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ON G-CORE-EP AND S-CORE-EP INVERSES IN $*$-RINGS WITH RELATED RELATIONS

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Abstract. The concepts of the G-core-EP and S-core-EP inverses are extended from the set of all $n \times n$ complex matrices to the set of all core-EP invertible elements in a $*$-ring. We investigate their properties and provide characterizations of these generalized inverses. Furthermore, by applying them, we introduce and study some new relations on $*$-rings and Rickart $*$-rings.

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1. Introduction

Let $M_{m,n}$ be the set of all $m \times n$ complex matrices. If $m = n$, that is, for the set of all $n \times n$ square complex matrices we write M_n instead of $M_{n,n}$. As usual, let $A^* \in M_{n,m}$ denote the conjugate transpose of $A \in M_{m,n}$ and let $\operatorname{rank}(A)$ denote its rank. A matrix $A \in M_n$ has a (unique) inverse A^{-1} (a matrix satisfying $AA^{-1} = A^{-1}A = I$, where I is the identity matrix) if and only if it is a square matrix with nonzero determinant. However, the need for some kind of a pseudoinverse of a more general class of matrices (even non-square ones) has arisen from applications. For given $A \in M_{m,n}$, any matrix $X \in M_{n,m}$ which is a solution to the equations

$$AXA = A, XAX = X, (AX)^* = AX, \quad \text{and} \quad (XA)^* = XA \tag{1}$$

is called *the Moore-Penrose* inverse of A and is denoted by $A^\dagger$. Recall that every matrix $A \in M_{m,n}$ has the unique Moore-Penrose inverse. If $X = A^- \in M_{n,m}$ satisfies the first equation in (1), i.e., $AA^-A = A$, then it is called *an inner generalized inverse* (or *a* 1*-inverse*) of A. An inner generalized inverse of $A \in M_{m,n}$ is not unique in general.

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Although generalized inverses and induced partial orders were often studied on sets of real or complex matrices, many of those concepts were introduced on more general settings or later extended from sets of matrices to rings or even semigroups [6,18,23,25,29]. In the present paper we follow this trend and generalize the G-core-EP and S-core-EP inverses [15] with related relations to rings.

Throughout the paper, the term ring means an associative ring with identity 1. Let $\mathcal{R}$ be a $*$-ring, i.e., a ring equipped with involution $*$. We define the unique Moore-Penrose inverse and an inner generalized inverse of $a \in \mathcal{R}$ in the same way as in the matrix case (see (1)). Since these inverses may or may not exist in $\mathcal{R}$, we say that $a \in \mathcal{R}$ is *regular* if it has an inner generalized inverse a^- and $*$-*regular* if it admits the Moore-Penrose inverse $a^\dagger$. We denote by $\mathcal{R}^{(1)}$, $\mathcal{R}^\dagger$, and $a\{1\}$ the sets of all regular and $*$-regular elements in $\mathcal{R}$, and the set of all inner generalized inverses of a, respectively. Another very well known generalized inverse is *the Drazin inverse* that has many applications in the theories of control theory [4], finite Markov chains [5], singular differential and difference equations [5], cryptography [14], and iterative methods in numerical analysis [22]. We say that an element $a \in \mathcal{R}$ has a Drazin inverse $x = a^D \in \mathcal{R}$ if

$$ax = xa, \qquad x = ax^2, \qquad a^k = a^{k+1}x \tag{2}$$

for some non-negative integer k. Note that for $k = 0$ we define $a^0 = 1$. If a has a Drazin inverse a^D, then we say that a is *Drazin invertible* and the smallest non-negative integer k in (2) is called *the Drazin index* $i(a)$ of a. It is well known that there is at most one $x = a^D$ such that (2) holds (see [10]). If $i(a) \leq 1$, the Drazin inverse x of a is the group inverse of a.

Another generalized inverse which is closely related to the group inverse is called *the core inverse* (see [2,25,27]). For a $*$-ring $\mathcal{R}$, we say that $a \in \mathcal{R}$ has the core inverse $a^{\textcircled{\#}}$ if $x = a^{\textcircled{\#}}$ is the unique solution to the following equations:

$$axa = a, \quad xax = x, \quad (ax)^* = ax, \quad xa^2 = a, \quad \text{and} \quad ax^2 = x. \tag{3}$$

The set of all core invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{\textcircled{\#}}$. Note (see [2]) that the core inverse of $A \in M_n$ exists if and only if the index of A is less than or equal to one. In [17] the notion of the core inverse was extended to the full matrix algebra M_n by introducing a new kind of matrix generalized inverse called *the core-EP inverse,* and as a generalization for both the core inverse in a $*$-ring and the core-EP inverse for complex matrices, the notion of *the pseudo-core inverse* was put forward in [11]. Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}$. If there exists $x = a^{\textcircled{\dagger}} \in \mathcal{R}$ such that

$$xa^{m+1} = a^m \text{ for some positive integer } m, \quad ax^2 = x, \quad \text{and} \quad (ax)^* = ax, \tag{4}$$

then we say that a is pseudo-core (or core-EP) invertible and call $a^{\textcircled{\dagger}}$ the pseudo-core (or core-EP) inverse of a. The set of all core-EP invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{\textcircled{\dagger}}$. It turns out (see [11, Theorem 2.2]) that $a^{\textcircled{\dagger}}$ is unique if it exists, and that in this case the smallest positive integer m in (4), which is called *the pseudo-core index* of a and denoted by $I(a)$, either equals the Drazin index $i(a)$ of a if $i(a) > 0$, or is 1 if $i(a) = 0$. When $I(a) = 1$, the pseudo-core inverse becomes the core inverse of a. Also, the core-EP inverse is *an outer inverse* (*a* 2*-inverse*), i.e., $a^{\textcircled{\dagger}}aa^{\textcircled{\dagger}} = a^{\textcircled{\dagger}}$ (see [11]).

For a $*$-ring $\mathcal{R}$, let $a \in \mathcal{R}^{\textcircled{\dagger}}$ with $I(a) = k$. By [12, Theorem 3.1], we may write $a = a_1 + a_2$, where

$$i(a_1) \leq 1, \quad a_2^k = 0, \quad \text{and} \quad a_1^* a_2 = a_2 a_1 = 0. \tag{5}$$

It turns out that this decomposition, which we call *the core-EP decomposition* in $\mathcal{R}$, is unique and that

$$a_1 = aa^{\textcircled{\dagger}}a \quad \text{and} \quad a_2 = a - aa^{\textcircled{\dagger}}a, \tag{6}$$

where $a^{\textcircled{\dagger}}$ is the core-EP inverse of a. We call a_2 *the nilpotent part* of the core-EP decomposition of a.

With all of the mentioned generalized inverses we may define relations on a ring or a $*$-ring $\mathcal{R}$. Let us present four very well known relations. Let $a, b \in \mathcal{R}$.

1. *The minus order* $\leq^-$ [13]: We write

$$a \leq^- b \quad \text{if} \quad a^- a = a^- b \quad \text{and} \quad aa^= = ba^= \tag{7}$$

   for some $a^-, a^= \in a\{1\}$. The minus relation is a partial order on $\mathcal{R}^{(1)}$ for any ring $\mathcal{R}$.
2. *The core order* $\leq^{\textcircled{\#}}$ [2,26]: We write

$$a \leq^{\textcircled{\#}} b \quad \text{if} \quad a^{\textcircled{\#}}a = a^{\textcircled{\#}}b \quad \text{and} \quad aa^{\textcircled{\#}} = ba^{\textcircled{\#}}.$$

3. *The star order* $\leq^*$ [10]: We write

$$a \leq^* b \quad \text{if} \quad a^* a = a^* b \quad \text{and} \quad aa^* = ba^*. \tag{8}$$

   It turns out that on $\mathcal{R}^\dagger$ we have $a \leq^* b$ if and only if $a^\dagger a = a^\dagger b$ and $aa^\dagger = ba^\dagger$.

4. *The core-EP preorder* $\leq^{\textcircled{d}}$ [12]: We write

$$a \leq^{\textcircled{d}} b \quad \text{if} \quad a^{\textcircled{d}}a = a^{\textcircled{d}}b \qquad \text{and} \qquad aa^{\textcircled{d}} = ba^{\textcircled{d}}. \tag{9}$$

The minus and core relations are partial orders on $\mathcal{R}^{(1)}$ and $\mathcal{R}^{\textcircled{\#}}$, respectively. The star order is a partial order for any proper $*$-ring $\mathcal{R}$ with "properness" as customarily defined via $aa^* = 0$ implies $a = 0$. The core-EP relation is not a partial order on $\mathcal{R}^{\textcircled{d}}$ but merely a preorder (it is reflexive and transitive [12, Theorem 4.2], however it is not antisymmetric [28, Example 4.1]).

Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where a_2 and b_2 are the nilpotent parts. It was proved in [12] that $a_1^{\textcircled{\#}} = a^{\textcircled{d}}$ and $a \leq^{\textcircled{d}} b$ if and only if $a_1 \leq^{\textcircled{\#}} b_1$. We observe that the core-EP preorder fails to be a partial order since it does not encode any information about the nilpotent parts of a and b. In [16], the authors introduced a new partial order on M_n based on the core-EP decomposition. Note first that every complex matrix has the core-EP inverse, i.e., ${M_n}^{\textcircled{d}} = M_n$, and recall that every $A \in M_n$ has an inner generalized inverse A^-. Let $A, B \in M_n$ and suppose $A = A_1 + A_2$ and $B = B_1 + B_2$ are the core-EP decompositions of A and B, respectively, where A_2 and B_2 are the nilpotent parts.

5. *The c-minus order* $\leq^{\textcircled{c}}$ [16]: We write

$$A \leq^{\textcircled{c}} B \quad \text{if} \quad A_1 \leq^{\textcircled{\#}} B_1 \quad \text{and} \quad A \leq^{-} B. \tag{10}$$

Does there exist a generalized inverse with which the c-minus partial order may be defined in a similar way as the four relations mentioned above? This question motivated Hua and Wang [15] to introduce a new pseudoinverse which they named *the generalized core-EP* (or *G-core-EP*) *inverse*.

Definition 1.1. Let $A \in M_n$ with $i(A) = k$. If $X \in M_n$ satisfies the following matrix equations

$$XA^{k+1} = A^k, \quad A^k \left(A^k\right)^\dagger X = A^{\textcircled{d}}, \quad AA^\dagger \left(A - AXA\right) = AA^{\textcircled{d}} \left(A - AXA\right),$$

then X is called a G-core-EP (or generalized core-EP) inverse of A and is denoted by A^{GC}.

Let $I \in M_n$ be the identity matrix. By replacing the last equation in Definition 1.1 another generalized inverse was introduced in [15].

Definition 1.2. Let $A \in M_n$ with $i(A) = k$. If $X \in M_n$ satisfies the following matrix equations

$$XA^{k+1} = A^k, \quad A^k \left(A^k\right)^\dagger X = A^{ⓓ}, \quad (A^* - X)\left(I - AA^{ⓓ}\right) = 0,$$

then X is called an S-core-EP (or star-core-EP) inverse of A and is denoted by A^{SC}.

The aim of this paper is to generalize Definitions 1.1 and 1.2 from M_n to rings, establish some interesting properties and characterizations of these generalized inverses and relations that are induced by them, and thus generalize some known results. In Section 2, we present some preliminary notions and briefly discuss the well known left-star partial order. The G-core-EP inverse and the induced G-core-EP partial order are defined and studied in Section 3. In Section 4, we introduce the S-core-EP inverse in ∗-rings and explore the star-core-EP and left-star-core-EP relations arising from this pseudoinverse.

2. Preliminaries

Let us now present some tools which will be useful throughout the paper. The equality $1 = e_1 + e_2 + \cdots + e_n$, where $e_1, e_2, \ldots, e_n$ are idempotent elements in a ring $\mathcal{R}$ and $e_ie_j = 0$ for $i \neq j$, is called a decomposition of the identity of $\mathcal{R}$. Let $1 = e_1 + \cdots + e_n$ and $1 = f_1 + \cdots + f_n$ be two decompositions of the identity of $\mathcal{R}$. We have

$$x = 1 \cdot x \cdot 1 = (e_1 + e_2 + \cdots + e_n)x(f_1 + f_2 + \cdots + f_n) = \sum_{i,j=1}^{n} e_ixf_j.$$

Then any $x \in \mathcal{R}$ can be uniquely represented in the following matrix form:

$$x = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}_{e \times f}, \tag{11}$$

where $x_{ij} = e_ixf_j \in e_i\mathcal{R}f_j$. If $x = (x_{ij})_{e\times f}$ and $y = (y_{ij})_{e\times f}$, then $x + y = (x_{ij} + y_{ij})_{e\times f}$. Moreover, if $1 = g_1 + \cdots + g_n$ is a decomposition of the identity of $\mathcal{R}$ and $z = (z_{ij})_{f\times g}$, then, by the orthogonality of the idempotents involved, $xz = \left(\sum_{k=1}^{n} x_{ik}z_{kj}\right)_{e\times g}$. Thus, if we have decompositions of the identity of $\mathcal{R}$, then the usual algebraic operations in $\mathcal{R}$ can be interpreted as simple operations between appropriate $n \times n$ matrices over $\mathcal{R}$. When $n = 2$ and $p, q \in \mathcal{R}$ are idempotent elements, we may write

$$x = pxq + px(1-q) + (1-p)xq + (1-p)x(1-q) = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}_{p\times q}.$$

Here $x_{11} = pxq$, $x_{12} = px(1-q)$, $x_{21} = (1-p)xq$, $x_{22} = (1-p)x(1-q)$.

If $\mathcal{R}$ is a $*$-ring, we may by (11) write

$$x^* = \begin{bmatrix} x_{11}^* & \cdots & x_{n1}^* \\ \vdots & \ddots & \vdots \\ x_{1n}^* & \cdots & x_{nn}^* \end{bmatrix}_{f^*\times e^*} \tag{12}$$

where this matrix representation of x^* is given relative to the decompositions of the identity $1 = f_1^* + \cdots + f_n^*$ and $1 = e_1^* + \cdots + e_n^*$.

Let $\mathcal{R}$ be a $*$-ring and $a \in \mathcal{R}^{@}$. Recall that $a^{@}$ is an outer inverse of a and therefore $p = aa^{@}$ (and $q = a^{@}a$) is an idempotent. Let $a = a_1 + a_2$ be the core-EP decomposition of a where a_2 is the nilpotent part. It was proved in [7] that for $p = aa^{@}$,

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} \quad \text{and} \quad a_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}_{p\times p}$$

where t is invertible in the ring $p\mathcal{R}p$, i.e., there exists $t^{-1} \in p\mathcal{R}p$ such that $tt^{-1} = t^{-1}t = p$. So,

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p}. \tag{13}$$

Note that $t = a^2a^{@}$ and $t^{-1} = a^{@}$ (see [8]). From now on a_2 denotes the nilpotent part in the core-EP decomposition of $a \in \mathcal{R}^{@}$.

2.1. The left-star order in Rickart $*$-rings. Consider a ring $\mathcal{R}$. For $a \in \mathcal{R}$, we denote by a° the right annihilator of a, i.e., the set $a^\circ = \{x \in \mathcal{R} : ax = 0\}$. Similarly, we denote the left annihilator ${}^\circ a$ of a, i.e., the set ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$. Observe (or see [6, Lemma 2.1]) that for an idempotent $p \in \mathcal{R}$, we have ${}^\circ p = \mathcal{R}(1-p)$ and $p^\circ = (1-p)\mathcal{R}$. A ring $\mathcal{R}$ is called a Rickart ring if for every $a \in \mathcal{R}$, there exist idempotent elements $p, q \in \mathcal{R}$ such that $a^\circ = p\mathcal{R}$ and ${}^\circ a = \mathcal{R}q$. Every Rickart ring $\mathcal{R}$ has the (multiplicative) identity 1 (see, e.g., [3]). A $*$-ring $\mathcal{R}$ is called a Rickart $*$-ring if the left annihilator ${}^\circ a$ of any element $a \in \mathcal{R}$ is generated by a (unique) self-adjoint idempotent $e \in \mathcal{R}$, i.e., ${}^\circ a = \mathcal{R}e$ where $e = e^* = e^2$. Note that the analogous property for right annihilators is fulfilled in case when $\mathcal{R}$ is a Rickart $*$-ring and that M_n is an example of a Rickart $*$-ring.

The left-star-core-EP relation that will be introduced and studied in Section 5 is closely related to the well known left-star partial order. Let us thus recall this notion. It was introduced in [1] on $M_{m,n}$ and was later extended in [21] to Rickart $*$-rings. The definition is as follows. Let $\mathcal{R}$ be a Rickart $*$-ring. For $a, b \in \mathcal{R}$, we write

$$a{*}{\leq}b \quad \text{if} \quad {}^{\circ}a = \mathcal{R}(1-p),\ a^{\circ} = (1-q)\mathcal{R},\ pa = pb,\ \text{and}\ \ aq = bq$$

where p is a self-adjoint idempotent and q is an idempotent in $\mathcal{R}$. We now present some characterizations of this relation.

Proposition 2.1. *Let $\mathcal{R}$ be a Rickart $*$-ring and $a, b \in \mathcal{R}$. The following statements are equivalent.*

(i) $a{*}{\leq}b$.

(ii) *There exist a self-adjoint idempotent p and an idempotent q in $\mathcal{R}$ such that ${}^{\circ}p \subseteq {}^{\circ}a$, $q^{\circ} \subseteq a^{\circ}$, $pa = pb$, and $aq = bq$.*

(iii) *There exist a self-adjoint idempotent p and an idempotent q in $\mathcal{R}$ such that $a = pb = bq$.*

(iv) $a^*a = a^*b$ *and* $a\mathcal{R} \subseteq b\mathcal{R}$.

Proof. Statement (i) is equivalent to statement (ii) according to [18, Theorem 5].

Suppose now that (ii) holds and let us show that then (iii) is satisfied. We have ${}^{\circ}p \subseteq {}^{\circ}a$ and hence $(1-p)a = 0$, i.e., $a = pa$. Thus $pa = pb$ yields $a = pb$. Similarly, $q^{\circ} \subseteq a^{\circ}$ and $aq = bq$ imply $a = bq$. Conversely, let (iii) hold. Then $a = pb = bq$ imply $pa = pb$ and $aq = bq$. Also, from $a = pb$ we have $a = p(pb) = pa$ and thus $(1-p)a = 0$, i.e., ${}^{\circ}p \subseteq {}^{\circ}a$. Similarly, $a = bq$ yields $q^{\circ} \subseteq a^{\circ}$ and thus (ii) is satisfied.

Let us now show that (iii) is equivalent to (iv). Applying the involution to $a = pb$, we get $a^* = b^*p$. Multiplying the latter from the right by a, we get $a^*a = b^*pa = b^*a = a^*b$. If $a = bq$, then $a\mathcal{R} \subseteq b\mathcal{R}$ and so (iii) implies (iv). Let us assume that (iv) holds. We may then find a self-adjoint idempotent p and an element z in $\mathcal{R}$ such that $a = pb = bz$. We observe that $az = p(bz) = pa = a$. Since $\mathcal{R}$ is a Rickart $*$-ring and thus a Rickart ring, there exist idempotents $r, s \in \mathcal{R}$ such that ${}^{\circ}a = {}^{\circ}r$ and $a^{\circ} = s^{\circ}$. It follows that

$$a = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{r\times s}.$$

Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{r\times s} \quad \text{and} \quad z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{s\times s}$$

and define

$$q = \begin{bmatrix} s & 0 \\ z_3 & 0 \end{bmatrix}_{s \times s}.$$

From $z_3 \in (1-s)\mathcal{R}s$ we have $z_3 s = z_3$ and thus we observe that $q^2 = q$, i.e., q is an idempotent. By

$$az = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{r \times s} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{s \times s} = \begin{bmatrix} az_1 & az_2 \\ 0 & 0 \end{bmatrix}_{r \times s}$$

and since $az = a$, we obtain $az_1 = a$, i.e., $1 - z_1 \in a^\circ = s^\circ$. Thus $s = sz_1$. Also, $az_2 = 0$, i.e., $z_2 \in a^\circ = s^\circ$ and so $sz_2 = 0$. It follows that $z_1 = szs = s(szs) = sz_1 = s$ and $z_2 = sz(1-s) = s(sz(1-s)) = sz_2 = 0$. From $a = bz$ we get then

$$\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{r \times s} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{r \times s} \begin{bmatrix} s & 0 \\ z_3 & z_4 \end{bmatrix}_{s \times s} = \begin{bmatrix} b_1 s + b_2 z_3 & b_2 z_4 \\ b_3 s + b_4 z_3 & b_4 z_4 \end{bmatrix}_{r \times s}.$$

Since $b_1, b_3 \in \mathcal{R}s$, it follows that $a = b_1 + b_2 z_3$ and $b_3 + b_4 z_3 = 0$. Now,

$$bq = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{r \times s} \begin{bmatrix} s & 0 \\ z_3 & 0 \end{bmatrix}_{s \times s} = \begin{bmatrix} b_1 + b_2 z_3 & 0 \\ b_3 + b_4 z_3 & 0 \end{bmatrix}_{r \times s} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}_{r \times s} = a$$

and so $a = pb = bq$ for a self-adjoint idempotent p and an idempotent q in $\mathcal{R}$ and thus statement (iii) holds. □

Unless stated otherwise, let $\mathcal{R}$ be a $*$-ring from now on.

3. The G-core-EP inverse

Let $A \in M_n$ with $\operatorname{rank}(A^k) = r$. Employing the core-EP decomposition of A, Hua and Wang [15] established that whenever there exists $X \in M_n$ satisfying the matrix equations given in Definition 1.1, there is a unitary matrix $U \in M_n$ such that

$$X = U \begin{bmatrix} T^{-1} & 0 \\ 0 & N^- \end{bmatrix} U^*$$

where $T \in M_r$ is nonsingular, $N \in M_{n-r}$, and $N^- \in N\{1\}$. We will now generalize this result. First, recall that every complex matrix is Moore-Penrose invertible which may not be the case for an element in a $*$-ring $\mathcal{R}$. Also, for $A \in M_n$ with $i(A) = k$, we have by [28, Corollary 3.3] $A^k \left(A^k\right)^\dagger = AA^{\textcircled{\dagger}}$. With an adjustment of the equations from Definition 1.1 we formulate the following result.

Theorem 3.1. *Let $a \in \mathcal{R}^{@}$ with $I(a) = k$ and suppose $a_2 \in \mathcal{R}^{(1)}$. Consider the following equations*

$$xa^{k+1} = a^k, \quad aa^{@}x = a^{@}, \quad a - axa = aa^{@}(a - axa). \tag{14}$$

Then there exists a solution of (14) *and it is of the form*

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \tag{15}$$

where $p = aa^{@}$, t is invertible in the ring $p\mathcal{R}p$, and $a_2^- \in a_2\{1\}$ is arbitrary.

Proof. Let $p = aa^{@}$. Write a in the matrix form (13). By $a_2^k = 0$, we observe that then

$$a^k = \begin{bmatrix} t^k & \widetilde{t} \\ 0 & 0 \end{bmatrix}_{p\times p} \qquad \text{and} \qquad a^{k+1} = \begin{bmatrix} t^{k+1} & t^k s + \widetilde{t}a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

where $\widetilde{t} = \sum_{i=0}^{k-1} t^i s a_2^{k-1-i}$. Suppose first

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p\times p}$$

is a solution of (14). So, $xa^{k+1} = a^k$ implies

$$\begin{bmatrix} x_1 t^{k+1} & x_1\left(t^k s + \widetilde{t}a_2\right) \\ x_3 t^{k+1} & x_3\left(t^k s + \widetilde{t}a_2\right) \end{bmatrix}_{p\times p} = \begin{bmatrix} t^k & \widetilde{t} \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

Thus $x_3 = 0$ and $x_1 = t^{-1}$. The equation $aa^{@}x = a^{@}$ then yields $px = t^{-1}$ and so

$$\begin{bmatrix} t^{-1} & x_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p},$$

i.e., $x_2 = 0$. By the last equation $a - axa = aa^{@}(a - axa)$, we obtain

$$\begin{bmatrix} 0 & -sx_4a_2 \\ 0 & a_2 - a_2x_4a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 0 & -sx_4a_2 \\ 0 & a_2 - a_2x_4a_2 \end{bmatrix}_{p\times p}$$

and therefore $a_2 - a_2x_4a_2 = 0$. So, $x_4 \in a_2\{1\}$ and by denoting $x_4 = a_2^-$, we get the desired form of x.

Conversely, let

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p}$$

where t is invertible in the ring $p\mathcal{R}p$, and $a_2^- \in a_2\{1\}$. Since $t^{k-1}s + t^{-1}\widetilde{t}a_2 = \widetilde{t}$, we get

$$xa^{k+1} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t^{k+1} & t^k s + \widetilde{t}a_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^k & \widetilde{t} \\ 0 & 0 \end{bmatrix}_{p\times p} = a^k.$$

Also, $aa^{@}x = px = t^{-1} = a^{@}$. Since $a_2a_2^-a_2 = a_2$,

$$\begin{aligned} a - axa &= \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} - \begin{bmatrix} t & s + sa_2^-a_2 \\ 0 & a_2a_2^-a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} 0 & -sa_2^-a_2 \\ 0 & 0 \end{bmatrix}_{p\times p} \\ &= \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 0 & -sa_2^-a_2 \\ 0 & 0 \end{bmatrix}_{p\times p} = p\,(a - axa)\,. \end{aligned}$$

It follows that $x = t^{-1} + a_2^-$ is a solution of (14) for every $a_2^- \in a_2\{1\}$. □

We mention that in some proofs we use ideas similar to the proofs of analogous results from [15] but since we can not use linear algebra techniques (such as singular value decomposition) we will not omit these proofs. With the following definition we then generalize Definition 1.1 to $*$-rings.

Definition 3.2. Let $\mathcal{R}$ be a $*$-ring and let $a \in \mathcal{R}^{@}$ with $I(a) = k$. The solution of (14), if it exists, is called the G-core-EP inverse of a and is denoted by a^{gc}.

Remark 3.3. Since $a_2\{1\}$ may not be a singleton, we observe by Theorem 3.1 that the G-core-EP inverse is not necessarily unique. We denote by $a\{gc\}$ the set of all G-core-EP inverses of $a \in \mathcal{R}^{@}$.

Remark 3.4. The last equation in (14) may be replaced with the following condition:

$$a - axa \in p\mathcal{R} \text{ where } p = aa^{@}.$$

Let $a \in \mathcal{R}^{@}$ be of the form (13) with $a_2 \in \mathcal{R}^{(1)}$. If $x \in a\{gc\}$, it follows by Theorem 3.1 that

$$axa = \begin{bmatrix} t & s + sa_2^-a_2 \\ 0 & a_2a_2^-a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s + sa_2^-a_2 \\ 0 & a_2 \end{bmatrix}_{p\times p}$$

and thus $axa = a$ if and only if $sa_2^-a_2 = 0$. We therefore get the following result.

Corollary 3.5. *Let $a \in \mathcal{R}^{@}$ be of the form* (13) *with $a_2 \in \mathcal{R}^{(1)}$. Suppose $x \in a\{gc\}$ with the form* (15). *Then $x \in a\{1\}$ if and only if $sa_2^-a_2 = 0$.*

We denote the set of all outer inverses of $a \in \mathcal{R}$ by $a\{2\}$. The set of all elements in $\mathcal{R}$ that are both inner and outer inverses of a is denoted by $a\{1, 2\}$.

Remark 3.6. Suppose $a \in \mathcal{R}$. Let $h \in a\{1\}$, $r = ah$, and $q = ha$. It is easy to check (see [24, page 1044]) that then

$$a^- \in a\{1\} \quad \text{if and only if} \quad a^- = \begin{bmatrix} hah & c \\ d & g \end{bmatrix}_{q\times r}$$

where $c \in q\mathcal{R}(1-r)$, $d \in (1-q)\mathcal{R}r$, $g \in (1-q)\mathcal{R}(1-r)$ are fixed but arbitrary.

Proposition 3.7. *Let $a \in \mathcal{R}^{@}$ be of the form* (13) *with $a_2 \in \mathcal{R}^{(1)}$. Suppose $x \in a\{gc\}$ with the form* (15). *Then the following statements hold.*

(i) *$x \in a\{2\}$ if and only if $sa_2^- = 0$ and $a_2^- \in a_2\{2\}$.*
(ii) *$x \in a\{1,2\}$ if and only if $x \in a\{2\}$.*

Proof. (i) By Theorem 3.1, we observe that $xax = x$ is equivalent to $a_2^- \in a_2\{2\}$ and $t^{-1}sa_2^- = 0$, i.e., $sa_2^- = 0$.

(ii) If $x \in a\{2\}$, then by the previous statement $sa_2^- = 0$ and so $sa_2^- a_2 = 0$. Thus, by Corollary 3.5, $x \in a\{1,2\}$. The converse implication is clear. □

Proposition 3.8. *If $a\{gc\} \cap a\{1\} \neq \emptyset$, then $a\{gc\} \cap a\{2\} \neq \emptyset$ and $a\{gc\} \cap a\{1,2\} \neq \emptyset$.*

Proof. Assume that $a\{gc\} \cap a\{1\} \neq \emptyset$. Let a be of the form (13) with $a_2 \in \mathcal{R}^{(1)}$. Note that by Corollary 3.5, every $x \in a\{gc\} \cap a\{1\}$ has the form

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p}$$

with $sa_2^- a_2 = 0$. Let $h \in a_2\{1\}$ and denote $r = a_2 h$, and $q = ha_2$. Then r and q are idempotents and by Remark 3.6,

$$a_2^- = \begin{bmatrix} ha_2h & c \\ d & g \end{bmatrix}_{q\times r}$$

where $c \in q\mathcal{R}(1-r)$, $d \in (1-q)\mathcal{R}r$, $g \in (1-q)\mathcal{R}(1-r)$. Note that $a_2 = ra_2q$ and let

$$s = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}_{r\times q}.$$

From

$$sa_2^- a_2 = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix}_{r\times q} \begin{bmatrix} ha_2h & c \\ d & g \end{bmatrix}_{q\times r} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}_{r\times q} = \begin{bmatrix} s_1ha_2ha_2 + s_2da_2 & 0 \\ s_3ha_2ha_2 + s_4da_2 & 0 \end{bmatrix}_{r\times q}$$

we observe that the solution of the equation $sa_2^- a_2 = 0$ does not depend on g. Fix $g = da_2c$, i.e.,

$$a_2^- = \begin{bmatrix} ha_2h & c \\ d & da_2c \end{bmatrix}_{q\times r}$$

and let $x = t^{-1} + a_2^-$. Hence,

$$a_2^- a_2 a_2^- = \begin{bmatrix} ha_2h & c \\ d & g \end{bmatrix}_{q\times r} \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}_{r\times q} \begin{bmatrix} ha_2h & c \\ d & g \end{bmatrix}_{q\times r} = \begin{bmatrix} ha_2h & ha_2c \\ da_2h & da_2c \end{bmatrix}_{q\times r}.$$

Since $c \in q\mathcal{R}(1-r)$ and $d \in (1-q)\mathcal{R}r$, and $r = a_2h$ and $q = ha_2$ are idempotents, it follows that $ha_2c = c$ and $da_2h = d$. Thus,

$$a_2^- a_2 a_2^- = \begin{bmatrix} ha_2h & c \\ d & da_2c \end{bmatrix}_{q\times r} = a_2^-.$$

So, from $sa_2^- a_2 = 0$ we obtain $sa_2^- = sa_2^- a_2 a_2^- = 0$ and thus by Proposition 3.7, $x \in a\{2\}$. We may conclude that $a\{gc\} \cap a\{2\} \neq \emptyset$ and $a\{gc\} \cap a\{1,2\} \neq \emptyset$. □

With [15, Example 2.1] it was shown that the G-core-inverse differs in general from some well known generalized inverses such as the Moore-Penrose inverse or the Drazin inverse. Write again $a \in \mathcal{R}^{\textcircled{\dagger}}$ in the matrix form (13) of the core-EP decomposition. The elements

$$a^{\textcircled{s}} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2 \end{bmatrix}_{p\times p} \quad \text{and} \quad a^{\textcircled{w}} = \begin{bmatrix} t^{-1} & t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p}$$

are called the C-S inverse (see [20, Definition 3 and Theorem 3.4]) and the WG inverse (see [19, pages 6, 15, 16]), respectively.

Remark 3.9. Recall that $a^{\textcircled{\dagger}} = t^{-1}$. For $a \in \mathcal{R}^{\textcircled{\dagger}}$ with $a_2 \in \mathcal{R}^{(1)}$ we thus have that $a^{\textcircled{\dagger}} = a^{gc}$ if and only if $a_2^- = 0$ and thus $a_2 = a_2 a_2^- a_2 = 0$. So, in this case $I(a) = 1$ and thus $a^{gc} = a^{\textcircled{\dagger}} = a^{\textcircled{\#}} = a^{\textcircled{s}}$. Moreover, we see that $a^{gc} = a^{\textcircled{s}}$ if and only if $a_2 = a_2^-$. So, $a_2 = a_2 a_2^- a_2 = a_2^3$ and then $a_2 = a_2^3 a_2^- a_2^3 = a_2^7$ etc. It follows that $a_2 = a_2^l$ for some $l > I(a)$, thus $a_2 = a_2^l = 0$. Finally, we observe that $a^{gc} = a^{\textcircled{w}}$ if and only if $s = 0$ and $a_2^- = 0$. We may conclude that when either $a^{\textcircled{\dagger}}$, $a^{\textcircled{s}}$, or $a^{\textcircled{w}}$ is an element of $a\{gc\}$, then $I(a) = 1$ and thus $a^{gc} = a^{\textcircled{\dagger}} = a^{\textcircled{\#}} = a^{\textcircled{s}}$.

We end this section by presenting several characterizations of a G-core-EP inverse. We thus generalize [15, Theorem 2.5].

Theorem 3.10. *Let $a \in \mathcal{R}^{@}$ and suppose $a_2 \in \mathcal{R}^{(1)}$. The following statements are equivalent.*

(i) $x \in a\{gc\}$.
(ii) $xa^{@} = \left(a^{@}\right)^2$, $xaa^{@} = aa^{@}x$, $a - axa = aa^{@}\left(a - axa\right)$.
(iii) $xaa^{@} = a^{@}$, $xaa^{@} = aa^{@}x$, $a - axa = aa^{@}\left(a - axa\right)$.
(iv) $a^{@}x = \left(a^{@}\right)^2$, $xaa^{@} = a^{@}$, $a - axa = aa^{@}\left(a - axa\right)$.

Proof. Assume first that $a \in \mathcal{R}^{@}$ is of the form (13) with $a_2 \in \mathcal{R}^{(1)}$ and let $x \in a\{gc\}$, i.e.,

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \quad \text{and} \quad x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p}$$

where $p = aa^{@}$. Since $a^{@} = t^{-1}$, we obtain

$$xa^{@} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-2} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \left(a^{@}\right)^2$$

and similarly, $a^{@}x = \left(a^{@}\right)^2$. Also,

$$xaa^{@} = xp = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = a^{@}$$

and similarly, $aa^{@}x = px = a^{@} = xaa^{@}$. From

$$a - axa = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} - \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} 0 & -sa_2^- a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

we observe that $a - axa = p\left(a - axa\right) = aa^{@}\left(a - axa\right)$. Thus, (i) implies (ii)–(iv).

Let us show that (ii) implies (i). Let

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p\times p}.$$

By $xa^{@} = \left(a^{@}\right)^2$ we obtain

$$\begin{bmatrix} x_1 t^{-1} & 0 \\ x_3 t^{-1} & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-2} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and thus $x_1 = t^{-1}$ and $x_3 = 0$. From $xaa^{@} = aa^{@}x$, we get

$$\begin{bmatrix} x_1 & 0 \\ x_3 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} x_1 & x_2 \\ 0 & 0 \end{bmatrix}_{p\times p}$$

and so $x_2 = 0$. Finally, $a - axa = aa^{\textcircled{d}}(a - axa)$ yields

$$\begin{aligned} a - axa &= \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} - \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \begin{bmatrix} t^{-1} & 0 \\ 0 & x_4 \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \\ &= \begin{bmatrix} 0 & -sx_4a_2 \\ 0 & a_2 - a_2x_4a_2 \end{bmatrix}_{p\times p} = aa^{\textcircled{d}}(a - axa) = \begin{bmatrix} 0 & -sx_4a_2 \\ 0 & 0 \end{bmatrix}_{p\times p}. \end{aligned}$$

So, $a_2 = a_2x_4a_2$, i.e., $x_4 = a_2^-$ for some $a_2^- \in a_2\{1\}$. It follows by Theorem 3.1 that $x \in a\{gc\}$. We prove similarly that both (iii) and (iv) imply (i). □

4. The G-core-EP order in ∗-rings

Recall that $\mathcal{R}$ stands for a $*$-ring. With [15, Definition 3.1] a new relation was introduced on M_n: For $A, B \in M_n$, we write $A \leq^{gc} B$ when $A^{gc_1}A = A^{gc_1}B$ and $AA^{gc_2} = BA^{gc_2}$ where $A^{gc_1}, A^{gc_2} \in A\{gc\}$. We now extend this definition to $*$-rings.

Definition 4.1. Let $a, b \in \mathcal{R}^{\textcircled{d}}$ with $a_2 \in \mathcal{R}^{(1)}$. We say that a is below b under the G-core-EP relation and write

$$a \leq^{gc} b \quad \text{when} \quad a^{gc_1}a = a^{gc_1}b \text{ and } aa^{gc_2} = ba^{gc_2}$$

where $a^{gc_1}, a^{gc_2} \in a\{gc\}$.

The next two results employ the core-EP decomposition to characterize the G-core-EP relation.

Theorem 4.2. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ where $a_2 \in \mathcal{R}^{(1)}$. Then*

$$a \leq^{gc} b \quad \textit{if and only if} \quad a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \textit{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \tag{16}$$

where $p = aa^{\textcircled{d}}$, t is invertible in the ring $p\mathcal{R}p$, and $a_2 \leq^- b_4$.

Proof. Let $a \in \mathcal{R}^{\textcircled{d}}$ be of the matrix form (13) with $a_2 \in \mathcal{R}^{(1)}$. Suppose first that $a \leq^{gc} b$ and let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p}.$$

There exist $a^{gc_1}, a^{gc_2} \in a\{gc\}$ such that $a^{gc_1}a = a^{gc_1}b$ and $aa^{gc_2} = ba^{gc_2}$. Also, there exist $a_2^-, a_2^= \in a_2\{1\}$ such that

$$a^{gc_1} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \quad \text{and} \quad a^{gc_2} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^= \end{bmatrix}_{p\times p}.$$

Since

$$a^{gc_1}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} p & t^{-1}s \\ 0 & a_2^-a_2 \end{bmatrix}_{p\times p}$$

equals

$$a^{gc_1}b = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1}b_1 & t^{-1}b_2 \\ a_2^-b_3 & a_2^-b_4 \end{bmatrix}_{p\times p},$$

it follows that $b_1 = t$, $b_2 = s$, and $a_2^-a_2 = a_2^-b_4$. We similarly show that $aa^{gc_2} = ba^{gc_2}$ yields $b_3 = 0$ and $a_2a_2^{=} = b_4a_2^{=}$. It follows that

$$b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \text{ and } a_2 \leq^- b_4.$$

Conversely, let a and b be of the matrix form from (16) with $a_2 \leq^- b_4$. Then there exist $a_2^-, a_2^{=} \in a_2\{1\}$ such that $a_2^-a_2 = a_2^-b_4$ and $a_2a_2^{=} = b_4a_2^{=}$. Let

$$a^{gc_1} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p}, \; a^{gc_2} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^{=} \end{bmatrix}_{p\times p}.$$

By Theorem 3.1, $a^{gc_1}, a^{gc_2} \in a\{gc\}$. Also,

$$a^{gc_1}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^- \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} = a^{gc_1}b$$

and similarly, $aa^{gc_2} = ba^{gc_2}$. Thus, $a \leq^{gc} b$. □

Theorem 4.3. *Let $a, b \in \mathcal{R}^{@}$ with $a_2 \in \mathcal{R}^{(1)}$. Then $a \leq^{gc} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$ with $e_1 = aa^{@}$, $e_2 = e_2^*$ such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \quad \textit{and} \quad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \tag{17}$$

where t is invertible in the ring $e_1\mathcal{R}e_1$, t_1 is invertible in the ring $e_2\mathcal{R}e_2$, $n_5^{I(b)} = 0 = (n_1 + n_2 + n_3 + n_4)^{I(a)}$, and $n_1 + n_2 + n_3 + n_4 \leq^- t_1 + z_1 + n_5$.

Proof. Let first that $a \leq^{gc} b$. By Theorem 4.2,

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} \text{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $p = aa^{@}$, t is invertible in the ring $p\mathcal{R}p$, $b_4 \in (1-p)\mathcal{R}(1-p)$, and a_2 is nilpotent with $a_2 \leq^- b_4$. Since $b \in \mathcal{R}^{@}$, it follows by [19, Lemma 2.3] that $b_4 \in \mathcal{R}^{@}$

so we may write b_4 in the matrix form of the core-EP decomposition

$$b_4 = \begin{bmatrix} t_1 & z_1 \\ 0 & n_5 \end{bmatrix}_{q\times q}$$

where $q = b_4 b_4^{\textcircled{\scriptsize @}}$, n_5 is the nilpotent part of the core-EP decomposition of b_4 and t_1 is invertible in the ring $q\mathcal{R}q$. Since $b_4 \in (1-p)\mathcal{R}$, we have $0 = pb_4 b_4^{\textcircled{\scriptsize @}} = pq$. Also, by (4), $q = q^*$ and $p = p^*$, and thus $qp = 0$. So,

$$\begin{aligned} b &= t + s + b_4 = t + s + t_1 + z_1 + n_5 \\ &= pbp + pb(1-p) + q(1-p)b(1-p)q + q(1-p)b(1-p)(1-q) \\ &\quad +(1-q)(1-p)b(1-p)(1-q) \\ &= pbp + pb(1-p) + qbq + qb(1-p-q) + (1-p-q)b(1-p-q). \end{aligned}$$

Observe that $t = pbp$, $t_1 = qbq$, $z_1 = qb(1-p-q)$, and $n_5 = (1-p-q)b(1-p-q)$. Denote $s_1 = pbq$ and $s_2 = pb(1-p-q)$. Thus, $s = pb(1-p) = pbq + pb(1-p-q) = s_1 + s_2$. It follows that for $e_1 = p$, $e_2 = q$, and $e_3 = 1-p-q$, we may represent b as

$$b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}.$$

Note that $e_2 = e_2^*$ and t_1 is invertible in the ring $e_2\mathcal{R}e_2$. Since n_5 is the nilpotent part in the core-EP decomposition of b_4, we have $n_5^{I(b_4)} = 0$, however by the proof of [19, Lemma 2.3], we observe that $I(b_4) = I(b)$ and so $n_5^{I(b)} = 0$. Let us now show that the form of a is as in (17). Recall that $a = t + s + a_2$ and thus on the one hand $s = pa(1-p) = paq + pa(1-p-q)$ but on the other hand $s = pbq + pb(1-p-q)$. So, $paq + pa(1-p-q) = pbq + pb(1-p-q)$ and thus $paq = pbq = s_1$ and $pa(1-p-q) = pb(1-p-q) = s_2$. Also,

$$a_2 = (1-p)a(1-p) = qaq + qa(1-p-q) + (1-p-q)aq + (1-p-q)a(1-p-q).$$

Denote $n_1 = qaq$, $n_2 = qa(1-p-q)$, $n_3 = (1-p-q)aq$, and $n_4 = (1-p-q)a(1-p-q)$. Thus,

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}.$$

Since a_2 is the nilpotent part of the core-EP decomposition of a, we have

$$(n_1 + n_2 + n_3 + n_4)^{I(a)} = 0.$$

Finally, since $a_2 \leq^- b_4$, we have $n_1 + n_2 + n_3 + n_4 \leq^- t_1 + z_1 + n_5$.

Conversely, let a and b be of the form (17). Note that by (5), $a = a_1 + a_2$ where $a_1 = t + s_1 + s_2$ and $a_2 = n_1 + n_2 + n_3 + n_4$ with $a_2^{I(a)} = 0$ is the core-EP decomposition of a. By assumption, $a_2 \leq^- t_1 + z_1 + n_5$. Denote $b_4 = t_1 + z_1 + n_5$. There exist $a_2^-, a_2^= \in a_2\{1\}$ such that $a_2^- a_2 = a_2^- b_4$ and $a_2 a_2^= = b_4 a_2^=$. Let $x_1 = t^{-1} + a_2^-$ and $x_2 = t^{-1} + a_2^=$. By Theorem 3.1, $x_1, x_2 \in a\{gc\}$. Following (17), note that $t^{-1} a_2 = 0 = t^{-1} b_4$ and $a_2^-(t + s_1 + s_2) = 0$. By $a_2^- a_2 = a_2^- b_4$, we thus have

$$x_1 a = \left(t^{-1} + a_2^-\right)(t + s_1 + s_2 + a_2) = \left(t^{-1} + a_2^-\right)(t + s_1 + s_2 + b_4) = x_1 b.$$

Similarly, by $a_2 a_2^= = b_4 a_2^=$, we obtain $a x_2 = b x_2$. Thus, $a \leq^{gc} b$. $\square$

The c-minus order (10) was generalized from M_n to $*$-rings in [9]. Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and let $a = a_1 + a_2$ and $b = b_1 + b_2$ be the core-EP decompositions of a and b, respectively, where as before a_2 and b_2 are the nilpotent parts. We say that a is below b under the c-minus relation and write

$$a \leq^{\copyright} b \quad \text{if} \quad a_1 \leq^{\textcircled{\dagger}} b_1 \quad \text{and} \quad a \leq^- b. \tag{18}$$

Note here that there are many equivalent definitions and generalizations of Hartwig's minus partial order (7). For example, Mitsch [23, Theorem 3] generalized the minus partial order to arbitrary semigroups and authors of [9] used Mitsch's definition in (18) and were thus not limited to regular rings, i.e., rings where $\mathcal{R} = \mathcal{R}^{(1)}$. Observe also (see [9, Theorem 4.2]) that the c-minus relation is a partial order on $\mathcal{R}^{\textcircled{d}}$. By [9, Theorem 4.2] and Theorem 4.3, it follows that for $a, b \in \mathcal{R}^{\textcircled{d}}$ with $a_2 \in \mathcal{R}^{(1)}$, we have $a \leq^{gc} b$ if and only if $a \leq^{\copyright} b$. So, at least for a regular ring $\mathcal{R}$, the c-minus and the G-core-EP relations are the same. We thus obtain the following result.

Theorem 4.4. *The G-core-EP relation is a partial order on $\mathcal{R}^{\textcircled{d}}$ where $\mathcal{R}$ is a regular $*$-ring.*

We end this section with another observation.

Proposition 4.5. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ with $a_2 \in \mathcal{R}^{(1)}$. Then $a \leq^{gc} b$ if and only if $a \leq^- b$, $a^{\textcircled{d}} a = a^{\textcircled{d}} b$, and $b a^{\textcircled{d}} a = a a^{\textcircled{d}} b$.*

Proof. Let first, $a \leq^{gc} b$. Thus, $a \leq^{\copyright} b$ and therefore by (18), $a_1 \leq^{\textcircled{\dagger}} b_1$ and $a \leq^- b$. Considering the core-EP decompositions $a = a_1 + a_2$ and $b = b_1 + b_2$ of a and b, respectively, it was proved in [12] that $a \leq^{\textcircled{d}} b$ if and only if $a_1 \leq^{\textcircled{\dagger}} b_1$. It follows that $a \leq^{\textcircled{d}} b$ and thus by (9), $a^{\textcircled{d}} a = a^{\textcircled{d}} b$ and $a a^{\textcircled{d}} = b a^{\textcircled{d}}$. By the latter equation, we obtain $a a^{\textcircled{d}} a = b a^{\textcircled{d}} a$ and so $b a^{\textcircled{d}} a = a\left(a^{\textcircled{d}} a\right) = a a^{\textcircled{d}} b$.

Conversely, let $a \leq^{-} b$, $a^{\textcircled{\dagger}}a = a^{\textcircled{\dagger}}b$, and $ba^{\textcircled{\dagger}}a = aa^{\textcircled{\dagger}}b$. Then $b\left(a^{\textcircled{\dagger}}aa^{\textcircled{\dagger}}\right) = aa^{\textcircled{\dagger}}ba^{\textcircled{\dagger}}$ and thus

$$ba^{\textcircled{\dagger}} = a\left(a^{\textcircled{\dagger}}b\right)a^{\textcircled{\dagger}} = aa^{\textcircled{\dagger}}aa^{\textcircled{\dagger}} = aa^{\textcircled{\dagger}}.$$

Hence, $a \leq^{\textcircled{\dagger}} b$ and so $a_1 \leq^{\textcircled{\#}} b_1$. By (18), we have $a \leq^{\textcircled{c}} b$, i.e., $a \leq^{gc} b$. □

5. The S-core-EP inverse and related relations in $*$-rings

In this section, we generalize the notions of the S-core-EP inverse and induced relations to $*$-rings. We commence with the following result which extends [15, Theorem 4.1] from M_n to the set of all core-EP invertible elements in a $*$-ring $\mathcal{R}$.

Theorem 5.1. *Let $a \in \mathcal{R}^{\textcircled{\dagger}}$ be of the matrix form (13) of its core-EP decomposition with $p = aa^{\textcircled{\dagger}}$ and let $I(a) = k$. Consider the following equations*

$$xa^{k+1} = a^k, \quad aa^{\textcircled{\dagger}}x = a^{\textcircled{\dagger}}, \quad (a^* - x)\left(1 - aa^{\textcircled{\dagger}}\right) = 0. \tag{19}$$

Then x is a solution of (19) if and only if

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p \times p}$$

where t and a_2 are as in (13).

Proof. Suppose first

$$x = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}_{p \times p}$$

is a solution of (19). As in the proof of Theorem 3.1, $xa^{k+1} = a^k$ implies $x_3 = 0$ and $x_1 = t^{-1}$. Also, by $aa^{\textcircled{\dagger}}x = a^{\textcircled{\dagger}}$, we get $x_2 = 0$. Observe that then

$$\begin{aligned} (a^* - x)\left(1 - aa^{\textcircled{\dagger}}\right) &= \left(\begin{bmatrix} t^* & 0 \\ s^* & a_2^* \end{bmatrix}_{p \times p} - \begin{bmatrix} t^{-1} & 0 \\ 0 & x_4 \end{bmatrix}_{p \times p} \right) \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{p \times p} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & (a_2^* - x_4)(1-p) \end{bmatrix}_{p \times p}. \end{aligned}$$

Since $a_2^*, x_4 \in \mathcal{R}(1-p)$, it follows by the last equation of (19) that $a_2^* - x_4 = 0$, i.e., $x_4 = a_2^*$. So,

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p \times p}.$$

Conversely, let

$$x = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p \times p}$$

where t and a_2 are as in (13). As in the proof of Theorem 3.1, we may check that the first two equations of (19) hold. Also,

$$(a^* - x)\left(1 - aa^{\textcircled{d}}\right) = \begin{bmatrix} t^* - t^{-1} & 0 \\ s^* & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} 0 & 0 \\ 0 & 1-p \end{bmatrix}_{p\times p} = 0$$

and thus x is a solution of (19). $\square$

Let us now extend Definition 1.2 from M_n to $\mathcal{R}^{\textcircled{d}}$.

Definition 5.2. Let $a \in \mathcal{R}^{\textcircled{d}}$ and $I(a) = k$. The solution of (19) is called the star-core-EP (or S-core-EP) inverse of a. We denote it by a^{sc}.

Remark 5.3. Clearly, by Theorem 5.1 and (6), we have $a^{sc} = a^{\textcircled{d}} + \left(a - aa^{\textcircled{d}}a\right)^*$.

The following result is a direct consequence of Theorem 5.1 and the uniqueness of the core-EP decomposition of $a \in \mathcal{R}^{\textcircled{d}}$.

Corollary 5.4. *The S-core-EP inverse a^{sc} exists and is unique for every $a \in \mathcal{R}^{\textcircled{d}}$ with $I(a) = k$.*

Corollary 5.5. *Let $a \in \mathcal{R}^{\textcircled{d}}$ be of the matrix form* (13) *of its core-EP decomposition with $p = aa^{\textcircled{d}}$. Then the following statements hold.*

(i) *$a^{sc} = a^{\textcircled{d}}a$ if and only if $p = t$ and $s = 0 = a_2$.*
(ii) *$a^{sc} = aa^{\textcircled{d}}$ if and only if $p = t$ and $a_2 = 0$.*
(iii) *$a^{sc} = a$ if and only if $t = t^{-1}$, $s = 0$, and $a_2 = a_2^*$.*
(iv) *$a^{sc} = a^*$ if and only if $t^* = t^{-1}$ and $s = 0$.*

Proof. To prove (i), observe that

$$a^{\textcircled{d}}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} p & t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p}.$$

It follows $a^{sc} = a^{\textcircled{d}}a$ if and only if $p = t^{-1}$, $t^{-1}s = 0$, and $a_2^* = 0$ which is equivalent to $p = t$ and $s = 0 = a_2$. Statements (ii)–(iv) may be proved similarly. $\square$

Remark 5.6. Let $a \in \mathcal{R}^{\textcircled{d}}$. If $I(a) = 1$, then $a^{\textcircled{d}} = a^{\textcircled{\#}}$ and $a_2 = 0$ where a_2 is the nilpotent part in the core-EP decomposition of a. Thus, in this case, $a^{sc} = a^{\textcircled{\#}} = a^{\textcircled{d}}$.

5.1. The SCE relation. By employing the S-core-EP inverse of $a \in \mathcal{R}^{\textcircled{d}}$ in a manner analogous to definitions of several well-known partial orders and preorders (see relations 1–4 in Section 1), a new relation was introduced on M_n in [15]. We now generalize this relation to $\mathcal{R}^{\textcircled{d}}$.

Definition 5.7. Let $a, b \in \mathcal{R}^{\textcircled{d}}$. We say that a is below b under the star-core-EP (or SCE) relation and write

$$a \leq^{sc} b \quad \text{when} \quad a^{sc}a = a^{sc}b \text{ and } aa^{sc} = ba^{sc}$$

where a^{sc} is the S-core-EP inverse of a.

Theorem 5.8. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$. Then*

$$a \leq^{sc} b \quad \text{if and only if} \quad a = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \text{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \tag{20}$$

where $p = aa^{\textcircled{d}}$, t is invertible in the ring $p\mathcal{R}p$, $n^{I(a)} = 0$, and $n \leq^* b_4$.

Proof. Suppose first $a \leq^{sc} b$ for $a, b \in \mathcal{R}^{\textcircled{d}}$. Let $p = aa^{\textcircled{d}}$ and write $a \in \mathcal{R}^{\textcircled{d}}$ in the matrix form (13) of its core-EP decomposition:

$$a = \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p}.$$

Then, by Theorem 5.1,

$$a^{sc} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p\times p}.$$

Let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p}.$$

We have

$$a^{sc}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} p & t^{-1}s \\ 0 & a_2^*a_2 \end{bmatrix}_{p\times p}$$

and

$$a^{sc}b = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p\times p} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1}b_1 & t^{-1}b_2 \\ a_2^*b_3 & a_2^*b_4 \end{bmatrix}_{p\times p}.$$

Since $a^{sc}a = a^{sc}b$, we obtain $b_1 = t$ and $b_2 = s$. Also, $a_2^*a_2 = a_2^*b_4$. Similarly, from $aa^{sc} = ba^{sc}$, we get $b_3 = 0$ and $a_2a_2^* = b_4a_2^*$. It follows that

$$b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

and $a_2 \leq^* b_4$. If we denote $n = a_2$, we get the desired form (20) of a and b.

Conversely, let a and b be as in (20). Note that $n^{I(a)} = 0$ and observe that a is written in the matrix form of the core-EP decomposition $a = a_1 + a_2$, i.e.,

$$a_1 = \begin{bmatrix} t & s \\ 0 & 0 \end{bmatrix}_{p\times p} \quad \text{and} \quad a_2 = n = \begin{bmatrix} 0 & 0 \\ 0 & n \end{bmatrix}_{p\times p}.$$

So, by Theorem 5.1,

$$a^{sc} = \begin{bmatrix} t^{-1} & 0 \\ 0 & n^* \end{bmatrix}_{p\times p}.$$

By assumption, $n^*n = n^*b_4$ and $nn^* = b_4n^*$. Therefore,

$$a^{sc}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & n^* \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & n^* \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} = a^{sc}b$$

and similarly, $aa^{sc} = ba^{sc}$. Thus, $a \leq^{sc} b$. □

Theorem 5.8 shows that the SCE relation is connected to the star partial order. Let us briefly study this connection.

Lemma 5.9. *Let $a, b \in \mathcal{R}^{@}$ with $a \leq^{sc} b$. Then $a^*a = a^*b$.*

Proof. By assumption and Theorem 5.8, we may write a and b in the matrix form (20) with $n \leq^* b_4$, i.e., $n^*n = n^*b_4$ and $nn^* = b_4n^*$. Then

$$a^*a = \begin{bmatrix} t^* & 0 \\ s^* & n^* \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} = \begin{bmatrix} t^*t & t^*s \\ s^*t & s^*s + n^*n \end{bmatrix}_{p\times p}$$

and

$$a^*b = \begin{bmatrix} t^* & 0 \\ s^* & n^* \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^*t & t^*s \\ s^*t & s^*s + n^*b_4 \end{bmatrix}_{p\times p}$$

and thus $a^*a = a^*b$. □

Proposition 5.10. *Let $a, b \in \mathcal{R}^{@}$ with $a \leq^{sc} b$. Suppose a and b are of the matrix form* (20) *with $n \leq^* b_4$. Then $a \leq^* b$ if and only if $n - b_4 \in {}^{\circ}(s^*)$.*

Proof. By assumption and Lemma 5.9, $a^*a = a^*b$. By (20), we have

$$aa^* = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \begin{bmatrix} t^* & 0 \\ s^* & n^* \end{bmatrix}_{p\times p} = \begin{bmatrix} tt^* + ss^* & sn^* \\ ns^* & nn^* \end{bmatrix}_{p\times p}$$

and

$$ba^* = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \begin{bmatrix} t^* & 0 \\ s^* & n^* \end{bmatrix}_{p\times p} = \begin{bmatrix} tt^* + ss^* & sn^* \\ b_4s^* & b_4n^* \end{bmatrix}_{p\times p}.$$

Since, by assumption, $nn^* = b_4n^*$ and $a^*a = a^*b$, we observe that $a \leq^* b$ if and only if $ns^* = b_4s^*$, i.e., $n - b_4 \in {}^{\circ}(s^*)$. □

Suppose $I(a) = 1$ for $a \in \mathcal{R}^{\textcircled{d}}$. Then, by Remark 5.6, $a^{sc} = a^{\textcircled{\#}} = a^{\textcircled{d}}$ and hence the SCE relation coincides with the core partial order on the set of elements of index one in $\mathcal{R}^{\textcircled{d}}$. Is the SCE relation a partial order on the full set $\mathcal{R}^{\textcircled{d}}$? The answer is no, since it fails to be transitive even in a special case when $\mathcal{R} = \mathcal{R}^{\textcircled{d}} = M_n$ (see [15, Theorem 4.11]). This observation led the authors of [15] to define the *left-star-core-EP* (or *LSC*) *order* on M_n that transforms the SCE relation so that it becomes a partial order. In the continuation, we extend the notion of the LSC order to Rickart $*$-rings.

5.2. The LSC order. It turns out (see Theorems 5.12 and 5.13) that the LSC order is related to the left-star partial order which we introduced in Subsection 2.1 on Rickart $*$-rings. From now on let $\mathcal{R}$ denote a Rickart $*$-ring.

Definition 5.11. Let $a, b \in \mathcal{R}^{\textcircled{d}}$. We say that a is below b under the left-star-core-EP (or LSC) relation and write

$$a \leq^{lsc} b \quad \text{when} \quad a^{sc}a = a^{sc}b,\ aa^{\textcircled{d}} = ba^{\textcircled{d}},\ \text{and } a\mathcal{R} \subseteq b\mathcal{R}.$$

We now give two characterizations of the LSC relation using 2×2 and 3×3 matrix representations of elements in $\mathcal{R}^{\textcircled{d}}$.

Theorem 5.12. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$. Then*

$$a \leq^{lsc} b \quad \text{if and only if} \quad a = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \text{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \tag{21}$$

where $p = aa^{\textcircled{d}}$, t is invertible in the ring $p\mathcal{R}p$, $n^{I(a)} = 0$, and $n {*\leq} b_4$.

Proof. Write $a \in \mathcal{R}^{\textcircled{d}}$ in the matrix form (13) of its core-EP decomposition with $p = aa^{\textcircled{d}}$. Then

$$a^{\textcircled{d}} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \quad \text{and} \quad a^{sc} = \begin{bmatrix} t^{-1} & 0 \\ 0 & a_2^* \end{bmatrix}_{p\times p}.$$

Suppose $a \leq^{lsc} b$ and let

$$b = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}_{p\times p}.$$

Since $a^{sc}a = a^{sc}b$, we have

$$\begin{bmatrix} p & t^{-1}s \\ 0 & a_2^*a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1}b_1 & t^{-1}b_2 \\ a_2^*b_3 & a_2^*b_4 \end{bmatrix}_{p\times p}$$

and thus $b_1 = t$, $b_2 = s$, and $a_2^* a_2 = a_2^* b_4$. By $aa^{@} = ba^{@}$, we obtain

$$\begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} b_1 t^{-1} & 0 \\ b_3 t^{-1} & 0 \end{bmatrix}_{p\times p}$$

and hence $b_3 = 0$. Therefore,

$$b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}.$$

Recall there exist the identity 1 in $\mathcal{R}$. So, from $a\mathcal{R} \subseteq b\mathcal{R}$, there exists $z \in \mathcal{R}$ such that $a = bz$. Let

$$z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{p\times p}$$

to obtain

$$\begin{bmatrix} t & s \\ 0 & a_2 \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{p\times p}$$

and therefore $a_2 = b_4 z_4$. So, $a_2\mathcal{R} \subseteq b_4\mathcal{R}$. This together with $a_2^* a_2 = a_2^* b_4$ implies by Proposition 2.1 that $a_2 {*}\!\leq b_4$. Denote $n = a_2$ to obtain the desired matrix form (21) of a and b with $n {*}\!\leq b_4$.

Conversely, let a and b be as in (21) where $p = aa^{@}$, t is invertible in the ring $p\mathcal{R}p$, $n^{I(a)} = 0$, and $n {*}\!\leq b_4$. Observe first that a is then in the matrix form of its core-EP decomposition with the nilpotent part $n = a_2$. Since by Proposition 2.1, $n^* n = n^* b_4$, we obtain

$$a^{sc} a = \begin{bmatrix} p & t^{-1}s \\ 0 & n^* n \end{bmatrix}_{p\times p} = \begin{bmatrix} p & t^{-1}s \\ 0 & n^* b_4 \end{bmatrix}_{p\times p} = a^{sc} b.$$

Also,

$$aa^{@} = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} = ba^{@}.$$

By $n\mathcal{R} \subseteq b_4\mathcal{R}$, there exists $z \in \mathcal{R}$ such that $n = b_4 z$. Without loss of generality we may assume that $z \in (1-p)\,\mathcal{R}\,(1-p)$. Denote

$$u = \begin{bmatrix} p & t^{-1}(s - sz) \\ 0 & z \end{bmatrix}_{p\times p}.$$

Then,

$$bu = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} \begin{bmatrix} p & t^{-1}(s - sz) \\ 0 & z \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s \\ 0 & b_4 z \end{bmatrix}_{p\times p} = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} = a$$

and therefore $a\mathcal{R} \subseteq b\mathcal{R}$. It follows that $a \leq^{lsc} b$. □

Theorem 5.13. *Let $a, b \in \mathcal{R}^{\textcircled{d}}$ and $p = aa^{\textcircled{d}}$. Then $a \leq^{lsc} b$ if and only if there exists a decomposition of the identity $1 = e_1 + e_2 + e_3$, with $e_1 = p$ and $e_2 = e_2^*$, such that*

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \quad \text{and} \quad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \tag{22}$$

where t is invertible in the ring $e_1\mathcal{R}e_1$, t_1 is invertible in the ring $e_2\mathcal{R}e_2$, $n_5^{I(b)} = 0 = (n_1 + n_2 + n_3 + n_4)^{I(a)}$, and $n_1 + n_2 + n_3 + n_4 {*}{\leq} t_1 + z_1 + n_5$.

Proof. Let $a \leq^{lsc} b$. Then, by Theorem 5.12,

$$a = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \quad \text{and} \quad b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $p = aa^{\textcircled{d}}$, t is invertible in the ring $p\mathcal{R}p$, $n^{I(a)} = 0$, and $n{*}{\leq} b_4$. Since $b \in \mathcal{R}^{\textcircled{d}}$, it follows that $b_4 \in \mathcal{R}^{\textcircled{d}}$ and hence we represent b_4 with the matrix form of its core-EP decomposition:

$$b_4 = \begin{bmatrix} t_1 & z_1 \\ 0 & n_5 \end{bmatrix}_{q\times q}$$

where $q = b_4 b_4^{\textcircled{d}}$, n_5 is the nilpotent part, and t_1 is invertible in the ring $q\mathcal{R}q$. We observe that $pq = qp = 0$. Denote $s_1 = pbq$, $s_2 = pb(1-p-q)$, $e_1 = p$, $e_2 = q$, and $e_3 = 1 - p - q$. Similarly, as in the proof of Theorem 4.3, we show that we may represent b as

$$b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e},$$

where $n_5^{I(b)} = 0$. To show that the form of a is as in (22), denote $n_1 = e_2 a e_2$, $n_2 = e_2 a e_3$, $n_3 = e_3 a e_2$, and $n_4 = e_3 a e_3$. Again, as in the proof of Theorem 4.3, we show that then

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e}.$$

Since $a = t + s + n$ is the core-EP decomposition of a where the nilpotent part is $a_2 = n = n_1 + n_2 + n_3 + n_4$, we have

$$(n_1 + n_2 + n_3 + n_4)^{I(a)} = 0.$$

By assumption, $n{*}{\leq} b_4$ and thus $n_1 + n_2 + n_3 + n_4 {*}{\leq} t_1 + z_1 + n_5$.

Conversely, let a and b be of the form (22). Then $e_1ae_1 = t = e_1be_1$ and

$e_1a(1-e_1) = e_1ae_2+e_1a(1-e_1-e_2) = s_1+s_2 = e_1be_2+e_1b(1-e_1-e_2) = e_1b(1-e_1)$.

Denote $s = e_1a(1-e_1) = e_1b(1-e_1)$ and $n = n_1 + n_2 + n_3 + n_4$. So,

$n = e_2ae_2+e_2a(1-e_1-e_2)+(1-e_1-e_2)ae_2+(1-e_1-e_2)a(1-e_1-e_2) = (1-e_1)a(1-e_1)$.

By assumption, $n^{I(a)} = 0$. Also, denote $b_4 = t_1 + z_1 + n_5$. Thus,

$b_4 = e_2be_2+e_2b(1-e_1-e_2)+(1-e_1-e_2)b(1-e_1-e_2) = (1-e_1)b(1-e_1)-(1-e_1-e_2)be_2$.

If we take into account that by (22), $(1 - e_1 - e_2)be_2 = 0$, we may conclude that $b_4 = (1 - e_1)b(1 - e_1)$. Since $e_1 = p = aa^{@}$, it follows that

$$a = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \text{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where t is invertible in the ring $p\mathcal{R}p$, $n^{I(a)} = 0$, and, by assumption, $n{*}{\leq}b_4$. It follows by Theorem 5.12 that $a \leq^{lsc} b$. □

Remark 5.14. Let the matrix form of $b \in \mathcal{R}^{@}$ be as in (22). Denote $b_1 = t + s_1 + s_2 + t_1 + z_1$ and $b_2 = n_5$. Then

$$b_2b_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & 0 \end{bmatrix}_{e\times e} = 0$$

and

$$b_1^*b_2 = \begin{bmatrix} t^* & 0 & 0 \\ s_1^* & t_1^* & 0 \\ s_2^* & z_1^* & 0 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} = 0.$$

Note that, by assumption, $b_2^{I(b)} = 0$ and that $i(b_1) \leq 1$ by [19, Lemma 2.2]. It follows by (5) that $b = b_1 + b_2$ is the core-EP decomposition of b where $b_2 = n_5$ is the nilpotent part.

We are now in position to prove that $\leq^{lsc}$ is a partial order on $\mathcal{R}^{@}$.

Theorem 5.15. *Let $\mathcal{R}$ be a Rickart ∗-ring. Then the left-star-core-EP relation $\leq^{lsc}$ is a partial order on $\mathcal{R}^{@}$.*

Proof. The LSC relation is clearly reflexive by Definition 5.11. To prove that it is antisymmetric, let for $a, b \in \mathcal{R}^{@}$, $a \leq^{lsc} b$ and $b \leq^{lsc} a$. Since $a \leq^{lsc} b$, we may write by Theorem 5.12,

$$a = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \text{ and } b = \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p}$$

where $p = aa^{@}$ and $n{*}{\leq}b_4$. It follows by Proposition 2.1 that there exist a self-adjoint idempotent $r \in \mathcal{R}$ and an idempotent $q \in \mathcal{R}$ such that $n = rb_4 = b_4q$. Since $b \leq^{lsc} a$, we have $b\mathcal{R} \subseteq a\mathcal{R}$ and so $b = az$ for some $z \in \mathcal{R}$. Let

$$z = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{p\times p}.$$

Then

$$az = \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}_{p\times p} = \begin{bmatrix} tz_1 + sz_3 & tz_2 + sz_4 \\ nz_3 & nz_4 \end{bmatrix}_{p\times p} = b$$

and thus $b_4 = nz_4$. Note that $rn = r(rb_4) = rb_4 = n$. It follows that

$$n = rb_4 = rnz_4 = nz_4 = b_4$$

and therefore $a = b$.

To finish the proof, let us show that $\leq^{lsc}$ is transitive. Let $a, b, c \in \mathcal{R}^{@}$ with $a \leq^{lsc} b$ and $b \leq^{lsc} c$. We will prove that then $a \leq^{lsc} c$. Since $a \leq^{lsc} b$, we may write by Theorem 5.13,

$$a = \begin{bmatrix} t & s_1 & s_2 \\ 0 & n_1 & n_2 \\ 0 & n_3 & n_4 \end{bmatrix}_{e\times e} \quad \text{and} \quad b = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e}$$

where $n_1 + n_2 + n_3 + n_4{*}{\leq}t_1 + z_1 + n_5$. Recall that by Remark 5.14, $b = (t + s_1 + s_2 + t_1 + z_1) + n_5$ is the core-EP decomposition of b where n_5 is the nilpotent part. Since $b \leq^{lsc} c$, it follows by Theorem 5.13 that c may be represented with the following 3×3 matrix form:

$$c = \begin{bmatrix} t & s_1 & s_2 \\ 0 & t_1 & z_1 \\ 0 & 0 & c_5 \end{bmatrix}_{e\times e}$$

for some $c_5 \in (1-e_1-e_2)\mathcal{R}(1-e_1-e_2)$ with $n_5{*}{\leq}c_5$. Taking into account Theorem 5.13 again and the 3×3 matrix representations of a and c, we observe that $a \leq^{lsc} c$, if $n_1 + n_2 + n_3 + n_4{*}{\leq}t_1 + z_1 + c_5$. Since the left-star order is transitive and since $n_1+n_2+n_3+n_4{*}{\leq}t_1+z_1+n_5$, it is enough to show that $t_1+z_1+n_5{*}{\leq}t_1+z_1+c_5$. From $n_5{*}{\leq}c_5$, we have by Proposition 2.1, $n_5^*n_5 = n_5^*c_5$ and $n_5\mathcal{R} \subseteq c_5\mathcal{R}$, i.e., $n_5 = c_5w$

for some $w \in \mathcal{R}(1-e_1-e_2)$. Without loss of generality we may assume that $w \in (1-e_1-e_2)\mathcal{R}(1-e_1-e_2)$. We have

$$\begin{aligned}
(t_1+z_1+n_5)^*(t_1+z_1+n_5) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1^* & 0 \\ 0 & z_1^* & n_5^* \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1^*t_1 & t_1^*z_1 \\ 0 & z_1^*t_1 & z_1^*z_1+n_5^*n_5 \end{bmatrix}_{e\times e} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1^*t_1 & t_1^*z_1 \\ 0 & z_1^*t_1 & z_1^*z_1+n_5^*c_5 \end{bmatrix}_{e\times e} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1^* & 0 \\ 0 & z_1^* & n_5^* \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & c_5 \end{bmatrix}_{e\times e} \\
&= (t_1+z_1+n_5)^*(t_1+z_1+c_5).
\end{aligned}$$

Let

$$y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & e_2 & t_1^{-1}(z_1 - z_1 w) \\ 0 & 0 & w \end{bmatrix}_{e\times e}.$$

Since $c_5 w = n_5$, we get

$$\begin{aligned}
t_1+z_1+n_5 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & n_5 \end{bmatrix}_{e\times e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & t_1 & z_1 \\ 0 & 0 & c_5 \end{bmatrix}_{e\times e} \begin{bmatrix} 0 & 0 & 0 \\ 0 & e_2 & t_1^{-1}(z_1 - z_1 w) \\ 0 & 0 & w \end{bmatrix}_{e\times e} \\
&= (t_1+z_1+c_5)\, y
\end{aligned}$$

and therefore $(t_1+z_1+n_5)\mathcal{R} \subseteq (t_1+z_1+c_5)\mathcal{R}$. Thus, by Proposition 2.1, as desired, $t_1+z_1+n_5 {*\leq} t_1+z_1+c_5$. □

We end the paper by comparing the LSC order to some other relations. For a Rickart $*$-ring $\mathcal{R}$, let $a, b \in \mathcal{R}^{@}$. Suppose $a \leq^{sc} b$. Since the star order $\leq^*$ implies the left-star order $*\leq$ (see, e.g., [21]), Theorems 5.8 and 5.12 imply $a \leq^{lsc} b$. It is also known (again see [21]) that the left-star order $*\leq$ implies the minus order $\leq^-$. If we compare Theorems 4.3 and 5.13, we see that the LSC order $\leq^{lsc}$ implies the G-core-EP order $\leq^{gc}$ (at least) when $\mathcal{R}$ is also regular. Note also that when

$a \leq^{lsc} b$, then $aa^{ⓓ} = ba^{ⓓ}$ and that by Theorem 5.12,

$$a^{ⓓ}a = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & n \end{bmatrix}_{p\times p} = \begin{bmatrix} p & t^{-1}s \\ 0 & 0 \end{bmatrix}_{p\times p} = \begin{bmatrix} t^{-1} & 0 \\ 0 & 0 \end{bmatrix}_{p\times p} \begin{bmatrix} t & s \\ 0 & b_4 \end{bmatrix}_{p\times p} = a^{ⓓ}b.$$

So, the LSC order $\leq^{lsc}$ also implies the core-EP relation $\overset{ⓓ}{\leq}$. We summarize these observations in the following proposition.

Proposition 5.16. *Let $a, b \in \mathcal{R}^{ⓓ}$. Then the following statements hold.*

(i) *If $a \leq^{sc} b$, then $a \leq^{lsc} b$.*
(ii) *If $a \leq^{lsc} b$, then $a \overset{ⓓ}{\leq} b$.*
(iii) *Suppose additionally that $\mathcal{R}$ is regular. If $a \leq^{lsc} b$, then $a \leq^{gc} b$.*

Corollary 5.17. *The SCE relation $\leq^{sc}$ is reflexive and antisymmetric on $\mathcal{R}^{ⓓ}$.*

Proof. The relation $\leq^{sc}$ is clearly reflexive by Definition 5.7. Suppose now that for $a, b \in \mathcal{R}^{ⓓ}$, $a \leq^{sc} b$ and $b \leq^{sc} a$. Then, by Proposition 5.16, $a \leq^{lsc} b$ and $b \leq^{lsc} a$ and hence by Theorem 5.15, $a = b$, i.e., the SCE relation is antisymmetric. □

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