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ESSENTIAL GRAPH OF AN ABELIAN GROUP

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Abstract. The essential graph of an abelian group G, denoted by $\varepsilon(G)$, is an undirected graph with vertex set as the collection of all non-trivial proper subgroups of G and any two distinct vertices A and B are adjacent if and only if AB is an essential subgroup of G. We discuss connectedness, traversability and planarity of $\varepsilon(G)$. The concept of a divisor subgroup with respect to a subgroup of G is introduced, whose effect is observed in some of the characterizations of $\varepsilon(G)$.

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1. Introduction

In this work, we denote G as a finite abelian group. A subgroup of G is called essential if it intersects non-trivially with any other subgroup of G. The symbols S^G and M^G stand for the sets of all simple and maximal subgroups of G, respectively. The *Frattini subgroup* $\Phi(G)$ is the intersection of all maximal subgroups of G [10].

Let K be a simple graph. If two vertices x and y are adjacent, we write $x \sim y$. We denote the number of edges of K by q and degree of a vertex v by $deg(v)$. The greatest and lowest degrees among all the degrees of the vertices are indicated by the symbols $\Delta(K)$ and $\delta(K)$, respectively. A degree-one vertex is a pendant vertex. A universal vertex of K is the vertex which is adjacent to all the remaining vertices of K. Furthermore, a graph with equal vertex degrees, say r, is called r-regular. A clique of K is a complete subgraph of K. We represent the clique number of K by $\omega(K)$, which is determined by the cardinality of the largest clique in K. The set of all vertices that are adjacent to a vertex v is called its neighborhood, and it is represented by the symbol $N(v)$. P_n represents a path of length n. For two distinct vertices, u and v, the shortest $u - v$ path is commonly used to describe a geodesic. The diameter, $diam(K)$, of a connected graph K is the length of any longest geodesic. For a disconnected graph K, $diam(K)$ is written as ∞. The girth, $girth(K)$, of K is the length of the shortest cycle in K, if any. A bipartite graph is a graph whose vertex set can be partitioned into two disjoint sets, where every edge connects a vertex in one set to a vertex of the other set. In a complete bipartite graph, $K_{m,n}$, all of the vertices in the vertex set of cardinality m are adjacent to all of the vertices in the vertex set of cardinality n. A tree is a connected acyclic graph. A cut vertex of K is one whose removal makes the graph disconnected. A graph K is said to be planar if it can be represented on a plane such that any two of its edges either do not meet at all or only meet at their end vertices. An independent set of K is a collection of vertices that are mutually non-adjacent. The independence number, $\alpha(K)$, is the cardinality of the largest independent vertex sets. The lower domination number of K is the number of elements in the smallest dominating set of K, where the dominating set is a subset of the vertex set of K and each vertex of K is either in the subset or has a neighbour in it. The total dominating set is a dominating set in which every member has a neighborhood within the set, whereas the connected domination number is the cardinality of the smallest connected dominating set. The cardinality of the smallest dominating set whose members are mutually non-adjacent is known as the independence domination number. The paper also looks at Hamiltonian, and Eulerian characters.

Over the past twenty years, the study of algebraic structures using graph properties has gained a lot of attention, producing a number of intriguing findings. Numerous papers have been written about matching a graph to a ring (see, for instance, [1,2,3,4,5]). In 1995, Sharma and Bhatwadekar defined a new graph whose vertices are the elements of a commutative ring R [15] with vertices a and b are adjacent if and only if $Ra+Rb = R$. Later, Maimani et al. in [12] termed this graph as the comaximal graph, denoted by $\Gamma(R)$. There has been a development in the idea of the sum essential graphs for rings and modules. After Amjadi introduced and studied it in [6], Barman and Rajkhowa studied the non-essential sum graph in an Artinian ring [7]. Matczuk and Majidinya [13] further developed the notion of sum essential graph into modules. For a better understanding of the essentiality concept in the case of a group, we define the essential graph. The *essential graph* of G, denoted by $\varepsilon(G)$, is a graph with vertex set A, the set of non-trivial proper subgroups of G and any two distinct vertices of the graph are adjacent if the product of the corresponding subgroups is an essential subgroup of G. We see that any two essential subgroups of G are adjacent.

We denote the comaximal graph of a group G by $\Gamma^{CM}(G)$, with the vertex as the all non-trivial proper subgroups of G and any two vertices H and K are adjacent if and only if $HK = G$. It is worth mentioning here that an abelian group G has no proper essential subgroup if and only if $\Gamma^{CM}(G)$ and $\varepsilon(G)$ are equal. Suppose G has no proper essential subgroup. Clearly, the vertex sets in both graphs are the same. Consider any two adjacent vertices H_1 and H_2 of $\varepsilon(G)$. As there is no proper essential subgroup of G, therefore, $H_1H_2 = G$ and thus, $H_1 \sim H_2$ in $\Gamma^{CM}(G)$ also. This implies that $\Gamma^{CM}(G)$ and $\varepsilon(G)$ are equal. To prove the other direction, if possible, assume that G possesses a proper essential subgroup, T, say. Then for any non-trivial proper subgroup K of G, KT is also essential. This implies that $T \sim K$. Thus, T is a universal vertex in $\varepsilon(G)$. If T is not simple, then there exists a proper non-trivial subgroup K of T such that $KT = T$. But then K and T are not adjacent in $\Gamma^{CM}(G)$ which makes T non-universal in $\Gamma^{CM}(G)$. Therefore, $\Gamma^{CM}(G)$ and $\varepsilon(G)$ are not equal. Now assume that T is simple. As G is abelian, T is of prime order, say p. If G has a composite order, then there exists another prime q for which we get another simple subgroup of order q, say S. But then $S \cap T = \{e\}$, a contradiction. Hence, G is of prime power order. This implies that every other non-trivial proper subgroup K of G contains T as a subgroup and $TK = K$. But $T \nsim K$ in $\Gamma^{CM}(G)$ as $TK = K \neq G$. Therefore, $\Gamma^{CM}(G)$ and $\varepsilon(G)$ are not equal. Henceforth, G has no proper essential subgroup. The unobviousness of being a comaximal graph makes the essential graph of G a significant one.

To continue this sequel, we introduce a special kind of subgroup of G with respect to a given subgroup of G. If H is a subgroup of G, then we define the subset $S = \{g : o(g)o(H) \text{ divides } o(G)\}$ of G, where $o(G)$ is the order of G. We claim that it is a subgroup of G. Clearly, the identity e of G is in S. Let $a, b \in S$. We know that $o(H)$ divides $o(G)$. Then we have an integer k such that $ko(H) = o(G)$. Since $o(a)o(H)|o(G)$, we have $o(a)|k$ and similarly, $o(b)|k$. Thus $lcm(o(a), o(b))|k$. We again claim that $o(ab)|lcm(o(a), o(b))$. If $o(a) = m$ and $o(b) = n$, then $(ab)^{lcm(m,n)} = a^{lcm(m,n)}b^{lcm(m,n)} = e$ as G is an abelian group. Thus $o(ab)|lcm(o(a), o(b))$. This implies that $o(ab)o(H)|o(G)$. Thus $ab \in S$. It is also clear that a^{-1} is also in S. Hence, S is a subgroup of G. We call this subgroup as *the divisor subgroup of G with respect to H*, which will be denoted by $d(H)$. It is observed that if H is a proper subgroup of G, then $Hd(H)$ is an essential subgroup of G. If not, then there exists a proper non-trivial subgroup K of G which intersects with $Hd(H)$ trivially. This implies that $K \cap H = \{e\}$ as $H \subset Hd(H)$. As $o(HK) = \frac{o(H)o(K)}{o(H\cap K)} = o(H)o(K)$ and HK is a subgroup of G as G is abelian, $o(H)o(K)$ divides $o(G)$, which implies that elements of K are in $d(H)$, which is a contradiction as $d(H) \cap K = \{e\}$ by the fact that $d(H) \subset Hd(H)$. Hence, $Hd(H)$ is an essential subgroup of G. The divisor subgroups play a very significant role in characterizing the essential graph of G.

Definition 1.1. [11] The socle of a group G, denoted by $Socle(G)$, consists of all $a \in G$ such that $o(a)$ is a square-free integer.

Remark 1.2. [11] $Socle(G)$ is a subgroup of G; it is $\{e\}$ if and only if G is torsion-free, and it is equal to G if and only if G is an elementary group in the sense that every element has a square-free order. It can also be seen that the socle of an abelian group is the subgroup generated by all elements of prime order.

Remark 1.3. [11] The socle is an essential subgroup of a torsion group. For elementary abelian groups, the group is its own socle. A subgroup is essential if and only if it contains the socle.

Remark 1.4. [14] The socle of a group is the product of minimal normal subgroups.

Some examples of essential graphs of some cyclic groups are shown below.

Example 1.5. We have shown the essential graph of $\mathbb{Z}_n$, where $n = p_1^3p_2^3$ and p_1 and p_2 are distinct, which can be seen in Figure 1.

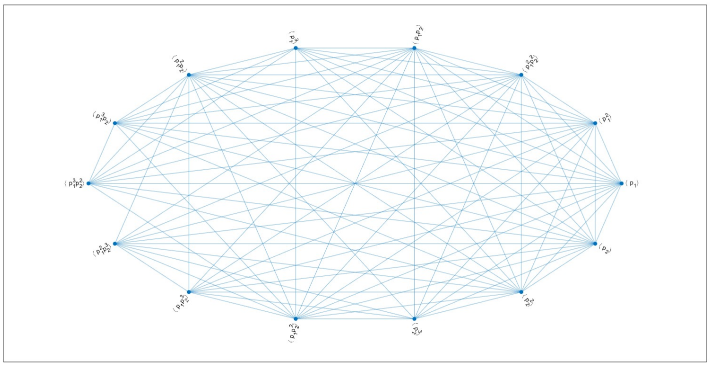

FIGURE 1. $\varepsilon(\mathbb{Z}_{p_1^3p_2^3})$.

It can be observed that the vertices $\langle p_1\rangle, \langle p_2\rangle, \langle p_1^2\rangle, \langle p_2^2\rangle, \langle p_1p_2\rangle, \langle p_1^2p_2\rangle, \langle p_1p_2^2\rangle$ and $\langle p_1^2p_2^2\rangle$ are universal here, as they are essential subgroups of $\mathbb{Z}_{p_1^3p_2^3}$.

One can find any ambiguous phrases in [8,9].

2. Connectedness of essential graph

In this section, we obtain some results related to connectedness, diameter, girth, completeness character. Here we start with a remark.

Remark 2.1. Let us consider a group G of order $\prod_{i=1}^{k} p_i$, where p_i's are distinct primes and k is any arbitrary positive integer. We take a subgroup $\left\langle \prod_{j=1}^{l} p_{i_j} \right\rangle$ of G, where $i_j \in \{1, 2, \ldots, k\}$ and l is an integer such that $1 \leq l < k$. Then we construct the subgroup $\left\langle \prod_{\alpha=1}^{\beta} p_{i_\alpha} \right\rangle$ in G, where $i_\alpha \in \{1, 2, \ldots, k\}$ and β is an integer such that $1 \leq \beta < k$ with $i_j \neq i_\alpha$ for all i_j and for all i_α. Since $\left\langle \prod_{j=1}^{l} p_{i_j} \right\rangle \bigcap \left\langle \prod_{\alpha=1}^{\beta} p_{i_\alpha} \right\rangle = \{e\}$, there is no non-trivial proper essential subgroup in $\varepsilon(G)$. In Theorem 2.2, we state all possible results related to the connectedness of $\varepsilon(G)$ for groups of this order.

Theorem 2.2. *For a group of order $\prod_{i=1}^{k} p_i$, where p_i's are distinct primes,*

(i) *$\varepsilon(G)$ is connected.*
(ii) *$deg\left(\left\langle \prod_{i=1}^{l} p_i \right\rangle\right) = 2^{k-l} - 1$, where $1 \leq l < k$.*
(iii) *$\varepsilon(G)$ is triangle-free if and only if $k \leq 2$.*
(iv) *$girth(\varepsilon(G)) = 3$ or ∞.*
(v) *$\omega(\varepsilon(G)) = k$.*
(vi) $2q = \sum_{l=1}^{k-1} \left(2^{k-l} - 1\right)^k C_l$.
(vii) *the number of pendant vertices is k.*
(viii) $\alpha(\varepsilon(G)) = 2^{k-1} - 2$.
(ix) *lower domination number, connected domination number, and total domination number are equal to k.*
(x) $\Delta(\varepsilon(G)) = 2^{k-1} - 1$.
(xi) *$diam(\varepsilon(G)) \leq 2$.*

Theorem 2.3. *$\varepsilon(G)$ is an empty graph if and only if G has exactly one proper subgroup.*

Proof. One part is trivial. To prove the converse part, let $\varepsilon(G)$ be an empty graph. Therefore, G cannot have more than one maximal subgroup as they will be adjacent otherwise. Consider K as the unique maximal subgroup of G. Let K be not essential. Then there exists a non-trivial proper subgroup H of G such that $H \cap K = \{e\}$. This is not possible as K is the only maximal subgroup of G. Therefore K is essential. If possible, assume that F is a proper subgroup of G that differs from K. Then $F \subsetneq K$ and therefore, $FK = K$ is essential. This implies that $F \sim K$, a contradiction. Hence, G has exactly one proper subgroup. $\square$

Theorem 2.4. *If the subgroups of G form a chain, then $\varepsilon(G)$ is a complete graph.*

Proof. Let $\{e\} < K_1 < K_2 < \cdots < K_n < G$ be the chain of G, where the K_i's are subgroups of G. It is easy to see that K_i is a non-trivial essential subgroup of G for all $i = 1, 2, \ldots, n$. If $1 \leq i < j \leq n$, then $K_i K_j = K_j$, which is essential. Therefore, all the vertices are adjacent to each other. Hence, $\varepsilon(G)$ is complete. □

Theorem 2.5. *Every proper essential subgroup of G is a universal vertex of $\varepsilon(G)$.*

Proof. Let E be a proper essential subgroup of G. For any subgroup K of G, $E \subseteq EK$. As $E \cap H \neq \{e\}$ for any non-trivial subgroup H of G, $H \cap EK \neq \{e\}$ also. Therefore, EK is also essential. Thus, $E \sim K$. Hence, E is a universal vertex of $\varepsilon(G)$. □

Lemma 2.6. *If there exists at least one non-trivial essential subgroup of G, then $\varepsilon(G)$ is connected.*

Remark 2.7. The converses of Theorem 2.4, Theorem 2.5 and Lemma 2.6 are not true. If we take the graph $\varepsilon(\mathbb{Z}_6)$, then it is a star graph but the subgroups of $\mathbb{Z}_6$ do not form a chain. Again, $\varepsilon(\mathbb{Z}_6)$ is connected and both vertices are universal vertices, but $\mathbb{Z}_6$ has no proper essential subgroup.

Lemma 2.8. *If G has no proper essential subgroup, then H is an isolated vertex in $\varepsilon(G)$ if and only if $H \subseteq \Phi(G)$.*

Proof. Let H be an isolated vertex. Let $H \nsubseteq \Phi(G)$. Then there exists a maximal subgroup of G, say M, that does not contain H. Therefore, $MH = G$. This implies that $H \sim M$ in $\varepsilon(G)$, a contradiction. Hence, $H \subseteq \Phi(G)$.

To prove the converse part, assume that $H \subseteq \Phi(G)$. Then for any other vertex K of G, HK is a subset of any maximal subgroup of G that contains K. As there is no proper essential subgroup, $H \nsim K$. Hence, H is an isolated vertex of $\varepsilon(G)$. □

Theorem 2.9. *$\varepsilon(G)$ is disconnected if and only if G has no proper essential subgroup and $\Phi(G) \neq \{e\}$.*

Proof. Let $\varepsilon(G)$ be disconnected. If there is any proper essential subgroup, then by Lemma 2.6, $\varepsilon(G)$ is connected. Therefore, G has no proper essential subgroup. Assume that $\Phi(G) = \{e\}$. Let H be a non-trivial proper subgroup of G. As $H \nsubseteq \Phi(G)$, there exists at least one maximal subgroup M' of G such that $M'H = G$. Thus, $M' \sim H$. This implies that every non-trivial proper subgroup of G is either maximal or adjacent to at least one maximal subgroup. As the maximal subgroups are adjacent to one another, any two distinct vertices are connected by a path containing some maximal subgroups. Therefore, $\varepsilon(G)$ is connected, a contradiction. Hence, $\Phi(G) \neq \{e\}$.

The converse is clear. $\square$

Lemma 2.10. *If $A(\neq \{e\}) \leq B \leq G$ and A is an essential subgroup of B, then B is an essential subgroup of G.*

Proof. Consider a non-trivial proper subgroup K of G. Then $(K \cap B) \cap A \neq \{e\}$ as A is essential in B. This implies that $K \cap (B \cap A) \neq \{e\}$. As $K \cap (B \cap A) \subseteq K \cap B$, $K \cap B \neq \{e\}$. Therefore, B is an essential subgroup of G. $\square$

Lemma 2.11. *If G has no proper essential subgroup, then G is a torsion group. Further $Socle(G) = G$.*

Proof. Consider that $x(\neq e) \in G$ has infinite order. Then $\langle x \rangle \cong \mathbb{Z}$. In $\mathbb{Z}$, every non-zero subgroup $k\mathbb{Z}$ satisfies $k\mathbb{Z} \cap m\mathbb{Z} = \mathrm{lcm}(k, m)\mathbb{Z} \neq 0$, so $k\mathbb{Z}$ is essential in $\mathbb{Z}$. If $\langle x \rangle = G$, then G has proper essential subgroups. Also, if $\langle x \rangle \neq G$, by Lemma 2.10, $\langle x \rangle$ is essential in G, contradicting that G has no proper essential subgroup. Hence every element of G has finite order. Hence, G is a torsion group.

To prove the second part, as G is a torsion group, by Remark 1.3, $Socle(G)$ is an essential subgroup of G. But G does not have any proper essential subgroup by hypothesis. Hence, $Socle(G) = G$. $\square$

Lemma 2.12. *If G has no proper essential subgroup, then $\Phi(G) = \{e\}$.*

Proof. Let x be any non-identity element of G. As G has no proper essential subgroup, by Lemma 2.11, $G = Socle(G)$. Therefore, every non-trivial element of G has prime order and consequently, $G = \bigoplus_p G_p$ for some primes p. Thus, there exists a prime p such that $o(x) = p$. As G is an abelian group, there exists a unique Sylow p-subgroup, say G_p and it is an elementary abelian p-group. Therefore, there exists a maximal subgroup M_p of G_p such that $x \notin M_p$. As $G = G_p \oplus \bigoplus_{q \neq p} G_q$, the subgroup $M = M_p \oplus \bigoplus_{q \neq p} G_q$ is a maximal subgroup of G and $x \notin M$. Since x is arbitrary, every non-trivial element of G is excluded from some maximal subgroup. Hence, $\Phi(G) = \{e\}$. $\square$

Lemma 2.13. *If G has only finitely many maximal subgroups and has no proper essential subgroup, then G is finite. Hence G has only finitely many subgroups.*

Proof. By Lemma 2.11 and Remark 1.2, G is an elementary abelian group. Therefore G can be written as $\prod_{i \in \Lambda} G_{p_i}$, where Λ is an index set and G_{p_i} is a p_i-group for each prime p_i. This G_{p_i} is a finite group of order p_i for each i. As G has a finite number of maximal subgroups, Λ is a finite set. Thus G is a finite product of finite groups, so G is finite. Further, as every finite group has only finitely many subsets, hence only finitely many subgroups. □

Theorem 2.14. *$\varepsilon(G)$ is finite if and only if every vertex of $\varepsilon(G)$ is of finite degree.*

Proof. If $\varepsilon(G)$ is finite, then the number of vertices of $\varepsilon(G)$ is finite and thus, every vertex of $\varepsilon(G)$ is of finite degree.

To prove the converse part, consider that every vertex of $\varepsilon(G)$ is of finite degree. We first assume that G has a proper essential subgroup, say E. Then E is adjacent to all other vertices of $\varepsilon(G)$. This implies that $\varepsilon(G)$ has a finite number of vertices and therefore, $\varepsilon(G)$ is a finite graph. Now we consider that G has no proper essential subgroup. By Lemma 2.12, $\Phi(G) = \{e\}$. Since every vertex of $\varepsilon(G)$ has finite degree and the maximal subgroups of G are adjacent to one another in $\varepsilon(G)$, therefore, the number of maximal subgroups of G is finite. Therefore, by Lemma 2.13, G has only finitely many subgroups. Hence, $\varepsilon(G)$ is a finite graph. □

Theorem 2.15. *$diam(\varepsilon(G))$ is $1, 2, 3$ or ∞.*

Proof. If every subgroup of G is maximal, then $diam(\varepsilon(G))$ is 1 as $M_1M_2 = G$ for any two maximal subgroups M_1 and M_2 of G. Now let $\Phi(G) \neq \{e\}$. If $\varepsilon(G)$ is complete, then $diam(\varepsilon(G))$ is 1. If $\varepsilon(G)$ is non-complete connected and has a proper essential subgroup, then $diam(\varepsilon(G))$ is 2. Let $\varepsilon(G)$ be a non-complete connected graph but does not have a proper essential subgroup. Then, as $\Phi(G) \neq \{e\}$, there exists a proper subgroup H of G such that $H \subseteq \Phi(G)$, which is an isolated vertex by Lemma 2.8. In this case, $diam(\varepsilon(G))$ is ∞. Assume that $\Phi(G) = \{e\}$. Let H be a non-trivial proper subgroup of G. As $H \nsubseteq \Phi(G)$, there exists at least one maximal subgroup M of G such that $MH = G$. Thus, M and H are adjacent in $\varepsilon(G)$. This implies that every non-trivial proper subgroup of G is either maximal or adjacent to at least one maximal subgroup. Let H_1 and H_2 be two non-adjacent vertices of $\varepsilon(G)$. Then both are not maximal subgroups of G. Let one of them be a maximal subgroup of G, say H_1. As $H_2 \nsim H_1$, there exists a maximal subgroup M' of G other than H_1 such that $H_2 \sim M'$. This implies that there exists a path $H_1 - M' - H_2$ in $\varepsilon(G)$. Now assume that both of H_1 and H_2 are non-maximal proper subgroups of G. Then there exist two maximal subgroups M_1 and M_2 of G such that $H_1 \sim M_1$ and $H_2 \sim M_2$. If $M_1 = M_2$, then $H_1 - M_1 - H_2$ is a path in $\varepsilon(G)$. If $M_1 \neq M_2$, then $H_1 - M_2 - M_1 - H_2$ is a path in $\varepsilon(G)$. Therefore, the diameter of $\varepsilon(G)$ is less than or equal to 3 in this case.

Hence, $diam(\varepsilon(G))$ is $1, 2, 3$ or ∞. □

Remark 2.16. The set $A = \left\{\left\langle \prod_{j=1}^{k} p_j^{i_j} \right\rangle \middle| 0 \leq i_j \leq \alpha_j - 1, \forall j = 1, 2, \ldots, k\right\}$ contains all the essential subgroups of a group of order $\prod_{j=1}^{k} p_j^{\alpha_j}$, where p_js are distinct primes $\forall j$. Also, all the proper subgroups of G in the set $B = \left\{\left\langle p_l^{\alpha_l} \prod_{i=1, i\neq l}^{k} p_i^{r_i} \right\rangle : 1 \leq l \leq k, 0 \leq r_i \leq \alpha_i\right\}$ are non-essential. We now prove Theorem 2.17.

Theorem 2.17. *Let G be a cyclic group. Then*

(i) $deg\left(\left\langle \prod_{j=1}^{k} p_j^{i_j} \right\rangle\right) = \left(\prod_{j=1}^{k} \alpha_j + 1\right) - 3$, *where* $0 \leq i_j \leq \alpha_j - 1, \forall j = 1, 2, \ldots, k$.

(ii) $deg\left(\left\langle \prod_{t=1}^{j} p_t^{\alpha_t} \prod_{t=j+1}^{k} p_t^{i_t} \right\rangle\right) = \left(\prod_{t=1}^{j} \alpha_t\right) \times \left(\prod_{t=j+1}^{k} \alpha_t + 1\right) - 1$, *where* $0 \leq i_j \leq \alpha_j - 1, \forall j = 1, 2, \ldots, t$.

(iii) $\omega(\varepsilon(G)) = \left(\prod_{i=1}^{k} \alpha_j\right) + k - 1$.

Proof. (i) The proof follows from Remark 2.16 and Theorem 2.5.

(ii) By Remark 2.16, the vertex $\left\langle \prod_{t=1}^{j} p_t^{\alpha_t} \prod_{t=j+1}^{k} p_t^{i_t} \right\rangle$ is adjacent to all the vertices of the type $\langle x \rangle$ such that x divides $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$, but x is not divisible by $p_1^{\alpha_1}, p_2^{\alpha_2}, \ldots, p_j^{\alpha_j}$. On counting, we get $\left[\prod_{t=1}^{j} \alpha_t \times \prod_{t=j+1}^{k} (\alpha_t + 1)\right] - 1$ such vertices.

(iii) The vertices in the set A defined in Remark 2.16 form a clique. Let us consider any two vertices $\langle x \rangle$ and $\langle y \rangle$ of B. If x divides y, then $\langle x \rangle \langle y \rangle = \langle x \rangle$, which is clearly not essential. Therefore, $\langle x \rangle \nsim \langle y \rangle$. Considering the fact in the proof of part (ii), we get that only a k number of subgroups of type $\left\langle p_m^{\alpha_m} \prod_{q=1, q\neq m}^{k} p_q^{i_q} \right\rangle, 0 \leq i_q < \alpha_q, 1 \leq m \leq k$ are in the largest clique along with the vertices in A. Hence, $\omega(\varepsilon(G)) = \left(\prod_{j=1}^{k} \alpha_j\right) + k - 1$. □

Lemma 2.18. *If I and J are two vertices of $\varepsilon(G)$ such that $I \subseteq J$, then it is easy to see that $deg(I) \leq deg(J)$.*

Theorem 2.19. *If G contains a non-trivial essential subgroup, then the following holds:*

(i) $\Delta(\varepsilon(G)) = deg(E)$, *where E is any essential subgroup of G.*

(ii) $\delta(\varepsilon(G)) = |S|$, *where* $|S| = min_{S_i \in S^G}\{deg(S_i)\}$.

Proof. (i) It is clear.
(ii) By Lemma 2.18, we have that $deg(S_i), S_i \in S^G$ is minimum among the degree of the vertices containing S_i. Now, every other non-simple subgroup of G contains at least one of the S_i's. Hence, $\delta(\varepsilon(G)) = |S|$. □

Theorem 2.20. *Assume that G has a proper essential subgroup. If $\varepsilon(G)$ has a cut vertex, then the number of proper essential subgroups of G is one.*

Proof. We assume that $\varepsilon(G)$ has a cut vertex I. Then there are vertices K and J, with I lying in every path from K to J. Without loss of generality, we can assume that $J - I - K$ is a path from J to K. Consider that G has more than one proper essential subgroup. Let I be not essential. If at least one of J and K is essential, say J, then J and K are adjacent and $\varepsilon(G) - I$ is connected, a contradiction. Consider that neither J nor K is essential. This implies that J and K are adjacent to all essential subgroups. Let E be an essential subgroup of G. Therefore, $J - E - K$ is a path, which implies that I is not a cut vertex. Therefore, I is essential. As J and K cannot be essential, we have another proper essential subgroup E' such that $J - E' - K$ is a path, which again leads to a contradiction. Therefore, G can have only one proper essential subgroup. □

Theorem 2.21. *If $\varepsilon(G)$ is an r-regular graph, then $|V(\varepsilon(G))| = r + 1$ or 2.*

Proof. If G has a proper essential subgroup I, say, then $deg(I) = |V(\varepsilon(G))| - 1$. Hence, $|V(\varepsilon(G))| = r + 1$. Now consider that G has no proper essential subgroup. As r is finite, by Theorem 2.14, $\varepsilon(G)$ is finite and has finite number of maximal subgroups. Hence, by Lemma 2.13, G is finite. Again, by Lemma 2.11, $G = Socle(G)$. By Remark 1.4, G is the product of all the minimal normal subgroups, which are also simple subgroups of G. As G is finite, $G = \prod_{i=1}^{n} S_i$, where $S_i \in S^G$ and $|S^G| = n$. If $n \geq 3$, then S_1 is adjacent to $\prod_{i=2}^{n} S_i$ only. This makes $deg(S_1) < deg(\prod_{i=2}^{n} S_i)$ as $\prod_{i=2}^{n} S_i$ is adjacent to S_1, S_1S_2 and $S_1S_2S_3$. Also, $\varepsilon(G)$ is a null graph for $n = 1$. Hence $n = 2$ and thus $\varepsilon(G) \cong K_2$. □

Theorem 2.22. *$\varepsilon(G)$ is a complete graph if and only if every non-trivial, non-essential subgroup of G is simple.*

Proof. Assume that every non-trivial, non-essential subgroup is a simple subgroup. Let I and J be two distinct non-trivial proper subgroups of G. If either I or J is essential, then clearly I and J are adjacent in $\varepsilon(G)$. Consider that neither I nor

J is essential. Then, I and J are simple. It follows from $I \subsetneq IJ$ that IJ is not simple. Hence, $\varepsilon(G)$ is a complete graph.

For the converse part, we take a non-trivial non-essential subgroup I of G which is not simple. Let J be a non-trivial proper subgroup of I. Since $\varepsilon(G)$ is a complete graph, we deduce that $IJ = I$ is an essential subgroup, which is a contradiction. Thus, every non-trivial, non-essential subgroup of G is simple. $\square$

Theorem 2.23. *Let G be a group. Let $N_i \trianglelefteq G$ for any $i \in \{1, 2, \ldots, r\}$. If $\left(\bigcap\limits_{i=1}^{r-1} N_i\right) N_r = G$ for any integer r, then $\frac{G}{\bigcap\limits_{i=1}^{r} N_i} \cong \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_r}$.*

Proof. We consider $r = 2$. Let us define a map $\phi : G \to \frac{G}{N_1} \times \frac{G}{N_2}$ by $\phi(g) = (gN_1, gN_2)$. Let $g_1, g_2 \in G$. Therefore $\phi(g_1g_2) = (g_1g_2N_1, g_1g_2N_2) = ((g_1N_1)(g_2N_1), (g_1N_2)(g_2N_2)) = (g_1N_1, g_1N_2)(g_2N_1, g_2N_2) = \phi(g_1)\phi(g_2)$. Therefore, ϕ is a homomorphism.

Now $Ker\phi = \{g \in G | \phi(g) = (N_1, N_2)\} = \{g \in G | (gN_1, gN_2) = (N_1, N_2)\} = \{g \in G | g \in N_1 \text{ and } g \in N_2\} = \{g \in G | g \in N_1 \cap N_2\} = N_1 \cap N_2$.

We now prove that ϕ is onto. Let $(aN_1, bN_2) \in \frac{G}{N_1} \times \frac{G}{N_2}$. As $N_1N_2 = G$, $a^{-1}b \in G = N_1N_2$. Therefore, there exist $n_1 \in N_1$ and $n_2 \in N_2$ such that $a^{-1}b = n_1n_2$. This implies that $b = an_1n_2$. Now we take $g = an_1$. Then $gN_1 = an_1N_1 = aN_1$ and $gN_2 = an_1N_2 = an_1n_2N_2 = bN_2$ as $n_2 \in N_2$. Thus $\phi(g) = (gN_1, gN_2) = (aN_1, bN_2)$. Hence ϕ is an epimorphism.

By the fundamental theorem of a group homomorphism, $\frac{G}{Ker\phi} \cong \frac{G}{N_1} \times \frac{G}{N_2}$. Hence, $\frac{G}{N_1 \cap N_2} \cong \frac{G}{N_1} \times \frac{G}{N_2}$.

Therefore, the statement is true for $r = 2$.

Let the statement be true for all $i \in \{1, 2, \ldots, r-1\}$. Therefore, $\frac{G}{\bigcap\limits_{i=1}^{r-1} N_i} \cong \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_{r-1}}$. Let $M = \bigcap\limits_{i=1}^{r-1} N_i$. By the hypothesis, $MN_r = G$. As the statement of the theorem is true for $r = 2$ and M is also a normal subgroup of G, we obtain $\frac{G}{M \cap N_r} \cong \frac{G}{M} \times \frac{G}{N_r}$. This implies that $\frac{G}{\left(\bigcap\limits_{i=1}^{r-1} N_i\right) \cap N_r} \cong \frac{G}{\bigcap\limits_{i=1}^{r-1} N_i} \times \frac{G}{N_r}$. Hence, $\frac{G}{\bigcap\limits_{i=1}^{r} N_i} \cong \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_r}$. $\square$

An immediate corollary of Theorem 2.23 is given below.

Corollary 2.24. *Let G be a group. Let $N_i \trianglelefteq G$ for any $i \in \{1, 2, \ldots, r\}$. If $\left(\bigcap\limits_{i=1}^{r-1} N_i\right) N_r = G$ for any integer r and $\bigcap\limits_{i=1}^{r} N_i = \{e\}$, then $G \cong \frac{G}{N_1} \times \frac{G}{N_2} \times \cdots \times \frac{G}{N_r}$.*

Theorem 2.25. *Let G be a cyclic group. If $\varepsilon(G)$ has a unique universal vertex, then G has exactly one proper essential subgroup. Moreover, this essential subgroup is a maximal subgroup of G.*

Proof. Let U be the unique universal vertex of $\varepsilon(G)$. Therefore, $o(G)$ is not prime. Consider a maximal subgroup M that contains U. If $U \subsetneq M$, then $UM = M$ is a universal vertex in $\varepsilon(G)$, a contradiction. Therefore, $U = M$. We claim that U is essential. If U is not simple, then there is a proper subgroup V of U. As $U \sim V$, $UV = U$ is an essential subgroup. We now assume that U is a simple subgroup of G. Since G is abelian, U is of prime order, say p. This implies that $d(U)$ has an element whose order is $\frac{o(G)}{p}$. This implies that $d(U)$ has order at least $\frac{o(G)}{p}$. That is, $o(d(U))$ is either $o(G)$ or $\frac{o(G)}{p}$. If $o(d(U)) = o(G)$, then $d(U) = G$. Let K be a subgroup of G of order $\frac{o(G)}{p}$. Clearly, K is maximal. As U is maximal, which is also simple, having order p, therefore, $K \cap U = \{e\}$. By Corollary 2.24, $G \cong \frac{G}{U} \times \frac{G}{K} \cong S_1S_2$, where $\frac{G}{U} = S_1$ and $\frac{G}{K} = S_2$ are simple subgroups of G as U and K are maximal. It gives $\varepsilon(G) = K_2$, a contradiction to the fact that $\varepsilon(G)$ has a unique universal vertex. Let $o(d(U)) = \frac{o(G)}{p}$. Therefore, $d(U)$ is a maximal subgroup of G. If $d(U) \neq U$, then $Ud(U)$ is essential as U is a universal vertex. Since U is maximal, $Ud(U)$ is either U or G. Let $Ud(U) = G$. As $d(U)$ is maximal having order $\frac{o(G)}{p}$ and U is simple as well as maximal having order p, $U \cap d(U) = \{e\}$. This asserts that $G \cong \frac{G}{U} \times \frac{G}{d(U)}$ by Corollary 2.24. As $\frac{G}{U} \times \frac{G}{d(U)} \cong S_1S_2$ for two simple subgroups S_1 and S_2, $\varepsilon(G) = K_2$, a contradiction to the fact that $\varepsilon(G)$ has a unique universal vertex. Therefore, $U = Ud(U)$. But as $Ud(U)$ is essential, U is essential.

As every proper essential subgroup is universal in $\varepsilon(G)$, G can not have a proper essential subgroup other than U, which is also maximal. □

Lemma 2.26. *Let G be a cyclic group. If G has a non-trivial maximal subgroup M which is not essential, then $G \cong \frac{G}{M} \times \frac{G}{d(M)}$.*

Proof. Since the maximal subgroup M is not essential and $Md(M)$ is always essential, $Md(M) = G$. Now, $d(M) \nsubseteq M$ as $Md(M) \neq M$. If $M \subseteq d(M)$, then $d(M) = G$ as M is maximal. Then for every element g of G, $o(g)o(M)$ divides $o(G)$. This implies that $o(g)$ is a factor of $\frac{o(G)}{o(M)}$. If $M \neq \{e\}$, we can not get any g in G such that $o(g) = o(G)$. This leads to a contradiction as G is cyclic. Therefore, $M = \{e\}$, which is again a contradiction. Hence, $M \cap d(M) = \{e\}$ as $Md(M) = G$. Thus, we have $G \cong \frac{G}{M} \times \frac{G}{d(M)}$ as desired. □

Theorem 2.27. *Let $M^G = \{M_1, M_2\}$. If $\varepsilon(G)$ is triangle-free, then M_1 and M_2 are non-essential subgroups of G and $\Phi(G) = \{e\}$. Moreover, if G is cyclic, then $G \cong S_1S_2$, where $S_1, S_2 \in S^G$ and $\varepsilon(G) = K_2$.*

Proof. Let $\varepsilon(G)$ be triangle-free. If possible, let M_i be essential for some i, say $i = 1$. Then $\Phi(G) \neq \{e\}$. Let M_2 also be an essential subgroup. It implies that the set $\{M_1, M_2, \Phi(G)\}$ forms a triangle, which is a contradiction. Henceforth, M_2 is not essential. Therefore, there exists a subgroup H of G such that $H \cap M_2 = \{e\}$ and $H \subset M_1$. In this case, $HM_2 = G$ and thus, $M_1 - H - M_2 - M_1$ is a triangle in $\varepsilon(G)$. This gives that M_1 is not essential. Similarly, M_2 is also not essential. Now if $\Phi(G) \neq \{e\}$, then by Theorem 2.9, $\varepsilon(G)$ is disconnected. This contradiction concludes that $\Phi(G) = \{e\}$.

Now as both of M_1 and M_2 are not essential and G is cyclic, therefore by Lemma 2.26, $G \cong \frac{G}{M_1} \times \frac{G}{d(M_1)} \cong \frac{G}{M_2} \times \frac{G}{d(M_2)}$, where $M_1 \cap d(M_1) = \{e\}$ and thus, $d(M_1) \subseteq M_2$. Let $d(M_1) \subsetneqq M_2$. As G is cyclic and M_1 is a maximal subgroup, $o\left(\frac{G}{M_1}\right)$ is a prime number, say p. We claim that $\frac{M_2}{d(M_1)}$ is the unique maximal subgroup of $\frac{G}{d(M_1)}$. Since M_2 is a maximal subgroup of G, therefore, $\frac{M_2}{d(M_1)}$ is a maximal subgroup of $\frac{G}{d(M_1)}$. Consider that $\frac{G}{d(M_1)}$ has another maximal subgroup, say $\frac{M'}{d(M_1)}$. Then M' is a maximal subgroup of G. As $M' \neq M_2$, $M' = M_1$. This implies that $\frac{M_1}{d(M_1)}$ is a maximal subgroup of $\frac{G}{d(M_1)}$, which is not possible as $M_1 \cap d(M_1) = \{e\}$. Therefore, $\frac{M_2}{d(M_1)}$ is the unique maximal subgroup of $\frac{G}{d(M_1)}$. This implies that $o\left(\frac{G}{d(M_1)}\right)$ is a prime power, say q^m. Then $G \cong \frac{G}{M_1} \times \frac{G}{d(M_1)} \cong \mathbb{Z}_p \times \mathbb{Z}_{q^m} \cong \mathbb{Z}_{pq^m}$. As $|M^G| = 2$, $p \neq q$. Here, the proper essential subgroups are $\mathbb{Z}_{pq^{m-1}}, \mathbb{Z}_{pq^{m-2}}, \dots, \mathbb{Z}_{pq}$. If $m \geq 2$, then the subgroups $\mathbb{Z}_{pq}, \mathbb{Z}_{pq^2}$ and $\mathbb{Z}_p$ make a triangle in $\varepsilon(G)$. It is noted that $m \neq 1$ as $d(M_1) \subsetneqq M_2$.

Next consider the case when $d(M_1) = M_2$. Then $G \cong \frac{G}{M_1} \times \frac{G}{d(M_1)} \cong \frac{G}{M_1} \times \frac{G}{M_2} = S_1S_2$, where $S^G = \{S_1, S_2\}$ and hence $\varepsilon(G) = K_2$. □

We end this section with a remark, where we observe some immediate results.

Remark 2.28. If I and J are two distinct vertices of $\varepsilon(G)$ such that $J \subset I$ and I is not essential, then $J \notin N(I)$. Also, if G has a non-trivial essential subgroup, then the lower domination number, connected domination number, and independence domination number of $\varepsilon(G)$ are 1, and the total domination number is 2.

3. Traversability and planarity of $\varepsilon(G)$

In this section, we first observe the Eulerian, and Hamiltonian characteristics of $\varepsilon(G)$, where G is a cyclic group, by emphasising the order of G as $\prod_{i=1}^{k} p_i^{\alpha_i}$, where

p_i's are distinct primes. We conclude the section with a discussion of the planarity of $\varepsilon(G)$.

Theorem 3.1. *$\varepsilon(G)$ is Eulerian if and only if $k = 1$ and $\alpha_1 \geq 4$ is even.*

Proof. If $k = 1$ and $\alpha_1 \geq 4$ is even, then by Theorem 2.4, $\varepsilon(G)$ is the complete graph $K_{\alpha_1 - 1}$, which is Eulerian in nature as $\alpha_1 - 1 \geq 3$ is odd.

To prove the converse part, we proceed as follows. Let $k > 1$. If $o(G) = \prod_{i=1}^{k} p_i$, then by Theorem 2.2, $\langle p_1 p_2 \dots p_{k-1} \rangle$ is a pendant vertex for $k \geq 2$. Now, let $o(G) = \prod_{i=1}^{k} p_i^{\alpha_i}$, $k > 1$ and $\alpha_i \geq 2$ for at least one i. Then, by Theorem 2.17, the degree of any vertex of the form $\left\langle p_1^{\alpha_1} p_2^{\alpha_2} \dots p_i^{\alpha_i} p_{i+1}^{t_{i+1}} \dots p_k^{t_k} \right\rangle$, where $0 \leq t_i \leq \alpha_i - 1, \forall i = 1, 2, \dots, k$ is $\left(\prod_{j=1}^{i} \alpha_j \right) \times \left(\prod_{j=i+1}^{k} \alpha_j + 1 \right) - 1$. This vertex degree is odd if and only if at least one α_j is even for $j = 1, \dots, i$ or at least one α_j is odd for $j = i + 1, \dots, k$. Thus $\varepsilon(G)$ is not Eulerian if $k \geq 2$.

Next consider that $k = 1$. If $\alpha_1 < 4$ or α_1 is odd, then $\varepsilon(G)$ is not Eulerian as $\varepsilon(G)$ is the complete graph $K_{\alpha_1 - 1}$. □

Theorem 3.2. *The following results on Hamiltonian character of $\varepsilon(G)$ hold:*

(i) *$\varepsilon(G)$ is not Hamiltonian if one of α_i is at least three and others are 1.*

(ii) *$\varepsilon(G)$ is Hamiltonian if $k = 2$, and $\alpha_1 = 2, \alpha_2 = 1$ or $\alpha_1 = 2, \alpha_2 = 2$ or $\alpha_1 \geq 3, \alpha_2 \geq 2$.*

(iii) *If $\alpha_i = 1$ for all i, then $\varepsilon(G)$ is not Hamiltonian.*

(iv) *For $k = 3,\ 4$, $\varepsilon(G)$ is Hamiltonian if $\alpha_i \geq 2$ for all i.*

(v) *For $k \geq 5$, $\varepsilon(G)$ is Hamiltonian if $\alpha_i \geq 2$ for at least $k - 1$ number of i.*

Proof. (i) Assume that α_i is at least three for $i = 1$. Here the vertices $\langle p_1 p_2 \dots p_k \rangle, \langle p_1^2 p_2 \dots p_k \rangle, \dots, \langle p_1^{\alpha_1 - 1} p_2 \dots p_k \rangle$ are adjacent to $\langle p_1 \rangle, \langle p_1^2 \rangle, \dots, \langle p_1^{\alpha_1 - 1} \rangle$ only. Thus, they cannot be a part of any Hamiltonian cycle.

(ii) Let $k = 2$. By Remark 2.16, there are $\alpha_1 \alpha_2 - 1$ number of universal vertices. The number of non-universal vertices is now $\alpha_1 + \alpha_2$. If $\alpha_1 = 2, \alpha_2 = 1$, then $\langle p_1 \rangle - \langle p_2 \rangle - \langle p_1 p_2 \rangle - \langle p_1^2 \rangle - \langle p_1 \rangle$ is the Hamiltonian cycle. If $\alpha_1 = 2, \alpha_2 = 2$, then $\langle p_1 \rangle - \langle p_2 \rangle - \langle p_1 p_2 \rangle - \langle p_1^2 \rangle - \langle p_1 p_2^2 \rangle - \langle p_1^2 p_2 \rangle - \langle p_2^2 \rangle - \langle p_1 \rangle$ is a Hamiltonian cycle. For $\alpha_1 \geq 3$ and $\alpha_2 \geq 2$, the non-universal vertices can be put in between the universal vertices to form the Hamiltonian cycle.

(iii) Here, G does not have any proper essential subgroup. By Theorem 2.2, any vertex $\langle p_1 p_2 \dots p_j \rangle$ has degree $2^{k-j} - 1$. Therefore, the vertex $\langle p_1 p_2 \dots p_{k-1} \rangle$ has degree 1, which is less than $\frac{|V(G)|}{2}$ for $k \geq 3$. Therefore, by Dirac's theorem, $\varepsilon(G)$

is not Hamiltonian. Further, it is obviously not Hamiltonian for $k = 1$ and 2.
(iv) For $k = 3$, 4, the number of universal vertices is at least the number of non-universal vertices of $\varepsilon(G)$. Therefore, we can embed a Hamiltonian cycle on it.
(v) For $k \geq 5$, proceeding in the same way as in (iv), we get the result. □

Now we characterize the planarity of the essential graphs.

Remark 3.3. If the number of essential subgroups is at least 5, then $\varepsilon(G)$ is not planar.

Theorem 3.4. (i) *$\varepsilon(G)$ is not planar for the group G of order $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_k^{\alpha_k}$, if $k \geq 4$, where $p_1, p_2, \dots, p_k$ are distinct primes.*

(ii) *Let G be a cyclic group of order $p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$, where $\alpha_i \geq 1$ $\forall i = 1, 2, 3$. Then $\varepsilon(G)$ is planar if and only if $\sum_{i=1}^{3} \alpha_i \leq 3$.*

(iii) *If G is a cyclic group of order $p_1^{\alpha_1}$, then $\varepsilon(G)$ is planar if and only if $\alpha_1 < 6$.*

(iv) *If G is a cyclic group of order $p_1^{\alpha_1}p_2^{\alpha_2}$, where p_1, p_2 are distinct prime numbers and $\alpha_1, \alpha_2 \geq 1$, then $\varepsilon(G)$ is planar if and only if $\sum_{i=1}^{2} \alpha_i \leq 3$.*

Proof. (i) Let $\alpha_i = 1, \forall i$. Then the sets of subgroups of orders $\left\{\frac{o(G)}{p_1}, \frac{o(G)}{p_2}, \frac{o(G)}{p_1p_2}\right\}$ and $\left\{\frac{o(G)}{p_3}, \frac{o(G)}{p_4}, \frac{o(G)}{p_3p_4}\right\}$ form $K_{3,3}$. If $\alpha_i > 1$ for at least one i, say for $i = 1$, then the vertices $\frac{o(G)}{p_1}, \frac{o(G)}{p_1^2}, \frac{o(G)}{p_2}, \frac{o(G)}{p_3}, \frac{o(G)}{p_4}$ form K_5, as the subgroup of order $\frac{o(G)}{p_1}$ is a proper essential subgroup of G. Therefore, $\varepsilon(G)$ is not planar.
(ii) If $\sum_{i=1}^{3} \alpha_i \leq 3$, then $o(G)$ is $p_1p_2p_3$ only. It can be easily seen from construction that $\varepsilon(G)$ is planar. To prove the converse, we first take $\sum_{i=1}^{3} \alpha_i \geq 5$. Then by Theorem 2.17, we have $\omega(\varepsilon(G)) \geq 5$. Therefore, $\varepsilon(G)$ is not planar. If $\sum_{i=1}^{3} \alpha_i = 4$, then one of $\alpha_i = 2$, say for $i = 1$. Then the sets of subgroups of orders $\{p_1p_2, p_1^2p_2, p_1p_2p_3\}$ and $\{p_1p_3, p_1^2p_3, p_3\}$ form $K_{3,3}$ as the subgroup of order $p_1p_2p_3$ is a proper essential subgroup of G.
(iii) If $o(G) = p_1^{\alpha_1}$, then by Theorem 2.4, $\varepsilon(G)$ is K_{α_1-1}. Therefore, $\varepsilon(G)$ is planar if and only if $\alpha_1 - 1 < 5$ or simply $\alpha_1 < 6$.
(iv) We have $o(G) = p_1^{\alpha_1}p_2^{\alpha_2}$. If $p_2 \geq 2$,$p_1 \geq 2$ or $p_1 \geq 4$,$p_2 \geq 1$, then by Theorem 2.17, $\omega(\varepsilon(G)) \geq 5$. Let $p_2 = 1$ and $p_1 = 3$. Then there is a bipartite graph $K_{3,3}$ with bipartition of subgroups of orders $\{p_1, p_1^2, p_1^3\}$ and $\{p_2, p_1p_2, p_1^2p_2\}$. Finally, for $p_2 = 1$ and $p_1 \leq 2$, we clearly see that the graphs are planar. □

We try to generalize the planarity of $\varepsilon(G)$ in Theorem 3.5, when G has exactly two maximal subgroups.

Theorem 3.5. *Suppose G has two maximal subgroups. Then, $\varepsilon(G)$ is planar if and only if G has at most four non-trivial proper subgroups.*

Proof. Let $M^G = \{M_1, M_2\}$. One direction is clear. To prove the converse part, we have the following cases:
Case 1: Assume that M_1 and M_2 are essential. We claim that $\Phi(G)$ is also an essential subgroup of G. As M_1 is essential, for any subgroup H of G, $H \cap M_1 \neq \{e\}$. Let $x(\neq e) \in H \cap M_1$. If $x \in M_2$, then we have $x \in H \cap (M_1 \cap M_2)$ and we are done. Therefore suppose $x \notin M_2$. As M_2 is maximal, $G = M_2\langle x\rangle$. Since G is abelian, $\frac{G}{M_2}$ is cyclic generated by xM_2. As M_2 is maximal, $\frac{G}{M_2}$ is simple and hence, $\frac{G}{M_2} \cong \mathbb{Z}_p$ for a prime p. This implies that $\frac{G}{M_2}$ has order p. This implies that $(xM_2)^p = M_2$. But $(xM_2)^p = x^pM_2$, which implies that $x^pM_2 = M_2 \implies x^p \in M_2$. Again $x \in M_1$ and thus $x^p \in M_1$. Hence, $x^p \in M_1 \cap M_2$. Also $x \in H$ implies that $x^p \in H$ and thus, $x^p \in H \cap (M_1 \cap M_2)$. Hence, $\Phi(G)$ is also an essential subgroup of G. If $|V(\varepsilon(G))| \geq 6$ and $X, Y, Z \in V(\varepsilon(G)) - \{M_1, M_2, \Phi(G)\}$, then there is an induced subgraph $K_{3,3}$ with bipartition $\{M_1, M_2, \Phi(G)\}$ and $\{X, Y, Z\}$, which is a contradiction. Thus, $|V(\varepsilon(G))| \leq 5$. Assume that $|\text{V } (\varepsilon(G))| = 5$. Let $V(\varepsilon(G)) = M_1, M_2, \Phi(G), I, J$. Since $\varepsilon(G)$ does not contain K_5, we have that IJ is not essential, implying that $IJ = I$ or $IJ = J$. Without loss of generality, we may assume that $IJ = I$. But then, we can see that I is also an essential subgroup of G, a contradiction. This concludes that at least one of the M_i is not essential.
Case 2: Assume that M_1 is not essential. Therefore, there exists a simple subgroup S of G such that $S \cap M_1 = \{e\}$ and $SM_1 = G$. As G is abelian, S is cyclic of prime order, say p. Suppose $\Phi(G)(= M_1 \cap M_2) \neq \{e\}$. Take $x(\neq e) \in M_1 \cap M_2$. Then $x \notin S$. Hence $S \subsetneq \langle x\rangle S$. As $S \nsubseteq M_1$, $\langle x\rangle S \nsubseteq M_1$. Therefore, $\langle x\rangle S \subseteq M_2$. But as $S \cap \langle x\rangle = \{e\}$, $\langle x\rangle S$ is not cyclic of prime power order. Hence, M_2 is not cyclic of prime power order. Therefore, M_2 has at least two proper maximal subgroups which can be extended to form distinct maximal subgroups of G as $G \cong M_1 \times M_2$. Therefore, we get at least three maximal subgroups of G, a contradiction. Hence, $\Phi(G) = \{e\}$. This implies that M_2 is also not essential. Hence, there exists no proper essential subgroup of G. If M_1 and M_2 both have at least two proper non-trivial subgroups, then we get a $K_{3,3}$ subgraph of $\varepsilon(G)$ made by the sets of subgroups of M_1 and M_2 together with M_1, M_2 as for any subgroup H_1 of M_1, $H_1M_2 = G$ and for any subgroup H_2 of M_2, $H_2M_1 = G$. Therefore, $|V(\varepsilon(G))| \leq 5$. Consider the case when $|V(\varepsilon(G))| = 5$. Without loss of generality, assume that M_1

has exactly two non-trivial proper subgroups, say H_1, H_1' and M_2 has exactly one non-trivial proper subgroup. Since G is abelian, all its subgroups and quotients are abelian. Thus, the simple subgroup H_2 must be an abelian simple group, forcing it to be isomorphic to $\mathbb{Z}_q$ for some prime q. A group whose only non-trivial proper subgroup is $\mathbb{Z}_q$ must be a cyclic group of order q^2. Hence $M_2 \cong \mathbb{Z}_{q^2}$. It is well known that an abelian group has exactly two non-trivial proper subgroups if and only if it is isomorphic to either $\mathbb{Z}_{p^3}$ (for any prime number p) or $\mathbb{Z}_{pr}$ (for two distinct prime numbers p and r). But if $M_1 \cong \mathbb{Z}_{pr}$, $G \cong M_1M_2$ will have three maximal subgroups, a contradiction. Hence, $M_1 \cong \mathbb{Z}_{p^3}$. Since $M_1 \cap M_2 = \{e\}$, $p \neq q$. As $G \cong M_1M_2$, $G \cong \mathbb{Z}_{p^3} \times \mathbb{Z}_{q^2} \implies G \cong \mathbb{Z}_{p^3q^2}$ as $p \neq q$. But then G has 10 non-trivial proper subgroups, contradicting the fact that $|V(\varepsilon(G))| = 5$. Hence, $|V(\varepsilon(G))| < 5$. □

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References

[1] S. Akbari, A. Alilou, J. Amjadi and S. M. Sheikholeslami, *The co-annihilating ideal graphs of commutative rings*, Canad. Math. Bull., 60(1) (2017), 3-11.

[2] S. Akbari, M. Habibi, A. Majidinya and R. Manaviyat, *A note on comaximal graph of non-commutative rings*, Algebr. Represent. Theory, 16(2) (2013), 303-307.

[3] A. Alilou and J. Amjadi, *The sum-annihilating essential ideal graph of a commutative ring*, Commun. Comb. Optim., 1(2) (2016), 117-135.

[4] A. Alilou, J. Amjadi, L. Asgharsharghi and S. M. Sheikholeslami, *On the sum-annihilating ideal graph of a commutative ring*, Indian J. Math., 58(2) (2016), 239-256.

[5] A. Alilou, J. Amjadi and S. M. Sheikholeslami, *A new graph associated to a commutative ring*, Discrete Math. Algorithms Appl., 8(2) (2016), 1650029 (13 pp).

[6] J. Amjadi, *The essential ideal graph of a commutative ring*, Asian-Eur. J. Math., 11(4) (2018), 1850058 (9 pp).

[7] B. Barman and K. K. Rajkhowa, *Nonessential sum graph of an Artinian ring*, Arab J. Math. Sci., 28(1) (2022), 37-43.

[8] P. J. Cameron, *Graphs defined on groups*, Int. J. Group Theory, 11(2) (2022), 53-107.

[9] D. S. Dummitt and R. M. Foote, Abstract Algebra, Third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[10] G. Frattini, *Intorno alla generazione dei gruppi di operazioni*, Rend. Accad. Lincei, 4 (1885), 281-285.

[11] L. Fuchs, Infinite Abelian Groups, I, Pure and Applied Mathematics, 36, Academic Press, New York-London, 1970.

[12] H. R. Maimani, M. Salimi, A. Sattari and S. Yassemi, *Comaximal graph of commutative rings*, J. Algebra, 319(4) (2008), 1801-1808.

[13] J. Matczuk and A. Majidinya, *Sum-essential graphs of modules*, J. Algebra Appl., 20(11) (2021), 2150211 (17 pp).

[14] D. J. S. Robinson, A Course in the Theory of Groups, Second edition, Graduate Texts in Mathematics, 80, Springer-Verlag, New York, 1996.

[15] P. K. Sharma and S. M. Bhatwadekar, *A note on graphical representation of rings*, J. Algebra, 176(1) (1995), 124-127.

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