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THE RATIO-COVARIETY OF NUMERICAL SEMIGROUPS HAVING MAXIMAL EMBEDDING DIMENSION WITH FIXED MULTIPLICITY AND FROBENIUS NUMBER

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Abstract. In this paper we will show that $\mathrm{MED}(F,m) = \{S \mid S$ is a numerical semigroup with maximal embedding dimension, Frobenius number F and multiplicity $m\}$ is a ratio-covariety. As a consequence, we present two algorithms: one that computes $\mathrm{MED}(F,m)$ and another one that calculates the elements of $\mathrm{MED}(F,m)$ with a given genus.

If $X \subseteq S\backslash(\langle m\rangle \cup \{F+1, \rightarrow\})$ for some $S \in \mathrm{MED}(F,m)$, then there exists the smallest element of $\mathrm{MED}(F,m)$ containing X. This element will be denoted by $\mathrm{MED}(F,m)[X]$ and we will say that X one of its $\mathrm{MED}(F,m)$-system of generators. We will prove that every element S of $\mathrm{MED}(F,m)$ has a unique minimal $\mathrm{MED}(F,m)$-system of generators and it will be denoted by $\mathrm{MED}(F,m)\mathrm{msg}(S)$. The cardinality of $\mathrm{MED}(F,m)\mathrm{msg}(S)$, will be called $\mathrm{MED}(F,m)$-rank of S. We will also see in this work, how all the elements of $\mathrm{MED}(F,m)$ with a fixed $\mathrm{MED}(F,m)$-rank are.

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1. Introduction

Denote by $\mathbb{Z}$ and $\mathbb{N}$ be the set of integers and nonnegative integers, respectively. A *numerical semigroup* S is a subset of $\mathbb{N}$ closed under additon, $0 \in S$ and $\mathbb{N}\backslash S$ is finite. The set $\mathbb{N}\backslash S$ is known as the set of *gaps* of S and its cardinality, denoted by $\mathrm{g}(S)$, is called the *genus* of S. The largest integer not belonging to S is called the *Frobenius number* of S and it will be denoted by $\mathrm{F}(S)$.

If A is a nonempty subset of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A, that is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\backslash\{0\}, \{a_1,\ldots,a_n\} \subseteq$

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A and $\{\lambda_1, \dots, \lambda_n\} \subseteq \mathbb{N}\}$. In [16, Lemma 2.1], it is shown that $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$.

If M is a submonoid of $(\mathbb{N}, +)$ and $M = \langle A \rangle$, then we say that A is a *system of generators* of M. Moreover, if $M \neq \langle B \rangle$ for all $B \subsetneq A$, then we will say that A is a *minimal system of generators* of M. In [16, Corollary 2.8] is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators, which in addition is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the *embedding dimension* of M and will be denoted by $\mathrm{e}(M)$. The multiplicity of S, denoted by $\mathrm{m}(S)$, is defined as the minimum of $S \backslash \{0\}$. In [16, Proposition 3.10], it is shown that if S is a numerical semigroup, then $\mathrm{e}(S) \leq \mathrm{m}(S)$.

In the literature one can find a long list of works dealing with the study of one dimensional analytically irreducible local domains via their value semigroups (see for instance [4,7,9,11,19]). One of the properties studied for this kind of rings using this approach is that of being of maximal embedding dimension (see [1,3,6,18]). The characterization of rings with maximal embedding dimension via their value semigroup, gave rise to the notion of numerical semigroups having maximal embedding dimension (see [17]), hereinafter MED-semigroup. A numerical semigroup is said to be a MED-semigroup if $\mathrm{e}(S) = \mathrm{m}(S)$.

If m and F are positive integers, we denote by

$$\mathrm{MED}(F, m) = \{S \mid S \text{ is a MED-semigroup, } \mathrm{F}(S) = F \text{ and } \mathrm{m}(S) = m\}.$$

The study of the set $\mathrm{MED}(F, m)$ is the aim of this work.

Let a and b be integers, we say that a *divides* b if there exists an integer c such that $b = ca$, and we denote this by $a \mid b$. Otherwise, a *does not divide* b, and we denote this by $a \nmid b$. Let S be a numerical semigroup such that $S \neq \mathbb{N}$, the *ratio* of S is defined as $\mathrm{r}(S) = \min\{s \in S \mid \mathrm{m}(S) \nmid s\}$. It is clear that $\mathrm{r}(S) = \min(\mathrm{msg}(S) \backslash \{\mathrm{m}(S)\})$.

Following the notation introduced in [12], a *ratio-covariety* is a nonempty family $\mathscr{R}$ of numerical semigroups fulfilling the following conditions:

(1) There is the minimum of $\mathscr{R}$, denoted by $\min(\mathscr{R})$, with respect to inclusion order.
(2) If $\{S, T\} \subseteq \mathscr{R}$, then $S \cap T \in \mathscr{R}$.
(3) If $S \in \mathscr{R}$ and $S \neq \min(\mathscr{R})$, then $S \backslash \{\mathrm{r}(S)\} \in \mathscr{R}$.

The paper is structured as follows. In Section 2, we show that if $m < F$ and $m \nmid F$, then $\mathrm{MED}(F, m)$ is a ratio-covariety. By using the results of [12], we will arrange the elements of $\mathrm{MED}(F, m)$ in a rooted tree and we present a characterization of the children of an arbitrary vertex in this tree. In Section 3, by taking advantage of the results obtained, we will present an algorithm which compute all the elements of $\mathrm{MED}(F,m)$.

In Section 4, we will see who are the maximal elements of $\mathrm{MED}(F,m)$. This fact will allows, in Section 5, to compute the set $\{\mathrm{g}(S) \mid S \in \mathrm{MED}(F,m)\}$, as well as, to present an algorithm which enables to calculate all the elements of $\mathrm{MED}(F,m)$ with a fixed genus.

We will say that a set X is a $\mathrm{MED}(F,m)$-*set* if $X \cap (\langle m \rangle \cup \{F+1, \rightarrow\}) = \emptyset$ and $X \subseteq S$ for some $S \in \mathrm{MED}(F,m)$, where the symbol $\rightarrow$ means that every integer greater than $F+1$ belongs to the set.

Section 6 is devoted to see that if X is a $\mathrm{MED}(F,m)$-set, then there is the smallest element of $\mathrm{MED}(F,m)$ containing X. This element will be denoted by $\mathrm{MED}(F,m)[X]$.

If $S = \mathrm{MED}(F,m)[X]$, then we will say that X is a $\mathrm{MED}(F,m)$-*system of generators* of S. The main aim of Section 6, will be to prove that every element of $\mathrm{MED}(F,m)$ admits a unique minimal $\mathrm{MED}(F,m)$-system of generators.

The $\mathrm{MED}(F,m)$-rank of an element of $\mathrm{MED}(F,m)$, will defined as the cardinality of its minimal $\mathrm{MED}(F,m)$-system of generators. In Section 7, we focus on getting all elements of $\mathrm{MED}(F,m)$ with a given $\mathrm{MED}(F,m)$-rank.

Throughout this paper, some examples are shown to illustrate the results proven. The computation of these examples are performed by using the GAP (see [10]) package `numericalsgps` ([8]).

2. The tree associated to the ratio-covariety $\mathrm{MED}(F,m)$

It is clear that if S is a numerical semigroup with multiplicity m and Frobenius number F, then $m-1 \leq F$ and $m \nmid F$. Moreover, if $F = m-1$, then $S = \{0, F+1, \rightarrow\}$.

Throughout this work, we suppose that m and F are positive integers such that $m < F$ and $m \nmid F$. Our first aim will be prove that $\mathrm{MED}(F,m)$ is a ratio-covariety.

Lemma 2.1. *With the above notation, we have that*

$$\Delta(F,m) = \langle m \rangle \cup \{F+1, \rightarrow\}$$

is the minimum of $\mathrm{MED}(F,m)$.

Proof. Clearly $\Delta(F,m)$ belongs to $\mathrm{MED}(F,m)$ and every element in this set contains $\Delta(F,m)$. □

The following result appears in [17, Proposition 3].

Lemma 2.2. *Let S and T be* MED*-semigroups such that* $\mathrm{m}(S) = \mathrm{m}(T)$. *Then* $S \cap T$ *is again a* MED*-semigroup with* $\mathrm{m}(S \cap T) = \mathrm{m}(S) = \mathrm{m}(T)$.

The next result is well known and easy to prove.

Lemma 2.3. *Let S and T be numerical semigroups and $x \in S$. Then the following conditions hold:*

(1) *$S \cap T$ is a numerical semigroup and* $\mathrm{F}(S \cap T) = \max\{\mathrm{F}(S), \mathrm{F}(T)\}$.
(2) *$S\backslash\{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

Lemma 2.4. *If $S \in \mathrm{MED}(F, m)$ and $S \neq \Delta(F, m)$, then $S\backslash\{\mathrm{r}(S)\} \in \mathrm{MED}(F, m)$.*

Proof. By applying (2) of Lemma 2.3, we easily deduce that $S\backslash\{\mathrm{r}(S)\}$ is a numerical semigroup having multiplicity m and Frobenius number F. Since $\mathrm{m}(S) + \mathrm{r}(S)$ can not be written as sum of two nonzero elements of $S\backslash\{\mathrm{r}(S)\}$, then $\mathrm{m}(S)+\mathrm{r}(S) \in \mathrm{msg}(S\backslash\{\mathrm{r}(S)\})$. Therefore, $(\mathrm{msg}(S)\backslash\{\mathrm{r}(S)\}) \cup \{\mathrm{m}(S) + \mathrm{r}(S)\} \subseteq \mathrm{msg}(S\backslash\{\mathrm{r}(S)\})$. As $\mathrm{e}(S) = m$, we have $\mathrm{e}(S\backslash\{\mathrm{r}(S)\}) \geq m$ and so $\mathrm{e}(S\backslash\{\mathrm{r}(S)\}) = m$. Hence, $S\backslash\{\mathrm{r}(S)\} \in \mathrm{MED}(F, m)$. □

The next proposition follows directly applying Lemmas 2.1, 2.2, 2.3 and 2.4.

Proposition 2.5. *With the above notation,* $\mathrm{MED}(F, m)$ *is a ratio-covariety and* $\Delta(F, m)$ *is its minimum.*

Define the graph $\mathrm{G}(\mathrm{MED}(F, m))$ as follows: $\mathrm{MED}(F, m)$ is the set of vertices and $(S, T) \in \mathrm{MED}(F, m) \times \mathrm{MED}(F, m)$ is an edge of $\mathrm{G}(\mathrm{MED}(F, m))$ if and only if $T = S\backslash\{\mathrm{r}(S)\}$.

From Proposition 2.5 and [12, Proposition 3], we have the next result.

Proposition 2.6. *With the above notation,* $\mathrm{G}(\mathrm{MED}(F, m))$ *is a tree with root* $\Delta(F, m)$.

We can recursively build a tree, starting from the root and connecting, through an edge, the vertices already built with their children. Hence, it is very interesting to characterize the children of an arbitrary vertex in the tree.

Following the terminology introduced in [14], we say that an integer z is a *pseudo-Frobenius number* of a numerical semigroup S if $z \notin S$ and $z + s \in S$ for all $s \in S\backslash\{0\}$. Denote by $\mathrm{PF}(S)$ the set formed by the pseudo-Frobenius numbers of S.

Let S be a numerical semigroup. The elements of the set $\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \in S\}$ will be called *special gaps* of S. The following result is [16, Proposition 4.33].

Lemma 2.7. *Let S be a numerical semigroup and $x \in \mathbb{N}\backslash S$. Then $x \in \mathrm{SG}(S)$ if and only if $S \cup \{x\}$ is a numerical semigroup.*

The next result is a consequence from Proposition 2.5 and [12, Proposition 4].

Proposition 2.8. *If $S \in \mathrm{MED}(F, m)$, then the set formed by the children of S in the tree $\mathrm{G}(\mathrm{MED}(F, m))$ is*

$$\{S \cup \{x\} \mid x \in \mathrm{SG}(S),\ \mathrm{m}(S) < x < \mathrm{r}(S) \text{ and } S \cup \{x\} \in \mathrm{MED}(F, m)\}.$$

3. An algorithm for computing $\mathrm{MED}(F, m)$

Let S be a numerical semigroup S and $n \in S\backslash\{0\}$, then we consider the *Apry set* of n in S (in recognition of [2]) as the set $\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}$. From [16, Lemma 2.4], it can deduced the following result.

Lemma 3.1. *Let S be a numerical semigroup and $n \in S\backslash\{0\}$. Then $\mathrm{Ap}(S, n)$ is a set with cardinality n. In addition, $\mathrm{Ap}(S, n) = \{0 = w(0), w(1), \ldots, w(n-1)\}$, where $w(i)$ is the least element of S congruent with i modulo n, for all $i \in \{0, \ldots, n-1\}$.*

Let S be a numerical semigroup. We define over $\mathbb{Z}$ the following order relation: $a \leq_S b$ if $b - a \in S$. The following result is Lemma 10 from [14].

Lemma 3.2. *If S is a numerical semigroup and $n \in S\backslash\{0\}$, then*

$$\mathrm{PF}(S) = \{w - n \mid w \in \mathrm{Maximals}_{\leq_S} \mathrm{Ap}(S, n)\}.$$

The next lemma has an immediate proof.

Lemma 3.3. *Let S be a numerical semigroup, $n \in S\backslash\{0\}$ and $w \in \mathrm{Ap}(S, n)$. Then $w \in \mathrm{Maximals}_{\leq_S}(\mathrm{Ap}(S, n))$ if and only if $w + w' \notin \mathrm{Ap}(S, n)$ for all $w' \in \mathrm{Ap}(S, n)\backslash\{0\}$.*

It is straightforward to prove the following result.

Lemma 3.4. *If S is a numerical semigroup and $S \neq \mathbb{N}$, then*

$$\mathrm{SG}(S) = \{x \in \mathrm{PF}(S) \mid 2x \notin \mathrm{PF}(S)\}.$$

Note 3.5. Observe that as a consequence of Lemmas 3.2, 3.3 and 3.4, if S is a numerical semigroup and we know $\mathrm{Ap}(S, n)$ for some $n \in S\backslash\{0\}$, then we can compute easily the set $\mathrm{SG}(S)$.

The following result has an easy proof.

Lemma 3.6. *If S is a numerical semigroup, $n \in S\backslash\{0\}$ and $x \in \mathrm{SG}(S)$, then $x + n \in \mathrm{Ap}(S, n)$. Moreover, $\mathrm{Ap}(S \cup \{x\}, n) = (\mathrm{Ap}(S, n)\backslash\{x + n\}) \cup \{x\}$.*

Note 3.7. Observe that as a consequence of Lemma 3.6, if we know $\mathrm{Ap}(S,n)$, then we can easily compute $\mathrm{Ap}(S\cup\{x\},n)$. In particular, Lemma 3.6 allows us to compute the set $\mathrm{Ap}(T,n)$ from $\mathrm{Ap}(S,n)$, for every child T of S in the tree $\mathrm{G}(\mathrm{MED}(F,m))$ (see Proposition 2.8).

We already have all the necessary knowledge to present the algorithm that gives title to this section.

Algorithm 3.8.

INPUT: Two positive integer F and m such that $m<F$ and $m\nmid F$.
OUTPUT: $\mathrm{MED}(F,m)$.

(1) Compute $\mathrm{Ap}(\Delta(F,m),m)$.
(2) $\mathrm{MED}(F,m)=\{\Delta(F,m)\}$ and $B=\{\Delta(F,m)\}$.
(3) For every $S\in B$ compute $\theta(S)=\{x\in \mathrm{SG}(S)\mid m<x<\mathrm{r}(S) \text{ and } S\cup\{x\}\in \mathrm{MED}(F,m)\}$.
(4) If $\bigcup_{S\in B}\theta(S)=\emptyset$, then return $\mathrm{MED}(F,m)$.
(5) $C=\bigcup_{S\in B}\{S\cup\{x\}\mid x\in\theta(S)\}$.
(6) $\mathrm{MED}(F,m)=\mathrm{MED}(F,m)\cup C$ and $B=C$.
(7) For every $S\in B$, compute $\mathrm{Ap}(S,m)$ and go to Step (3).

Next we illustrate this algorithm with an example.

Example 3.9. We are going to compute $\mathrm{MED}(12,5)$ by using Algorithm 3.8.

- $\mathrm{Ap}(\Delta(12,5),5)=\{0,13,14,16,17\}$, $\mathrm{MED}(12,5)=\{\Delta(12,5)\}$ and $B=\{\Delta(12,5)\}$.
- $\theta(\Delta(12,5))=\{9,11\}$ and $C=\{\Delta(12,5)\cup\{9\},\Delta(12,5)\cup\{11\}\}$.
- $\mathrm{MED}(12,5)=\{\Delta(12,5),\Delta(12,5)\cup\{9\},\Delta(12,5)\cup\{11\}\}$, $B=\{\Delta(12,5)\cup\{9\},\Delta(12,5)\cup\{11\}\}$.
- $\mathrm{Ap}\left(\Delta(12,5)\cup\{9\},5\right)=\{0,9,13,16,17\}$ and $\mathrm{Ap}\left(\Delta(12,5)\cup\{11\},5\right)=\{0,11,13,14,17\}$.
- $\theta(\Delta(12,5)\cup\{9\})=\emptyset$ and $\theta(\Delta(12,5)\cup\{11\})=\{8,9\}$.
- $C=\{\Delta(12,5)\cup\{8,11\},\Delta(12,5)\cup\{9,11\}\}$.
- $\mathrm{MED}(12,5)=\{\Delta(12,5),\Delta(12,5))\cup\{9\},\Delta(12,5)\cup\{11\},\Delta(12,5)\cup\{8,11\},\Delta(12,5)\cup\{9,11\}\}$ and $B=\{\Delta(12,5)\cup\{8,11\},\Delta(12,5)\cup\{9,11\}\}$.
- $\mathrm{Ap}\left(\Delta(12,5)\cup\{8,11\},5\right)=\{0,8,11,14,17\}$ and $\mathrm{Ap}\left(\Delta(12,5)\cup\{9,11\},5\right)=\{0,9,11,13,17\}$.
- $\theta(\Delta(12,5)\cup\{8,11\})=\emptyset$ and $\theta(\Delta(12,5)\cup\{9,11\})=\emptyset$.

- The Algorithm returns

$$\mathrm{MED}(12,5) =$$

$$\{\Delta(12,5), \Delta(12,5) \cup \{9\}, \Delta(12,5) \cup \{11\}, \Delta(12,5) \cup \{8,11\}, \Delta(12,5) \cup \{9,11\}\}.$$

4. Maximal elements of $\mathrm{MED}(F,M)$

Let A and B be nonempty subsets of $\mathbb{Z}$, denote by $A+B = \{a+b \mid a \in A, b \in B\}$. As a consequence from Propositions 2 and 9 of [13], we have the following result.

If a and b are positive integers, we denote by

$$\mathscr{L}(a,b) = \{S \mid S \text{ is a numerical semigroup, } b \in S \text{ and } \mathrm{F}(S) = a\}.$$

Proposition 4.1. *With the above notation, it is verified that* $\mathrm{MED}(F,m) =$

$$\{(\{m\}+T) \cup \{0\} \mid T \textit{ is a numerical semigroup, } m \in T \textit{ and } \mathrm{F}(T) = F-m\}.$$

As a direct consequence of previous proposition, we have the following result.

Proposition 4.2. *Let* $T \in \mathscr{L}(F-m,m)$. *Then* $S = (\{m\}+T) \cup \{0\}$ *is a maximal element of* $\mathrm{MED}(F,m)$ *if and only if* T *is a maximal element of* $\mathscr{L}(F-m,m)$.

A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. This concept was introduced in [15] and the following characterization also appears there.

Lemma 4.3. *Let* S *be a numerical semigroup. Then* S *is irreducible if and only if* S *is a maximal element in the set of numerical semigroups with same Frobenius number.*

The irreducible numerical semigroups with Frobenius number odd (respectively, even) are called *symmetric numerical semigroups* (respectively, *pseudo-symmetric numerical semigroups*). This kind of numerical semigroups has been widely studied because one dimensional analytically irreducible local ring is Gorenstein (respectively, Kunz) if and only if its value semigroup is symmetric (respectively, pseudo-symmetric), as it can be seen in [11] and [3].

If $q \in \mathbb{Q}$, then $\lceil q \rceil = \min\{z \in \mathbb{Z} \mid q \le z\}$ and $\lfloor q \rfloor = \max\{z \in \mathbb{Z} \mid z \le q\}$. The following result is deduced from [16, Lemma 2.14 and Corollary 4.5].

Lemma 4.4. *Let* S *be a numerical semigroup. Then the following conditions hold.*

(1) $\dfrac{\mathrm{F}(S)+1}{2} \le \mathrm{g}(S) \le \mathrm{F}(S)$.

(2) S *is symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+1}{2}$.

(3) *S is pseudo-symmetric if and only if* $\mathrm{g}(S) = \dfrac{\mathrm{F}(S)+2}{2}$.

(4) *S is irreducible if and only if* $\mathrm{g}(S) = \left\lceil \dfrac{\mathrm{F}(S)+1}{2} \right\rceil$.

Denote by $\mathrm{Max}(\mathscr{L}(a,b))$ the set formed by the maximal elements of $\mathscr{L}(a,b)$. This set is described in the next proposition.

Proposition 4.5. *If a and b are positive integers, then*

$$\mathrm{Max}(\mathscr{L}(a,b)) = \{S \in \mathscr{L}(a,b) \mid S \text{ is irreducible}\}.$$

Proof. Let $S \in \mathrm{Max}(\mathscr{L}(a,b))$, then $S \in \mathscr{L}(a,b)$. If S is not irreducible, then by Lemma 4.3, we deduce that there is a numerical semigroup T such that $S \subsetneq T$ and $\mathrm{F}(T) = a$. Therefore, $T \in \mathscr{L}(a,b)$ and $S \subsetneq T$ contradicting that $S \in \mathrm{Max}(\mathscr{L}(a,b))$. If $S \in \mathscr{L}(a,b)$ and S is irreducible, then by Lemma 4.3, we have $S \in \mathrm{Max}(\mathscr{L}(a,b))$. □

In [5], an algorithm appears which allows to compute all the irreducible numerical semigroups with a fixed Frobenius number. Hence, we have an algorithm to compute the set $\mathrm{Max}(\mathscr{L}(a,b))$. And by applying Proposition 4.2, we have an algorithm to calculate the set

$$\mathrm{Max}(\mathrm{MED}(F,m)) = \{S \mid S \text{ is a maximal element in } \mathrm{MED}(F,m)\}.$$

This procedure will be shown in the next example.

Example 4.6. We are going to calculate the set $\mathrm{Max}(\mathrm{MED}(17,6))$. By applying Propositions 4.1 and 4.5, we have

$$\mathrm{Max}(\mathrm{MED}(17,6)) = \{(\{6\}+T) \cup \{0\} \mid T \in \mathrm{Max}(\mathscr{L}(11,6))\}.$$

By Proposition 4.5, we know that $\mathrm{Max}(\mathscr{L}(11,6)) = \{S \in \mathscr{L}(11,6) \mid S$ is irreducible$\}$. By using the algorithm described in [5], we obtain that the set of all the irreducible numerical semigroups with Frobenius number 11 is

$$\{\langle 6,7,8,9,10\rangle, \langle 3,7\rangle, \langle 4,6,9\rangle, \langle 5,7,8,9\rangle, \langle 2,13\rangle, \langle 4,5\rangle\}.$$

Therefore, $\mathrm{Max}(\mathscr{L}(11,6)) = \{\langle 6,7,8,9,10\rangle, \langle 3,7\rangle, \langle 4,6,9\rangle, \langle 2,13\rangle\}$.

Finally, $\mathrm{Max}(\mathrm{MED}(17,6)) = \{(\{6\}+\langle 6,7,8,9,10\rangle)\cup\{0\}, (\{6\}+\langle 3,7\rangle)\cup\{0\}, (\{6\}+\langle 4,6,9\rangle)\cup\{0\}, (\{6\}+\langle 2,13\rangle)\cup\{0\}\} = \{\langle 6,13,14,15,16,23\rangle, \langle 6,9,13,16,20,23\rangle, \langle 6,10,14,15,19,23\rangle, \langle 6,8,10,19,21,23\rangle\}$.

5. The elements of $\mathrm{MED}(F, m)$ with fixed genus

In the following proposition, as a consequence of Lemma 4.4 and Proposition 4.5, we characterize the elements of $\mathrm{Max}(\mathscr{L}(a,b))$.

Proposition 5.1. *Let a and b be positive integers and $S \in \mathscr{L}(a,b)$. Then $S \in \mathrm{Max}(\mathscr{L}(a,b))$ if and only if $\mathrm{g}(S) = \lceil \frac{a+1}{2} \rceil$.*

As a consequence from [13, Proposition 9], we obtain the following lemma will be helpful in order to give a characterization of $\mathrm{Max}(\mathrm{MED}(F,m))$.

Lemma 5.2. *Let S be a numerical semigroup, $x \in S\backslash\{0,1\}$ and $S' = (\{x\} + S) \cup \{0\}$, then $\mathrm{F}(S') = \mathrm{F}(S) + x$ and $\mathrm{g}(S') = \mathrm{g}(S) + x - 1$.*

Proposition 5.3. *Let $S \in \mathrm{MED}(F,m)$. Then $S \in \mathrm{Max}(\mathrm{MED}(F,m))$ if and only if $\mathrm{g}(S) = \lceil \frac{F-m+1}{2} \rceil + m - 1$.*

Proof. *Necessity.* If $S \in \mathrm{Max}(\mathrm{MED}(F,m))$, then by Proposition 4.2, there exists $T \in \mathrm{Max}(\mathscr{L}(F-m,m))$ such that $S = (\{m\} + T) \cup \{0\}$. In this conditions, we know that T is irreducible, by Proposition 4.5 and $\mathrm{g}(T) = \lceil \frac{F-m+1}{2} \rceil$, by Lemma 4.4. Hence, Lemma 5.2 asserts that $\mathrm{g}(S) = \mathrm{g}(T) + m - 1$. That is, $\mathrm{g}(S) = \lceil \frac{F-m+1}{2} \rceil + m - 1$.

Sufficiency. If $S \in \mathrm{MED}(F,m)$, then by Proposition 4.1, there exists $T \in \mathscr{L}(F-m,m)$ such that $S = (\{m\} + T) \cup \{0\}$. By Lemma 5.2, we know that $\mathrm{g}(S) = \mathrm{g}(T) + m - 1$ and thus $\lceil \frac{F-m+1}{2} \rceil + m - 1 = \mathrm{g}(T) + m - 1$, and consequently, $\mathrm{g}(T) = \lceil \frac{F-m+1}{2} \rceil$. By applying Proposition 4.5 we have that T is irreducible. Next, Proposition 4.5 asserts that $T \in \mathrm{Max}(\mathscr{L}(F-m,m))$. Finally, by Proposition 4.2 we obtain that $S \in \mathrm{Max}(\mathrm{MED}(F,m))$. $\square$

Note 5.4. Observe that all the elements of $\mathrm{Max}(\mathrm{MED}(17,6))$ which we have seen in Example 4.6, have genus $\lceil \frac{17-6+1}{2} \rceil + 6 - 1 = 6 + 5 = 11$.

Let S be a numerical semigroup such that $S \neq \mathbb{N}$, then the *ratio-sequence associated* to S is recurrently defined as: $S_0 = S$ and $S_{n+1} = S_n \backslash \{\mathrm{r}(S_n)\}$ for all $n \in \mathbb{N}$.

If S is a numerical semigroup, then we denote by $\mathrm{A}(S) = \{x \in S \mid x < \mathrm{F}(S) \text{ and } \mathrm{m}(S) \nmid x\}$. The cardinality of $\mathrm{A}(S)$ will be denoted by $\mathrm{a}(S)$.

Let S be a numerical semigroup and $\{S_n\}_{n\in\mathbb{N}}$ be the ratio-sequence associated to S, then the set $\mathrm{Rat\text{-}Cad}(S) = \{S_0, S_1, \ldots, S_{\mathrm{a}(S)}\}$ is called the *ratio-chain associated* to S. It is clear that $S_{\mathrm{a}(S)} = \Delta(\mathrm{F}(S), \mathrm{m}(S))$.

The following result is easy to prove and it appears in [12, Lemma 11].

Lemma 5.5. *With the above notation, it is verified that* $\mathrm{g}(\Delta(F,m)) = F - \left\lfloor \dfrac{F}{m} \right\rfloor$.

Proposition 5.6. *With above notation, it is verified that*

$$\{\mathrm{g}(S) \mid S \in \mathrm{MED}(F,m)\} = \left\{ x \in \mathbb{N} \mid \left\lceil \frac{F-m+1}{2} \right\rceil + m - 1 \leq x \leq F - \left\lfloor \frac{F}{m} \right\rfloor \right\}.$$

Proof. As $\Delta(F,m)$ is the minimum of $\mathrm{MED}(F,m)$, by applying Lemma 5.5, we have that $F - \left\lfloor \dfrac{F}{m} \right\rfloor = \max\{\mathrm{g}(S) \mid S \in \mathrm{MED}(F,m)\}$. Let $p = \min\{\mathrm{g}(S) \mid S \in \mathrm{MED}(F,m)\}$ and let $T \in \mathrm{MED}(F,m)$ such that $\mathrm{g}(T) = p$. It is obvious that $T \in \mathrm{Max}(\mathrm{MED}(F,m))$ and so, by Proposition 5.3, we have that $p = \mathrm{g}(T) = \lceil \frac{F-m+1}{2} \rceil + m - 1$. In order to conclude the proof, it suffices to note that if $\text{Rat-Cad}(T) = \{T_0, T_1, \ldots, T_{\mathrm{a}(T)}\}$, then $\{\mathrm{g}(T_0), \mathrm{g}(T_1), \ldots, \mathrm{g}(T_{\mathrm{a}(T)})\} = \left\{ x \in \mathbb{N} \mid \left\lceil \dfrac{F-m+1}{2} \right\rceil + m - 1 \leq x \leq F - \left\lfloor \dfrac{F}{m} \right\rfloor \right\}$. □

Example 5.7. As a direct application of Proposition 5.6, we have that

$$\{\mathrm{g}(S) \mid S \in \mathrm{MED}(12,5)\} = \\ = \left\{ x \in \mathbb{N} \mid \left\lceil \frac{12-5+1}{2} \right\rceil + 5 - 1 \leq x \leq 12 - \left\lfloor \frac{12}{5} \right\rfloor \right\} = \{8,9,10\}.$$

Our next aim is to present an algorithm which computes the set $\{S \in \mathrm{MED}(F,m) \mid \mathrm{g}(S) = g\}$. Observe that by Proposition 5.6, we know that this set is not empty if and only if $\lceil \frac{F-m+1}{2} \rceil + m - 1 \leq g \leq F - \left\lfloor \dfrac{F}{m} \right\rfloor$.

Algorithm 5.8.

INPUT: Positive integers F, m and g such that $m < F$, $m \nmid F$ and $\lceil \frac{F-m+1}{2} \rceil + m - 1 \leq g \leq F - \left\lfloor \dfrac{F}{m} \right\rfloor$.

OUTPUT: $\{S \in \mathrm{MED}(F,m) \mid \mathrm{g}(S) = g\}$.

(1) $H = \{\Delta(F,m)\}$, $i = F - \left\lfloor \dfrac{F}{m} \right\rfloor$.
(2) If $i = g$, return H.
(3) For every $S \in H$, compute $\theta(S) = \{x \in \mathrm{SG}(S) \mid m < x < \mathrm{r}(S) \text{ and } S \cup \{x\} \in \mathrm{MED}(F,m)\}$.
(4) $H = \displaystyle\bigcup_{S \in H} \{S \cup \{x\} \mid x \in \theta(S)\}$, $i = i - 1$ and go to Step (2).

Next we will show how Algorithm 5.8 works.

Example 5.9. We will compute the set $\{S \in \mathrm{MED}(12,5) \mid \mathrm{g}(S) = 9\}$, by using Algorithm 5.8. Observe that $5 < 12$, $5 \nmid 12$ and $\lceil \frac{12-5+1}{2} \rceil + 5 - 1 \leq 9 \leq 12 - \left\lfloor \dfrac{12}{5} \right\rfloor$.

- $H = \{\Delta(12,5)\}$, $i = 10$.
- $\theta(\Delta(12,5)) = \{9, 11\}$.
- $H = \{\Delta(12,5) \cup \{9\}, \Delta(12,5) \cup \{11\}\}$, $i = 9$.
- The Algorithm returns

$$\{S \in \mathrm{MED}(12,5) \mid \mathrm{g}(S) = 9\} = \{\Delta(12,5) \cup \{9\}, \Delta(12,5) \cup \{11\}\}.$$

6. $\mathrm{MED}(F,m)$-system of generators

A set X is said a $\mathrm{MED}(F,m)$-*set*, if it verifies the following conditions:

(1) $X \cap \Delta(F,m) = \emptyset$.
(2) There is $S \in \mathrm{MED}(F,m)$ such that $X \subseteq S$.

If X is a $\mathrm{MED}(F,m)$-set, then we will denote by $\mathrm{MED}(F,m)[X]$ the intersection of all elements of $\mathrm{MED}(F,m)$ containing X. As $\mathrm{MED}(F,m)$ is a finite set, by applying Proposition 2.5, we have that the intersection of elements of $\mathrm{MED}(F,m)$ is again an element of $\mathrm{MED}(F,m)$. Hence, we have the following result.

Proposition 6.1. *Let X be a $\mathrm{MED}(F,m)$-set. Then $\mathrm{MED}(F,m)[X]$ is the smallest element of $\mathrm{MED}(F,m)$ containing X.*

If X is a $\mathrm{MED}(F,m)$-set and $S = \mathrm{MED}(F,m)[X]$, then we will say that X is a $\mathrm{MED}(F,m)$-*system of generators* of S. In addition, if $S \neq \mathrm{MED}(F,m)[Y]$ for all $Y \subsetneq X$, then X will be called a *minimal* $\mathrm{MED}(F,m)[X]$-*system of generators* of S.

Our next aim will be to prove that every element of $\mathrm{MED}(F,m)$ admits a unique minimal $\mathrm{MED}(F,m)$-system of generators.

The following result can be deduced from [16, Proposition 3.12].

Lemma 6.2. *Let S be a numerical semigroup. Then S is a MED-semigroup if and only if $x + y - \mathrm{m}(S) \in S$ for all $\{x, y\} \subseteq S \backslash \{0\}$.*

Lemma 6.3. *If $\{S, T\} \subseteq \mathrm{MED}(F,m)$, $S \subsetneq T$ and $x = \max(T \backslash S)$, then $S \cup \{x\} \in \mathrm{MED}(F,m)$.*

Proof. As $x = \max(T \backslash S)$ and $S \subseteq T$, we have $2x \in S$ and $x + S \subseteq S$, consequently, $S \cup \{x\}$ is a numerical semigroup with multiplicity m. Moreover, $m < x < F$ and thus $S \cup \{x\}$ is a numerical semigroup with multiplicity m and Frobenius number F. To conclude the proof, we will see that $S \cup \{x\}$ is a MED-semigroup. For this, we will show, by using Lemma 6.2, that if $\{a, b\} \subseteq (S \cup \{x\}) \backslash \{0\}$, then $a + b - m \in S \cup \{x\}$. We distinguish three cases:

(1) If $\{a, b\} \subseteq S$, then $a + b - m \in S \subseteq S \cup \{x\}$.

(2) If $a=b=x$, then $2x-m\in T$ and $2x-m>x$. Therefore, $2x-m\in S\subseteq S\cup\{x\}$.
(3) If $a=x$ and $b\in S$, then $x+b-m\in T$ and $x+b-m\geq x$. Thus, $x+b-m\in S\cup\{x\}$. □

Lemma 6.4. *If* $S\in \mathrm{MED}(F,m)$, *then* $X=\{x\in \mathrm{msg}(S)\mid S\backslash\{x\}\in \mathrm{MED}(F,m)\}$ *is a* $\mathrm{MED}(F,m)$*-set and* $\mathrm{MED}(F,m)[X]=S$.

Proof. It is clear that X is a $\mathrm{MED}(F,m)$-set and $X\subseteq S$. Therefore, by Proposition 6.1, we have that $\mathrm{MED}(F,m)[X]\subseteq S$.

If $T=\mathrm{MED}(F,m)[X]$ and $T\subsetneq S$, then there is $a=\min(S\backslash T)$. It is easy to deduce that $a\in \mathrm{msg}(S)$ and $m<a<F$. Note that if $S=T\cup\{s_1>s_2>\cdots>s_n\}$, then $s_n=a$. By applying repeatedly Lemma 6.3, we have that T, $T\cup\{s_1\}$, $T\cup\{s_1,s_2\},\cdots,T\cup\{s_1,s_2,\cdots,s_{n-1}\}$ are elements of $\mathrm{MED}(F,m)$. Hence, $S\backslash\{a\}=T\cup\{s_1,s_2,\cdots,s_{n-1}\}\in \mathrm{MED}(F,m)$. As $a\in \mathrm{msg}(S)$ and $S\backslash\{a\}\in \mathrm{MED}(F,m)$, we have $a\in X$. Therefore, $a\in X\subseteq T$, which is absurd. □

We end this section by giving a characterization of the minimal $\mathrm{MED}(F,m)$-system of generators of an element of $\mathrm{MED}(F,m)$.

Proposition 6.5. *If* $S\in \mathrm{MED}(F,m)$, *then* $X=\{x\in \mathrm{msg}(S)\mid S\backslash\{x\}\in \mathrm{MED}(F,m)\}$ *is the unique minimal* $\mathrm{MED}(F,m)$*-system of generators of* S.

Proof. By Lemma 6.4, we have X is a $\mathrm{MED}(F,m)$-set and $S=\mathrm{MED}(F,m)[X]$. We will finish the proof by seeing that if Y is a $\mathrm{MED}(F,m)$-set and $S=\mathrm{MED}(F,m)[Y]$, then $X\subseteq Y$. Indeed, if $X\not\subseteq Y$, then there is $x\in X\backslash Y$. Thus, $S\backslash\{x\}\in \mathrm{MED}(F,m)$ and $Y\subseteq S\backslash\{x\}$. Consequently, $S=\mathrm{MED}(F,m)[Y]\subseteq S\backslash\{x\}$, which is absurd. □

If $S\in \mathrm{MED}(F,m)$, then we denote by $\mathrm{MED}(F,m)\mathrm{msg}(S)$, the minimal $\mathrm{MED}(F,m)$-system of generators of S. The cardinality of $\mathrm{MED}(F,m)\mathrm{msg}(S)$ is called the $\mathrm{MED}(F,m)$-*rank* of S and it will be denoted by $\mathrm{MED}(F,m)\mathrm{rank}\,(S)$.

These concepts will be illustrate in the next example.

Example 6.6. Let $S=\langle 5,8,11,14,17\rangle\in \mathrm{MED}(12,5)$. It is easy to check that $\{x\in \mathrm{msg}(S)\mid S\backslash\{x\}\in \mathrm{MED}(12,5)\}=\{8\}$. By applying Proposition 6.5, we have $\mathrm{MED}(12,5)\mathrm{msg}(S)=\{8\}$ and so $\mathrm{MED}(12,5)\mathrm{rank}\,(S)=1$.

The above calculations can be performed using the gap order, `MEDRank`, which we have implemented:

```
gap> MEDRank([5,8,11,14,17],12,5);
MEDmsg= [ 8 ]
MEDrank= 1
```

7. The elements of $\mathrm{MED}(F,m)$ with fixed $\mathrm{MED}(F,m)$-rank

From [12, Proposition 11; and Lemmas 14 and 15], we can deduce the following proposition.

Proposition 7.1. *Let $S \in \mathrm{MED}(F,m)$. Then the following claims hold:*

(1) $\mathrm{MED}(F,m)\mathrm{rank}\,(S) = 0$ *if and only if* $S = \Delta(F,m)$.
(2) *If* $S \neq \Delta(F,m)$, *then* $\mathrm{r}(S) \in \mathrm{MED}(F,m)\mathrm{msg}(S)$.
(3) $\mathrm{MED}(F,m)\mathrm{rank}\,(S) = 1$ *if and only if* $\mathrm{MED}(F,m)\mathrm{msg}(S) = \{\mathrm{r}(S)\}$.

Our next purpose is to describe how we can build all the elements $S \in \mathrm{MED}(F,m)$ such that $\mathrm{MED}(F,m)\mathrm{rank}\,(S) = 1$.

Lemma 7.2. *If $r \in \mathbb{Z}$, $m < r < F$ and $F - m \notin \langle m, r-m\rangle$, then $S = \langle m, r-m\rangle \cup \{F-m+1, \rightarrow\}$ is the smallest element of $\mathscr{L}(F-m,m)$ containing $\{r-m\}$.*

Proof. It is clear that S is a numerical semigroup, $\mathrm{F}(S) = F-m$ and $m \in S$. Therefore, $S \in \mathscr{L}(F-m,m)$. If $T \in \mathscr{L}(F-m,m)$ and $\{r-m\} \subseteq T$, then $\langle m, r-m\rangle \subseteq T$ and $\{F-m+1, \rightarrow\} \subseteq T$. Thus, $S \subseteq T$ and consequently, S is the smallest element of $\mathscr{L}(F-m,m)$ containing $\{r-m\}$. □

With all these results we obtain the following characterization for the elements of $\mathrm{MED}(F,m)$ with $\mathrm{MED}(F,m)$-rank equal to 1.

Proposition 7.3. *If $r \in \mathbb{Z}$, $m < r < F$, $m \nmid r$ and $F - m \notin \langle m, r-m\rangle$, then $(\{m\} + (\langle m, r-m\rangle \cup \{F-m+1, \rightarrow\})) \cup \{0\}$ is an element belonging to $\mathrm{MED}(F,m)$ with $\mathrm{MED}(F,m)$-rank equal to 1. Moreover, every element of $\mathrm{MED}(F,m)$ with $\mathrm{MED}(F,m)$-rank equal to 1 has this form.*

Proof. Let $S = \langle m, r-m\rangle \cup \{F-m+1, \rightarrow\}$ and $T = (\{m\}+S) \cup \{0\}$. By Lemma 7.2, we know that S is the smallest element of $\mathscr{L}(F-m,m)$ which contains $\{r-m\}$. By applying Proposition 4.1, we deduce that T is the smallest element of $\mathrm{MED}(F,m)$ that contains $\{r\}$. Then, by Proposition 6.1, we have that $T = \mathrm{MED}(F,m)[\{r\}]$ and so $\mathrm{MED}(F,m)\mathrm{rank}\,(T) = 1$.

Let $P \in \mathrm{MED}(F,m)$ such that $\mathrm{MED}(F,m)\mathrm{rank}\,(P) = 1$. Then by Proposition 7.1, we know that $P = \mathrm{MED}(F,m)[\{\mathrm{r}(P)\}]$. It is clear that $m < \mathrm{r}(P) < F$ and $m \nmid \mathrm{r}(P)$. By Proposition 4.1, there exists $L \in \mathscr{L}(F-m,m)$ such that $P = (\{m\} + L) \cup \{0\}$. As P is the smallest element of $\mathrm{MED}(F,m)$ containing $\{\mathrm{r}(P)\}$, we deduce, by applying again Proposition 4.1, that L is the smallest element of $\mathscr{L}(F-m,m)$ containing $\{\mathrm{r}(P)-m\}$. Therefore, we conclude that $F-m \notin \langle m, \mathrm{r}(P)-m\rangle$ and $L = \langle m, \mathrm{r}(P)-m\rangle \cup \{F-m+1, \rightarrow\}$. □

In the following example we will illustrate the content of the previous proposition.

Example 7.4. If we consider $m = 5$, $r = 9$ and $F = 12$ in Proposition 7.3, then $(\{5\} + (\langle 4, 5\rangle \cup \{8, \rightarrow\})) \cup \{0\} = \langle 5, 9, 13, 16, 17\rangle$ is an element of $\mathrm{MED}(12, 5)$ with $\mathrm{MED}(12, 5)$-rank equal to 1.

Our next purpose will be to describe how the elements of $\mathrm{MED}(F, m)$ with $\mathrm{MED}(F, m)$-rank greater than or equal to two are.

The following result has an immediate proof.

Lemma 7.5. *If $\{n_1, \cdots, n_p\} \subseteq \mathbb{Z}$, $m < n_1 < \cdots < n_p < F$ and $F - m \notin \langle m, n_1 - m, \cdots, n_p - m\rangle$, then $\langle m, n_1 - m, \cdots, n_p - m\rangle \cup \{F - m + 1, \rightarrow\}$ is the smallest element of $\mathscr{L}(F - m, m)$ which contains $\{n_1 - m, \cdots, n_p - m\}$.*

By Propositions 4.1 and 6.1 and Lemma 7.5, we deduce the next result.

Proposition 7.6. *If $\{n_1, \cdots, n_p\} \subseteq \mathbb{Z}$, $m < n_1 < \cdots < n_p < F$ and $F - m \notin \langle m, n_1 - m, \cdots, n_p - m\rangle$, then $(\{m\} + (\langle m, n_1 - m, \cdots, n_p - m\rangle \cup \{F - m + 1, \rightarrow\})) \cup \{0\} = \mathrm{MED}(F, m)[\{n_1, \cdots, n_p\}]$.*

The next lemma has an easy proof.

Lemma 7.7. *Let X and Y be $\mathrm{MED}(F, m)$-sets with $X \subseteq Y$. Then $\mathrm{MED}(F, m)[X] \subseteq \mathrm{MED}(F, m)[Y]$.*

Lemma 7.8. *Let X be a $\mathrm{MED}(F, m)$-set and $S = \mathrm{MED}(F, m)[X]$. Then X is the minimal $\mathrm{MED}(F, m)$-system of generators of S if and only if $x \notin \mathrm{MED}(F, m)[X\backslash\{x\}]$ for every $x \in X$.*

Proof. *Necessity.* If $x \in \mathrm{MED}(F, m)[X\backslash\{x\}]$, then every element of $\mathrm{MED}(F, m)$ which contains $X\backslash\{x\}$, also contains to X. Therefore, $\mathrm{MED}(F, m)[X\backslash\{x\}] = \mathrm{MED}(F, m)[X]$. Accordingly X is not the minimal MED-system of generators of S.

Sufficiency. If $X \neq \mathrm{MED}(F, m)\mathrm{msg}(S)$, then there is $Y \subsetneq X$ such that $\mathrm{MED}(F, m)[Y] = S$. Let $x \in X\backslash Y$, then by applying Lemma 7.7, we have $x \in \mathrm{MED}(F, m)[Y] \subseteq \mathrm{MED}(F, m)[X\backslash\{x\}]$. □

With all this information, we get a new characterization of the minimal $\mathrm{MED}(F, m)$-system of generators of $\mathrm{MED}(F, m)[\{n_1, \cdots, n_p\}]$.

Proposition 7.9. *Let $\{n_1, \cdots, n_p\} \subseteq \mathbb{Z}$, $m < n_1 < \cdots < n_p < F$ and $F - m \notin \langle m, n_1 - m, \cdots, n_p - m\rangle$. Then $\{n_1, \cdots, n_p\}$ is the minimal $\mathrm{MED}(F, m)$-system of generators of $\mathrm{MED}(F, m)[\{n_1, \cdots, n_p\}]$ if and only if $\{n_1 - m, \cdots, n_p - m\} \subseteq \mathrm{msg}\left(\langle m, n_1 - m, \cdots, n_p - m\rangle\right)$.*

Proof. As a direct consequence of Lemma 7.8, we have that $\{n_1, \cdots, n_p\}$ is not the minimal $\mathrm{MED}(F, m)$-system of generators of $\mathrm{MED}(F, m)[\{n_1, \cdots, n_p\}]$ if and only if, there is $i \in \{1, \cdots, p\}$ such that $n_i \in \mathrm{MED}(F, m)[\{n_1, \cdots, n_p\} \backslash \{n_i\}]$. But, by applying now Proposition 7.6, we obtain that $n_i \in \mathrm{MED}(F, m)[\{n_1, \cdots, n_p\} \backslash \{n_i\}]$ if and only if $n_i \in \{m\} + \langle m, n_1 - m, \cdots, n_{i-1} - m, n_{i+1} - m, \cdots, n_p - m\rangle$.

Finally, it is clear that $n_i \in \{m\} + \langle m, n_1 - m, \cdots, n_{i-1} - m, n_{i+1} - m, \cdots, n_p - m\rangle$ if and only if $n_i - m \notin \mathrm{msg}\left(\langle m, n_1 - m, \cdots, n_p - m\rangle\right)$. □

Example 7.10. Taking $m = 10 < n_1 = 15 < n_2 = 17 < F = 21$, in Proposition 7.9, we have $F - m = 11 \notin \langle m, n_1 - m, n_2 - m\rangle = \langle 10, 5, 7\rangle$. As $\{5, 7\} \subseteq \mathrm{msg}(\langle 10, 5, 7\rangle)$, by Proposition 7.9, we can assert that $\{15, 17\}$ is the minimal $\mathrm{MED}(21, 10)$-system of generators of $\mathrm{MED}(21, 10)[\{15, 17\}]$.

As a consequence of Proposition 7.9, next we present a characterization of the elements of $\mathrm{MED}(F, m)$ with $\mathrm{MED}(F, m)$-rank equal to p.

Corollary 7.11. *Let* $\{n_1, \cdots, n_p\} \subseteq \mathbb{Z}$, $m < n_1 < \cdots < n_p < F$ *and* $F - m \notin \langle m, n_1 - m, \cdots, n_p - m\rangle$ *and* $\{n_1 - m, \cdots, n_p - m\} \subseteq \mathrm{msg}\left(\langle m, n_1 - m, \cdots, n_p - m\rangle\right)$. *Then* $\mathrm{MED}(F, m)[\{n_1, \cdots, n_p\}]$ *is an element of* $\mathrm{MED}(F, m)$ *with* rank *equal to* p. *Moreover, every element of* $\mathrm{MED}(F, m)$ *with* rank *equal to* p *has this form.*

Proof. The first part of Corollary follows of Proposition 7.9. Let S be an element of $\mathrm{MED}(F, m)$ such that $\mathrm{MED}(F, m)\mathrm{rank}\,(S) = p$. Then there exists a $\mathrm{MED}(F, m)$-set, X with cardinality p which verifies that X is the minimal $\mathrm{MED}(F, m)$-system of generators of S. Suppose that $X = \{n_1 < \cdots < n_p\}$. Then $m < n_1 < \cdots < n_p < F$, $F - m \notin \langle m, n_1 - m, \cdots, n_p - m\rangle$ and by applying Proposition 7.9, we conclude that $\{n_1 - m, \cdots, n_p - m\} \subseteq \mathrm{msg}\left(\langle m, n_1 - m, \cdots, n_p - m\rangle\right)$. □

The following result appears in [16, Corollary 3.2].

Lemma 7.12. *Let* S *be a* MED-*semigroup. Then* $\mathrm{F}(S) = \max(\mathrm{msg}(S)) - \mathrm{m}(S)$.

Proposition 7.13. *If* $S \in \mathrm{MED}(F, m)$, *then* $\mathrm{MED}(F, m)\mathrm{rank}\,(S) \leq m - 2$.

Proof. By Proposition 6.5, we know that $\mathrm{MED}(F, m)\mathrm{rank}\,(S)$ is the least cardinality of the set $\{x \in \mathrm{msg}(S) \mid S \backslash \{x\} \in \mathrm{MED}(F, m)\}$. As $\mathrm{e}(S) = m$, $\{m, F + m\} \subseteq \mathrm{msg}(S)$, $S \backslash \{m\} \notin \mathrm{MED}(F, m)$ and $S \backslash \{F + m\} \notin \mathrm{MED}(F, m)$, we have $\mathrm{MED}(F, m)\mathrm{rank}\,(S) \leq m - 2$. □

Remark 7.14. The previous result is also a consequence of [12, Proposition 12], where it is given an upper bound for $\mathscr{R}(F, m)(S)$.

The following example shows that the bound given in Proposition 7.13 can be achieved.

Example 7.15. Let $S = \langle 3, 7, 11 \rangle$. Then it is clear that $S \in \mathrm{MED}(8, 3)$. By applying Proposition 6.5, we obtain that $\{7\}$ is the minimal $\mathrm{MED}(8, 3)$-system of generators of S, and thus $\mathrm{MED}(8, 3)\mathrm{rank}\,(S) = 1 = \mathrm{m}(S) - 2$.

The above calculations can be performed using the following gap orders:

```
gap> IsMEDfm([3,7,11],8,3);
true
gap> MEDRank([3,7,11],8,3);
MEDmsg= [ 7 ]
MEDrank= 1
```

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