 0 & b_{41} & b_{42} \\ 0 & 0 & 0 \end{bmatrix}_{e \times e},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $b_{41} \in (e_2\mathcal{R}e_2)^{-1}$.

3. Additive results for core-EP and Drazin inverses

In this section, we present additive results for the core-EP inverse and the Drazin inverse in a ring with involution.

The equalities $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$ can be connected to one-sided core-EP orthogonality in the next manner.

Lemma 3.1. *For $a, b \in \mathcal{R}^{\mathfrak{D}}$ and $k = \text{ind}(a)$, we have*

- (i) $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ if and only if $a \perp_{\mathfrak{D}, l}(b+a)$ if and only if $(a+b)\mathcal{R} \subseteq [(a^k)^*]^{\circ}$ if and only if $aa^{\mathfrak{D}}(b+a) = 0$ if and only if $aa^{\mathfrak{D}} \perp_{\oplus, l}(b+a)$;
- (ii) $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$ if and only if $a \perp_{\mathfrak{D}, r}(b+a)$ if and only if $a^k\mathcal{R} \subseteq (a+b)^{\circ}$ if and only if $(b+a)a^{\mathfrak{D}}a = 0$;
- (iii) $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$ if and only if $a \perp_{\mathfrak{D}}(b+a)$.

We consider matrix forms of elements a and b in the case that $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$.

Lemma 3.2. *For $a, b \in \mathcal{R}^{\mathfrak{D}}$, $k = \text{ind}(a)$ and $p = aa^{\oplus}$, we have*

- (i) $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} -a_1 & -a_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $a_3^k = 0$.

(ii) $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ such that

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} -a_1 & b_2 \\ 0 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$, $a_3^k = 0$ and $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$. In addition,

$$b^{\mathfrak{D}} = \begin{bmatrix} -a_1^{-1} & a_1^{-1}b_2b_4^{\mathfrak{D}} \\ 0 & b_4^{\mathfrak{D}} \end{bmatrix}_{p \times p}.$$

(iii) $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ and $e_2 = e_2^*$ such that

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} -a_1 & -a_2 \\ 0 & b_4 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$, $a_3^k = 0$ and $b_4 \in ((1-p)\mathcal{R}(1-p))^{\mathfrak{D}}$. In addition,

$$b^{\mathfrak{D}} = \begin{bmatrix} -a_1^{-1} & -a_1^{-1}a_2b_4^{\mathfrak{D}} \\ 0 & b_4^{\mathfrak{D}} \end{bmatrix}_{p \times p}.$$

Proof. Using [2] and [11, Lemma 2.3], a and $a^{\mathfrak{D}}$ are represented as:

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad a^{\mathfrak{D}} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

where $a_1 \in (p\mathcal{R}p)^{-1}$ and $a_3^k = 0$. Set

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p \times p}.$$

The rest follows by [11, Lemma 2.3] as in the proof of [16, Lemma 3.2]. \square

Under assumptions $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, we develop characterizations of $(a+b)^{\mathfrak{D}} = a^{\mathfrak{D}} + b^{\mathfrak{D}}$.

Theorem 3.3. *If $a, b \in \mathcal{R}^{\mathfrak{D}}$ satisfy $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a+b)^{\mathfrak{D}} = a^{\mathfrak{D}} + b^{\mathfrak{D}}$;
- (ii) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a+b)^{\mathfrak{D}} = b^{\mathfrak{D}}(1 - aa^{\mathfrak{D}})$;
- (iii) $a + b \in \mathcal{R}^{\mathfrak{D}}$, $(a+b)^{\mathfrak{D}} = (1 - aa^{\mathfrak{D}})b^{\mathfrak{D}}$ and $a^{\mathfrak{D}} = -aa^{\mathfrak{D}}b^{\mathfrak{D}}$;
- (iv) $a + b \in \mathcal{R}^{\mathfrak{D}}$, $(a+b)^{\mathfrak{D}} = (1 - aa^{\mathfrak{D}})b^{\mathfrak{D}}$ and $(a^{\mathfrak{D}})^2 = -a^{\mathfrak{D}}b^{\mathfrak{D}}$.

Proof. Applying Lemma 3.2, we show this result as [16, Theorem 3.2]. \square

We obtain additive results for the core inverse as a consequence of Theorem 3.3.

Corollary 3.4. *If $a, b \in R^\oplus$ satisfy $a^\oplus b + a^\oplus a = 0$ and $ba^\oplus + aa^\oplus = 0$, the next assertions are equivalent:*

- (i) $a + b \in R^\oplus$ and $(a + b)^\oplus = a^\oplus + b^\oplus$;
- (ii) $a + b \in R^\oplus$ and $(a + b)^\oplus = b^\oplus(1 - aa^\oplus)$;
- (iii) $a + b \in R^\oplus$ and $a^\oplus = -aa^\oplus b^\oplus$;
- (iv) $a + b \in R^\oplus$ and $(a^\oplus)^2 = -a^\oplus b^\oplus$.

Under the same assumptions as in Theorem 3.3, we obtain new additive results for the Drazin inverse.

Theorem 3.5. *If $a, b \in \mathcal{R}^\mathfrak{D}$ satisfy $a^\mathfrak{D}b + a^\mathfrak{D}a = 0$ and $ba^\mathfrak{D} + aa^\mathfrak{D} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^D$ and $(a + b)^D = a^D + b^D$;
- (ii) $a + b \in \mathcal{R}^D$ and $(a + b)^D = b^D(1 - aa^D)$;
- (iii) $a + b \in \mathcal{R}^D$, $(a + b)^D = (1 - aa^\mathfrak{D})b^D$ and $a^D = -aa^\mathfrak{D}b^D$;
- (iv) $a + b \in \mathcal{R}^D$, $(a + b)^D = (1 - aa^\mathfrak{D})b^D$ and $(a^D)^2 = -a^\mathfrak{D}b^D$;
- (v) $a + b \in \mathcal{R}^D$ and $((a + b)^D - b^D)(1 - aa^D) = 0$.

Proof. Lemma 3.2 implies that there is a decomposition of the identity $1 = e_1 + e_2$ with $e_1 = p$ and $e_2 = e_2^*$ such that

$$a = \begin{bmatrix} a_1 & a_2 \\ 0 & a_3 \end{bmatrix}_{p \times p} \quad \text{and} \quad b = \begin{bmatrix} -a_1 & -a_2 \\ 0 & b_4 \end{bmatrix}_{p \times p},$$

where $k = \text{ind}(a)$, $a_1 \in (p\mathcal{R}p)^{-1}$, $a_3^k = 0$, and $b_4 \in ((1-p)\mathcal{R}(1-p))^\mathfrak{D}$. We deduce that $b_4 \in ((1-p)\mathcal{R}(1-p))^D$,

$$a^D = \begin{bmatrix} a_1^{-1} & s_1 \\ 0 & 0 \end{bmatrix}_{p \times p}, \quad b^D = \begin{bmatrix} -a_1^{-1} & s_2 \\ 0 & b_4^D \end{bmatrix}_{p \times p},$$

$$a^\mathfrak{D} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p \times p} \quad \text{and} \quad b^\mathfrak{D} = \begin{bmatrix} -a_1^{-1} & -a_1^{-1}a_2b_4^\mathfrak{D} \\ 0 & b_4^\mathfrak{D} \end{bmatrix}_{p \times p},$$

for some elements $s_1, s_2 \in \mathcal{R}$. Thus,

$$a^D + b^D = \begin{bmatrix} 0 & s_1 + s_2 \\ 0 & b_4^D \end{bmatrix}_{p \times p}.$$

Because $a + b = \begin{bmatrix} 0 & 0 \\ 0 & a_3 + b_4 \end{bmatrix}_{p \times p}$ is Drazin invertible, it follows that $a_3 + b_4$ is also Drazin invertible and

$$(a + b)^D = \begin{bmatrix} 0 & 0 \\ 0 & (a_3 + b_4)^D \end{bmatrix}_{p \times p}.$$

We observe that $(a + b)^D = a^D + b^D$ is equivalent to

$$0 = s_1 + s_2, \quad (2)$$

$$(a_3 + b_4)^D = b_4^D. \quad (3)$$

(i) \Leftrightarrow (ii): Since

$$b^D(1 - aa^D) = b^D \begin{bmatrix} 0 & -a_1 s_1 \\ 0 & 1 - p \end{bmatrix}_{p \times p} = \begin{bmatrix} 0 & s_1 + s_2 \\ 0 & b_4^D \end{bmatrix}_{p \times p},$$

$(a + b)^D = b^D(1 - aa^D)$ if and only if (2) and (3) hold. Hence, (i) and (ii) are equivalent.

(i) \Leftrightarrow (iii): By

$$(1 - aa^\ominus)b^D = \begin{bmatrix} 0 & 0 \\ 0 & 1 - p \end{bmatrix}_{p \times p} \begin{bmatrix} -a_1^{-1} & s_2 \\ 0 & b_4^D \end{bmatrix}_{p \times p} = \begin{bmatrix} 0 & 0 \\ 0 & b_4^D \end{bmatrix}_{p \times p},$$

$(a + b)^D = (1 - aa^\ominus)b^D$ is equivalent to (3). From

$$a^D + aa^\ominus b^D = \begin{bmatrix} 0 & s_1 + s_2 \\ 0 & 0 \end{bmatrix}_{p \times p},$$

we deduce that $a^D = -aa^\ominus b^D$ if and only if $a^D + aa^\ominus b^D = 0$ if and only if (2) holds. So, (i) is equivalent to (iii).

(i) \Leftrightarrow (iv): Note that the conditions $a^D = -aa^\ominus b^D$ and $(a^D)^2 = -a^\ominus b^D$ are equivalent by known equality $a^D = a^D aa^\ominus$.

(i) \Leftrightarrow (v): We have, by

$$\begin{aligned} ((a + b)^D - b^D)(1 - aa^D) &= \begin{bmatrix} a_1^{-1} & -s_2 \\ 0 & (a_3 + b_4)^D - b_4^D \end{bmatrix}_{p \times p} \begin{bmatrix} 0 & -a_1 s_1 \\ 0 & 1 - p \end{bmatrix}_{p \times p} \\ &= \begin{bmatrix} 0 & -s_1 - s_2 \\ 0 & (a_3 + b_4)^D - b_4^D \end{bmatrix}_{p \times p}, \end{aligned}$$

that $((a + b)^D - b^D)(1 - aa^D) = 0$ is equivalent to (2) and (3). \square

Theorem 3.5 gives additive results for the group inverse.

Corollary 3.6. *If $a, b \in \mathcal{R}^\oplus$ satisfy $a^\oplus b + a^\oplus a = 0$ and $ba^\oplus + aa^\oplus = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^\#$ and $(a + b)^\# = a^\# + b^\#$;
- (ii) $a + b \in \mathcal{R}^\#$ and $(a + b)^\# = b^\#(1 - aa^\#)$;
- (iii) $a + b \in \mathcal{R}^\#$ and $a^\# = -aa^\oplus b^\#$;
- (iv) $a + b \in \mathcal{R}^\#$ and $(a^\#)^2 = -a^\oplus b^\#$;
- (v) $a + b \in \mathcal{R}^\#$ and $((a + b)^\# - b^\#)(1 - aa^\#) = 0$.

Also, we can check the following results.

Theorem 3.7. *If $a, b \in \mathcal{R}^{\mathfrak{D}}$ satisfy $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a + b)^{\mathfrak{D}} = a^{\mathfrak{D}} + b^{\mathfrak{D}}$;
- (ii) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a + b)^{\mathfrak{D}} = b^{\mathfrak{D}}(1 - aa^{\mathfrak{D}})$;
- (iii) $a + b \in \mathcal{R}^{\mathfrak{D}}$, $(1 - aa^{\mathfrak{D}})((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}}) = 0$ and $aa^{\mathfrak{D}}((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}})(1 - aa^{\mathfrak{D}}) = 0$;
- (iv) $a + b \in \mathcal{R}^{\mathfrak{D}}$, $(1 - aa^{\mathfrak{D}})((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}}) = 0$ and $a^{\mathfrak{D}}((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}})(1 - aa^{\mathfrak{D}}) = 0$.

Corollary 3.8. *If $a, b \in \mathcal{R}^{\oplus}$ satisfy $ba^{\oplus} + aa^{\oplus} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\oplus}$ and $(a + b)^{\oplus} = a^{\oplus} + b^{\oplus}$;
- (ii) $a + b \in \mathcal{R}^{\oplus}$ and $(a + b)^{\oplus} = b^{\oplus}(1 - aa^{\oplus})$;
- (iii) $a + b \in \mathcal{R}^{\oplus}$, $(1 - aa^{\oplus})((a + b)^{\oplus} - b^{\oplus}) = 0$ and $bb^{\oplus}((a + b)^{\oplus} - b^{\oplus})(1 - aa^{\oplus}) = 0$;
- (iv) $a + b \in \mathcal{R}^{\oplus}$, $(1 - aa^{\oplus})((a + b)^{\oplus} - b^{\oplus}) = 0$ and $a^{\oplus}((a + b)^{\oplus} - b^{\oplus})(1 - aa^{\oplus}) = 0$.

Theorem 3.9. *If $a, b \in \mathcal{R}^{\mathfrak{D}}$ satisfy $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a + b)^{\mathfrak{D}} = a^{\mathfrak{D}} + b^{\mathfrak{D}}$;
- (ii) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a + b)^{\mathfrak{D}} = b^{\mathfrak{D}}(1 - aa^{\mathfrak{D}})$;
- (iii) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}})(1 - aa^{\mathfrak{D}}) = 0$.

Corollary 3.10. *If $a, b \in \mathcal{R}^{\oplus}$ satisfy $ba^{\oplus} + aa^{\oplus} = 0$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\#}$ and $(a + b)^{\#} = a^{\#} + b^{\#}$;
- (ii) $a + b \in \mathcal{R}^{\#}$ and $(a + b)^{\#} = b^{\#}(1 - aa^{\#})$;
- (iii) $a + b \in \mathcal{R}^{\#}$ and $((a + b)^{\#} - b^{\#})(1 - aa^{\#}) = 0$.

As in Theorem 3.5, we show the following additive results under the assumption $a \perp_{\mathfrak{D}, r} b$.

Theorem 3.11. *If $a, b \in \mathcal{R}^{\mathfrak{D}}$ satisfy $a \perp_{\mathfrak{D}, r} b$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $(a + b)^{\mathfrak{D}} = a^{\mathfrak{D}} + b^{\mathfrak{D}}$;
- (ii) $a + b \in \mathcal{R}^{\mathfrak{D}}$ and $((a + b)^{\mathfrak{D}} - b^{\mathfrak{D}})(1 - aa^{\mathfrak{D}}) = 0$.

Corollary 3.12. *If $a, b \in \mathcal{R}^{\oplus}$ satisfy $a \perp_{\oplus, r} b$, the next assertions are equivalent:*

- (i) $a + b \in \mathcal{R}^{\#}$ and $(a + b)^{\#} = a^{\#} + b^{\#}$;
- (ii) $a + b \in \mathcal{R}^{\#}$ and $((a + b)^{\#} - b^{\#})(1 - aa^{\#}) = 0$.

In the case that $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, we study equivalent conditions for $a^2 \leq^{\mathfrak{D}} b^2$.

Theorem 3.13. *If $a, b \in \mathcal{R}^{\mathfrak{D}}$ satisfy $a^{\mathfrak{D}}b + a^{\mathfrak{D}}a = 0$ and $ba^{\mathfrak{D}} + aa^{\mathfrak{D}} = 0$, the next assertions are equivalent:*

- (i) $a^2 \leq^{\mathfrak{D}} b^2$;
- (ii) $aa^{\mathfrak{D}}(a^2 + ab) = 0$;
- (iii) $a^{\mathfrak{D}}(a^2 + ab) = 0$;
- (iv) $aa^{\mathfrak{D}}(ab - ba) = 0$;
- (v) $a^{\mathfrak{D}}(ab - ba) = 0$.

Proof. Using Lemma 3.2 and [11, Lemma 2.3], we complete this proof as in [16, Theorem 3.4]. \square

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Olivera Stanimirović (Corresponding Author)

University of Niš

Faculty of Technology

Department of General Technical Sciences

Bulevar oslobođenja 124, 16000 Leskovac, Serbia

e-mail: olivera-stanimirovic@tf.ni.ac.rs

Dijana Mosić

University of Niš

Faculty of Sciences and Mathematics

P.O. Box 224, 18000 Niš, Serbia

e-mail: dijana@pmf.ni.ac.rs