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PSOS-GROUPS REVISITED

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Abstract. A finite group G is said to be a PSOS-group if for every subgroup H of G, the cardinality of the set $\{K \leq G \mid |K| = |H|\}$ divides $|G|$. In the current note, this property is studied for two other important classes of finite groups: generalized dihedral groups and (generalized) dicyclic groups.

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1. Introduction

Let G be a finite group and $x \in G$. Then the set of all elements of G having the same order as x is called the *order subset* of G determined by x (see [2]). We say that G is a *group with perfect order subsets*, or briefly a *POS-group*, if the number of elements in each order subset of G is a divisor of $|G|$. Many recent works, such as [1], [3], [6], [8], [11] and [14], deal with the structure of POS-groups.

Now, let H be a subgroup of G. Inspired by the above notions, in [12] we called the *subgroup order subset* of G determined by H the set of all subgroups of G having the same order as H. We also said that G is a *group with perfect subgroup order subsets*, or briefly a *PSOS-group*, if the number of elements in each subgroup order subset of G is a divisor of $|G|$.

The study of PSOS-groups was then developed in [12] and [13]. The containment of several finite groups to this new class was investigated, for example:

- A finite p-group is a PSOS-group if and only if it is cyclic (see [12, Theorem 1]).
- The dihedral group D_{2n} is a PSOS-group if and only if n is odd (see [12, Proposition 2]).
- The generalized quaternion group Q_{4n} is a PSOS-group if and only if n is odd (see [12, Proposition 3]).
- A non-abelian P-group is a PSOS-group if and only if it is of order pq, where p, q are primes and q divides $p - 1$ (see [12, Proposition 4]).

- The symmetric group S_n is a PSOS-group if and only if $n \leq 3$, while the alternating group A_n is a PSOS-group if and only if $n \leq 5$ (see [12, Proposition 5]).

Note that the class of PSOS-groups is not closed under subgroups and all subgroups of a finite group G are PSOS-groups if and only if G is a ZM-group, as shown in [12, Theorem 6]. Recall that a ZM-group is a finite group with all Sylow subgroups cyclic (see e.g. [4, Ch. IV, Satz 2.11]). Also, the direct product of two finite groups G_1 and G_2 of odd orders such that $\gcd(|G_1|, |G_2|) \neq 1$ is not a PSOS-group by [12, Theorem 7]. This remains not true if G_1 and G_2 are of even orders, as it is shown in [13].

In what follows, we will study the PSOS property for generalized dihedral groups and (generalized) dicyclic groups.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on group theory can be found in [5]. For subgroup lattice notions, we refer the reader to [10].

2. Generalized dihedral groups

Given an abelian group A, the *generalized dihedral group* of A, written $Dih(A)$, is the semidirect product of A and C_2, with C_2 acting on A by inverting elements. If x is the generator of C_2, then

$$Dih(A) = A \rtimes C_2 = \langle A, x \mid x^2 = 1,\, xax^{-1} = a^{-1},\, \forall\, a \in A \rangle.$$

In what follows, we will deal only with the case when A is finite. Note that the standard dihedral group D_{2n} is obtained by taking $A \cong C_n$.

The subgroups of $Dih(A)$ can be easily described (see e.g. [9, Theorem 7.2]).

Lemma 2.1. *For every subgroup A_1 of A, $Dih(A)$ has one subgroup isomorphic to A_1 and $[A : A_1]$ subgroups isomorphic to $Dih(A_1)$.*

Let $|A| = n$ and $k_d(A)$ be the number of subgroups of order d of A, for every divisor d of n. Then $Dih(A)$ has

- $k_d(A)$ subgroups of order d
- $\dfrac{n}{d}\, k_d(A)$ subgroups of order $2d$,

for every divisor d of n.

Assume now that $Dih(A)$ is a PSOS-group. We distinguish the following two cases:

Case 1. n is even

Then we obtain that $Dih(A)$ has $k_2(A)+n$ subgroups of order 2. It follows that $k_2(A)+n$ divides $2n$ and so there exists $\alpha \in \mathbb{N}^*$ such that

$$2n = \alpha(k_2(A)+n).$$

Clearly, we have $\alpha = 1$ and therefore $k_2(A) = n$. This means that all elements of A are of order 2, a contradiction.

Case 2. n is odd

Let d be a prime divisor of n. Then $\frac{n}{d}\,k_d(A)$ divides $2n$, or equivalently $k_d(A)$ divides $2d$:

$$2d = \beta k_d(A)$$

with $\beta \in \mathbb{N}^*$. Since A is abelian, it has a unique Sylow d-subgroup A_d. It follows that all subgroups of order d of A are contained in A_d and so

$$k_d(A) = k_d(A_d).$$

Thus

$$k_d(A) \equiv 1 \,(\mathrm{mod}\, d).$$

If $k_d(A) \neq 1$, then

$$k_d(A) \geq d+1,$$

which leads to $\beta = 1$ and

$$k_d(A) = 2d \equiv 0 \,(\mathrm{mod}\, d),$$

a contradiction. Thus $k_d(A) = 1$ for all prime divisors d of n, implying that all Sylow subgroups of A are cyclic. Since a finite abelian group is the direct product of its Sylow subgroups, we infer that A itself is cyclic.

Consequently, we proved the following theorem.

Theorem 2.2. *Let A be a finite abelian group. Then the generalized dihedral group $Dih(A)$ is a PSOS-group if and only if A is cyclic of odd order.*

3. (Generalized) Dicyclic groups

First of all, we will focus on the dicyclic group Dic_{4n}. Recall that, for a positive integer $n \geq 1$, the structure of this group is

$$Dic_{4n} = \langle a, \gamma \mid a^{2n} = 1, \gamma^2 = a^n, a^\gamma = a^{-1} \rangle.$$

The subgroup structure of Dic_{4n} is well-known. For every divisor d of $2n$, it has one subgroup isomorphic to C_d, namely $\langle a^{\frac{2n}{d}} \rangle$. Also, for every divisor d of n, it possesses $\frac{n}{d}$ subgroups isomorphic to Dic_{4d}, namely $\langle a^{\frac{n}{d}}, a^{i-1}\gamma \rangle$, where $i = 1, \ldots, \frac{n}{d}$.

If n is even, then Dic_{4n} has $n+1$ subgroups of order 4. By assuming that Dic_{4n} is a PSOS-group, it follows that $n+1$ divides $4n$, which leads to $n \in \{1,3\}$, a contradiction.

If n is odd and $d_1, \ldots, d_k$ are its divisors, then Dic_{4n} has:

- $\frac{n}{d_i}$ subgroups of order $4d_i$
- one subgroup of order d_i
- one subgroup of order $2d_i$,

for any $i = 1, \ldots, k$. Since all these numbers of subgroups divide $4n$, we conclude that in this case Dic_{4n} is a PSOS-group.

Thus the following theorem holds.

Theorem 3.1. *The dicyclic group Dic_{4n} is a PSOS-group if and only if n is odd.*

Next, we will focus on the generalized dicyclic group $Dic_{4n}(A)$, where n is an even positive integer and A is a finite abelian group of order $2n$. The structure of this group is

$$Dic_{4n}(A) = \langle A, \gamma \mid \gamma^4 = 1, \gamma^2 \in A \setminus \{1\}, g^\gamma = g^{-1}, \forall g \in A \rangle.$$

Note that $Dic_{4n} = Dic_{4n}(C_{2n})$.

In what follows, we will treat the particular case $A = C_2 \times C_n$, where $n = 2^m m'$, $m \in \mathbb{N}^*$ and m' is an odd positive integer. Let a and b be the generators of the cyclic groups C_n and C_2, respectively. Since $\gamma^2 \in A \setminus \{1\}$ and $\gamma^4 = 1$, we infer that

$$\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}.$$

Assume that $Dic_{4n}(A)$ is a PSOS-group. By [7, Theorem 2.4], we have the following three cases:

Case 1. ($m \geq 1$, $m' \neq 1$, $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$) or ($m \geq 2$, $m' = 1$, $\gamma^2 \in \{b, a^{\frac{n}{2}}b\}$)

Then $L(Dic_{4n}(A))$ consists of the subgroups of A and of $\frac{n}{d}$ subgroups of order $4d$ (namely $\langle a^{\frac{n}{d}}, a^{i-1}\gamma \rangle$, where $i = 1, \ldots, \frac{n}{d}$), for every divisor d of n. Since

$$A \cong C_2 \times C_{2^m} \times C_{m'},$$

we infer that A has one subgroup of order 4 for $m = 1$ and 3 subgroups of order 4 for $m \geq 2$. Thus $Dic_{4n}(A)$ has $n+1$ subgroups of order 4 for $m = 1$ and $n+3$ subgroups of order 4 for $m \geq 2$. By hypothesis, we obtain

$$n+1 \mid 4n \text{ for } m = 1$$

and

$$n+3 \mid 4n \text{ for } m \geq 2.$$

These conditions lead either to $n \in \{1,3\}$ for $m = 1$ or to $n \in \{1,3,9\}$ for $m \geq 2$, which contradicts the fact that n is even.

Case 2. $m \geq 2$, $m'=1$, $\gamma^2 = a^{\frac{n}{2}}$

Then $Dic_{4n}(A)$ is isomorphic to $C_2 \times Q_{2^{m+1}}$ and so it has 3 subgroups of order 2. It follows that 3 divides $4n$, i.e., 3 divides 2^{m+2}, a contradiction.

Case 3. $m=m'=1$, $\gamma^2 \in \{a^{\frac{n}{2}}, b, a^{\frac{n}{2}}b\}$

Then $Dic_{4n}(A)$ is isomorphic to $C_2 \times C_4$ and so it has 3 subgroups of order 2. We get that 3 divides 8, a contradiction.

Hence we get the following theorem.

Theorem 3.2. *Let n be an even positive integer and $A = C_2 \times C_n$. Then the generalized dicyclic group $Dic_{4n}(A)$ is not a PSOS-group.*

Inspired by the above results, we came up with the next conjecture.

Conjecture 3.3. *Let n be a positive integer and A be a finite abelian group of order $2n$. Then the generalized dicyclic group $Dic_{4n}(A)$ is a PSOS-group if and only if n is odd.*

Finally, we mention that the problem of deciding whether a finite group is a PSOS-group can be also studied for other important classes of groups.

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