International Electronic Journal of Algebra
Published Online: August 31, 2026
DOI: 10.24330/ieja.2029225

DOUBLE COSET SCHUR RINGS AND DIAGONAL SUBGROUPS

Mahir Bilen Can and Julianne Rainbolt

Received: 19 March 2026; Revised: 29 June 2026; Accepted: 2 July 2026
Communicated by A. Çiğdem Özcan Çevik

Abstract. We investigate the structure of Schur rings arising from group triples (G,H,K), where $K \trianglelefteq H \leq G$, building on foundational work by Brender and Travis. Central to our approach is the interplay between Schur rings and (strong) Gelfand pairs. We introduce certain double coset algebras, and illustrate our findings with explicit computations for the general linear group $GL(2,\mathbb{F}_q)$, and its Borel subgroup. Finally, we bridge our algebraic results with combinatorial design theory by demonstrating that the structure constants of these double coset algebras yield the exact intersection numbers for novel, coarser (in the sense of fewer and larger classes) association schemes.

Mathematics Subject Classification (2020): 20C15, 16S34, 05E30
Keywords: Schur ring (S-ring), double coset algebra, Hecke algebra, association scheme, Bose-Mesner algebra

1. Introduction

Schur rings, introduced by Schur in his study of permutation groups [16], offer a powerful algebraic framework for capturing symmetries in finite groups. Roughly speaking, a Schur ring over a finite group G is a subring of the group algebra $\mathbb{C}[G]$ that reflects a partition of G into symmetric subsets closed under taking inverses. This perspective connects naturally with association schemes, algebraic graph theory, and representation theory [1,20]. Recall, for a subgroup K of G, a pair (G,K) is called a *Gelfand pair* if the convolution algebra of K-bi-invariant functions on G is commutative. In this article, we introduce a natural family of Schur rings associated to any finite group G, constructed via the diagonal subgroup of $G \times G$. This approach is motivated by the structure theory of Gelfand pairs, which play a central role in harmonic analysis on finite groups.

Let K and H be subgroups of a finite group G, with K normal in H. In [2], Brender associates to the triple (G,H,K) a Schur ring $\mathbb{C}[G]_K^H$ with a specified basis $\{\tau_i\}_{i=1}^s$, satisfying the following properties:

(a) $\mathbb{C}[G]_K^H$ centralizes every element of H, and
(b) the identity element of $\mathbb{C}[G]_K^H$ is given by $\frac{\tau_1}{|K|} = \sum_{k\in K} \frac{k}{|K|}$.

Moreover, any Schur algebra over G that satisfies properties (a) and (b) is a Schur subalgebra of $\mathbb{C}[G]_K^H$.

Building on Brender's result, and using the fact that for any finite group G, the pair $(G \times G, G^*)$, where G^* denotes the diagonal subgroup of $G \times G$, is a Gelfand pair, we establish the following theorem, which constitutes the main result of this article.

Theorem 1.1. *Let M be a proper subgroup of $G \times G$ such that G^* is normal in M. Then $\mathbb{C}[G \times G]_{G^*}^M$ is isomorphic to a unital central subalgebra of $\mathbb{C}[G]$.*

Needless to say, the unital central subalgebra of $\mathbb{C}[G]$ referenced in the theorem is itself a Schur ring. An interesting consequence of this result is the following.

Corollary 1.2. *Let G be a finite group. If M is a subgroup of $G \times G$ such that the diagonal subgroup G^* of $G \times G$ is a normal subgroup of M, then M/G^* is an abelian group.*

In the following, the double coset algebra $\mathscr{D}(G)$ refers to the convolution algebra of complex-valued functions $f : G \times G \to \mathbb{C}$ that are constant on each double coset of G^* in $G \times G$. The algebra $\mathscr{C}(G)$, on the other hand, denotes the convolution algebra of complex-valued functions $g : G \to \mathbb{C}$ that are constant on each conjugacy class of G.

As previously noted, the proof of Theorem 1.1 relies on the fact that $(G \times G, G^*)$ forms a Gelfand pair. A closely related, and perhaps widely known though often unstated, observation is the following.

Proposition 1.3. *Let G be a finite group. Then the double coset algebra, $\mathscr{D}(G)$, of the Gelfand pair $(G \times G, G^*)$ is isomorphic to $\mathscr{C}(G)$.*

Another purpose of this article is to document applications of our results with some important and concrete examples. Applications of the above results are demonstrated when $G = GL(2, \mathbb{F}_q)$, the general linear group where $\mathbb{F}_q$ is a finite field with q elements (where q is a power of a prime). Specifically, we determine all distinct (M, G^*) double cosets in $G \times G$, where M is the subgroup of $G \times G$ generated by the elements of the form

$$(g, g) \quad \text{and} \quad (z, e), \qquad \text{where } g \in G,\ z \in Z(G).$$

Here $Z(G)$ denotes the center of G and, as above, G^* denotes the diagonal subgroup of $G \times G$. Note that M contains G^* as a normal subgroup. In addition, we also consider the example where instead $G = B$, the Borel subgroup of $GL(2, \mathbb{F}_q)$ consisting of upper triangular invertible 2×2 matrices with entries from $\mathbb{F}_q$. For this family of groups, we work out in detail not only all distinct (M, B^*) double cosets in $B \times B$ (where now M is the subgroup of $B \times B$ generated by the elements of the form (g, g) and (z, e) where $g \in B$, $z \in Z(B)$) but also the structure constants (in Proposition 6.6) of the Schur ring $\mathbb{C}[B \times B]_{B^*}^{M}$.

We conclude this introduction by outlining the structure of this article. In Section 2, we review the necessary background on Schur rings, Gelfand pairs, and double coset algebras. Section 3 forms the core of our work: we revisit and extend some results of Brender [2] and Travis [18], prove Proposition 1.3, and highlight the role of strong Gelfand pairs in our framework. (By a *strong Gelfand pair*, we mean a pair of finite groups (G, K) with $K \leq G$, such that for every irreducible representation V of K, the induced representation $\mathrm{ind}_K^G(V)$ is multiplicity-free, that is, each irreducible representation W of G appears at most once as a summand in this induced representation.) In this case we call K a strong Gelfand subgroup. We note in passing that the concept of strong Gelfand subgroups traces back to the foundational work of Gelfand and Cetlin [6,7]. In Section 4, we carry out a detailed example computation for the group $GL(2, \mathbb{F}_q)$. Section 5 contains the proof of Theorem 1.1 and its Corollary 1.2. In Section 6, we then analyze the structure of the double coset algebra $\mathbb{C}[B \times B]_{B^*}^{M}$ in depth. The article concludes in Section 7 where we bridge our algebraic results with combinatorial design theory. Specifically, we demonstrate that the structure constants of the double coset algebras computed in Section 6 yield the exact intersection numbers for a novel, coarser (in the sense of fewer and larger classes) association scheme on the Borel subgroup. This explicit construction provides a natural algebraic mechanism for generating commutative Bose-Mesner algebras from the representation theory of finite groups, complementing other foundational works (for example in [19]).

2. Preliminaries

We begin with Tamaschke's definition [17] of a Schur ring. Let G be a finite group. For a subset S of G, let S^{-1} denote the set $\{g^{-1} \mid g \in S\}$. A *Schur ring on* G is a subring T of the group ring $\mathbb{Z}[G]$ such that there exists a decomposition of G into disjoint nonempty subsets,

$$G = \mathscr{T}_1 \sqcup \cdots \sqcup \mathscr{T}_t,$$

for which the following properties hold:

(1) For every $i \in \{1, \ldots, t\}$, there exists a $j \in \{1, \ldots, t\}$ such that $\mathscr{T}_i^{-1} = \mathscr{T}_j$.
(2) The elements $\tau_1, \ldots, \tau_t$, where $\tau_i := \sum_{g \in \mathscr{T}_i} g$, form a $\mathbb{Z}$-basis for T.

As a convention, it is assumed that $\mathscr{T}_1$ contains the identity element of G.

As an important note, the more modern definition of Schur ring includes an additional third property (see for example [1] and [15, p. 4]). Namely the property

(3) $\mathscr{T}_1 = \{e\}$ where e denotes the identity element in G.

Even though this third property is assumed in most modern articles about Schur rings, we will not assume it in the following, as it is not needed to prove the results presented in this article. We note that algebraic structures satisfying only properties (1) and (2) are also frequently referred to in the literature as *Hecke algebras* or *adjacency algebras of association schemes* (see, for example, Evdokimov and Ponomarenko [5]). Furthermore, even though we do not assume property (3), we always have that $\mathscr{T}_1$ is a subgroup of G. (This fact was observed by Tamaschke in [17].)

Let F be a field. Let T be a Schur ring over G. We call $R := F \otimes_{\mathbb{Z}} T$ a *Schur F-algebra on G*. If F is evident from the context, we call R simply a *Schur algebra on G*.

We now discuss a special family of Schur algebras.

Let H be a subgroup of G. Let us denote by $\mathbb{C}[G]^H$ the centralizer in $\mathbb{C}[G]$ of $\mathbb{C}[H]$, that is,

$$\mathbb{C}[G]^H := \{x \in \mathbb{C}[G] \mid \ xy = yx \text{ for every } y \in \mathbb{C}[H]\}.$$

We state some important remarks.

Remark 2.1. Let H be a subgroup of a finite group G.

(1) The centralizer $\mathbb{C}[G]^H$ is a $\mathbb{C}$-subalgebra of $\mathbb{C}[G]$.
(2) If $H = G$, then $\mathbb{C}[G]^H$ is nothing but the center of the group ring. In other words, we have $\mathbb{C}[G]^G = Z(\mathbb{C}[G])$.
(3) The inclusion $\mathbb{C}[G]^G \subseteq \mathbb{C}[G]^H$ always holds, though the reverse inclusion is not always true.

More generally, if $\Gamma \leq \operatorname{Aut}(G)$, then Γ acts on G, and the sums of the Γ-orbits span a Schur algebra, often called the *orbit Schur ring* associated with Γ. In the present situation, the subgroup H acts on G by inner automorphisms, equivalently through the automorphic subgroup $\operatorname{Inn}(H) \leq \operatorname{Aut}(G)$. The algebra $\mathbb{C}[G]^H$ is the fixed subalgebra for this action, hence is exactly the orbit Schur algebra for $\operatorname{Inn}(H)$.

The following lemma is well known (see, for example, [15]), however a sketch of the proof is included for completeness.

Lemma 2.2. *Let H be a subgroup of a finite group G. Then $\mathbb{C}[G]^H$ is a Schur algebra relative to G.*

Proof. By the preceding paragraph, the basic sets are the H-conjugacy orbits $K(g) = \{hgh^{-1} \mid h \in H\}$ in G. These orbits are stable under inversion, since $K(g)^{-1} = K(g^{-1})$. Their orbit sums form a basis of the fixed subalgebra: an element $\sum_{g\in G} a_g g \in \mathbb{C}[G]$ is fixed by conjugation by H if and only if the coefficient a_g is constant on each H-conjugacy orbit. Thus $\mathbb{C}[G]^H$ is the orbit Schur algebra for $\mathrm{Inn}(H)$. □

2.1. Gelfand pairs and the double coset algebra. Given a subgroup H of G, recall the pair (G, H) is called a *Gelfand pair* if the restriction of any irreducible representation of G to H contains the trivial representation of H with multiplicity at most one. Equivalently, (G, H) is a Gelfand pair if the trivial representation of H induced to G is multiplicity-free. If every irreducible character of H induces to a multiplicity-free character of G, the pair (G, H) is called a *strong Gelfand pair*.

Given a subgroup H, partition G into its double cosets of H in G:

$$G = \mathscr{T}_1 \sqcup \cdots \sqcup \mathscr{T}_s,$$

where we will take $\mathscr{T}_1 = HeH = H$. Note that if the sums $\tau_1, \ldots, \tau_t$, where $\tau_i := \sum_{g\in\mathscr{T}_i} g$, form a $\mathbb{Z}$-linear basis for a subring T of $\mathbb{Z}G$, then T is a Schur ring on G. Although the double coset algebra is defined over $\mathbb{Z}$, it is more convenient to work with rational coefficients. In fact, we will work with the normalized basis

$$\frac{1}{|H|}\tau_1, \ldots, \frac{1}{|H|}\tau_t. \tag{2.1}$$

Lemma 2.3. *Let H be a subgroup of the finite group G. Then the sums over the double H-cosets in G, equivalently the normalized sums in Equation* (2.1)*, span a Schur algebra on G. If, in addition, (G, H) is a Gelfand pair, then this double coset algebra is commutative.*

Proof. The double H-cosets partition G, contain H as the identity-containing part, and are closed under inversion since $(HxH)^{-1} = Hx^{-1}H$. It remains to check closure under multiplication. For $x, y \in G$, write

$$\widehat{HxH}\widehat{HyH} = \sum_{z\in G} m_z z,$$

where m_z counts the number of pairs $(a, b) \in HxH \times HyH$ with $ab = z$. Since $HxHHyH$ is a union of double H-cosets, $m_z = 0$ outside that union. Moreover, if $z' = h_1 z h_2$ with $h_1, h_2 \in H$, then $(a, b) \mapsto (h_1 a, b h_2)$ gives a bijection between the pairs contributing to m_z and those contributing to $m_{z'}$. Hence m_z is constant on double H-cosets, and

$$\widehat{HxH}\widehat{HyH} = \sum_z m_z \widehat{HzH},$$

where z runs over double H-coset representatives. Thus the double coset sums span a Schur algebra, and the normalized sums in Equation (2.1) span the same algebra over $\mathbb{C}$. The Gelfand-pair hypothesis is only needed for commutativity, which follows from the usual multiplicity-free criterion for the double coset algebra. $\square$

Note that the double coset algebra is commutative when (G, H) is a Gelfand pair, but this does not necessarily imply that the double coset algebra is in the center of the group algebra.

3. Travis and Brender's works

We begin with summarizing some important results of Travis [18] and Brender [2]. These results will be used below to relate double coset algebras to multiplicity-free restrictions.

As above, let K and H be subgroups of G such that $K \trianglelefteq H \subseteq G$. Note that the semidirect product $K \rtimes H$ acts on G by $(k,h)g = khgh^{-1}$ for all $k \in K, h \in H, g \in G$. Let (g) denote the orbit of $g \in G$ under this action and partition G by these orbits:

$$G = \sqcup_{i=1}^{s}(g_i)$$

where s is the number of orbits and $\{g_i \mid 1 \leq i \leq s\}$ is a set of orbit representatives. Each orbit will be both a union of H conjugacy classes in G and a union of right cosets of K in G. For $i = 1, \dots, s$, let

$$\tau_i = \sum_{g \in (g_i)} g.$$

The following is Proposition 2.3 of [2].

Proposition 3.1. *Using the notation in the preceding paragraph, the τ_i's span a Schur ring T. Furthermore the following properties hold.*

(a) *The Schur algebra $\mathbb{C}T$ centralizes every element in H. In other words, we have*

$$\mathbb{C}T \subseteq \mathbb{C}[G]^H.$$

(b) *The identity element of $\mathbb{C}T$ is $\frac{\tau_1}{|K|} = \sum_{k \in K} \frac{k}{|K|}$.*

Furthermore, every Schur algebra over G for which both (a) and (b) hold is a Schur subalgebra of $\mathbb{C}T$.

The Schur algebra described in Proposition 3.1 will be denoted by $\mathbb{C}[G]^H_K$.

With this notation, one may also view Brender's algebra as the intersection

$$\mathbb{C}[G]^H_K = \mathbb{C}[G]^H \cap \mathbb{C}[G]^K_K.$$

Indeed, the basic sets of $\mathbb{C}[G]_K^H$ are precisely the common unions determined by the H-conjugation action and the K-double-coset/right-coset structure; equivalently, they are the orbit sums for the combined action used by Brender. Thus $\mathbb{C}[G]_K^H$ may be viewed as the intersection of these two Schur algebras.

We proceed with presenting some basic observations regarding the relationship between various subalgebras of $\mathbb{C}[G]$. We summarize the orbit-refinement consequences in the following proposition.

Proposition 3.2. *Let $K_1 \subseteq H_1$ and $K_2 \subseteq H_2$ be two nested pairs of subgroups of G. Assume that each algebra $\mathbb{C}[G]_K^H$ displayed below is defined; in particular, the relevant subgroup K is normal in H. Then the following assertions hold:*

(1) *If $K_1 \subseteq K_2$, then $\mathbb{C}[G]_{K_2}^{H_2} \subseteq \mathbb{C}[G]_{K_1}^{H_2}$ as subalgebras. In particular, $\mathbb{C}[G]_{K_2}^{H_2}$ is contained in the centralizer of H_2 in $\mathbb{C}[G]$.*
(2) *If $H_1 \subseteq H_2$, then $\mathbb{C}[G]_{K_1}^{H_2} \subseteq \mathbb{C}[G]_{K_1}^{H_1}$. In particular, $\mathbb{C}[G]_{H_1}^{H_2}$ is contained in the double coset algebra $\mathbb{C}[G]_{H_1}^{H_1}$.*
(3) *The algebra $\mathbb{C}[G]_{H_i}^{G}$ $(i \in \{1, 2\})$ is contained in the center of $\mathbb{C}[G]$.*

Proof. By the intersection and orbit-refinement description above, enlarging K or enlarging H coarsens the relevant orbit partition, and the sums over coarser orbits are sums of the finer orbit sums. Thus, if $K_1 \subseteq K_2$, the partition defining $\mathbb{C}[G]_{K_2}^{H_2}$ is coarser than the partition defining $\mathbb{C}[G]_{K_1}^{H_2}$, giving $\mathbb{C}[G]_{K_2}^{H_2} \subseteq \mathbb{C}[G]_{K_1}^{H_2}$. The containment $\mathbb{C}[G]_{K_2}^{H_2} \subseteq \mathbb{C}[G]^{H_2}$ follows at once from $\mathbb{C}[G]_{K_2}^{H_2} = \mathbb{C}[G]^{H_2} \cap \mathbb{C}[G]_{K_2}^{K_2}$.

Similarly, if $H_1 \subseteq H_2$, then the partition defining $\mathbb{C}[G]_{K_1}^{H_2}$ is coarser than the partition defining $\mathbb{C}[G]_{K_1}^{H_1}$, so $\mathbb{C}[G]_{K_1}^{H_2} \subseteq \mathbb{C}[G]_{K_1}^{H_1}$. Taking $K_1 = H_1$ gives the stated containment in the double H_1-coset algebra. Finally, $\mathbb{C}[G]_{H_i}^{G} \subseteq \mathbb{C}[G]^G = Z(\mathbb{C}[G])$ by the same intersection description. $\square$

Remark 3.3. In [2, Proposition 2.4], it is observed by Brender that if K_1 is a normal subgroup of K_2, then $\mathbb{C}[G]_{K_2}^{H_2}$ is a two-sided ideal of $\mathbb{C}[G]_{K_1}^{H_2}$. It follows from this observation at once that the double coset algebra $\mathbb{C}[G]_H^H$ is a two-sided ideal of $\mathbb{C}[G]_K^H$.

It is well-known that the set of all irreducible complex linear representations of G is in a bijection with the set of all simple (left) $\mathbb{C}G$-modules. Following an article of Karlof [12], we briefly review the theory of "irreducible representations" of $\mathbb{C}[G]^H$. Let $\{M_1, \dots, M_s\}$ be the set of all simple $\mathbb{C}G$-modules. Let χ_j $(1 \leq j \leq s)$ denote the character of the corresponding irreducible representation of G. Similarly, let $\{N_1, \dots, N_t\}$ be the set of all simple $\mathbb{C}H$-modules. Let ϕ_i $(1 \leq i \leq t)$ denote the characters of the corresponding irreducible representations of H. For $j \in \{1, \dots, s\}$

and $i \in \{1, \dots, t\}$, we define the integer c_{ij} by the restriction of χ_j to H:

$$\operatorname{res}^G_H(\chi_j) := \sum_{i=1}^{t} c_{ij}\phi_i. \tag{3.1}$$

Let $\{e_1, \dots, e_t\}$ be a set of primitive orthogonal idempotents of $\mathbb{C}H$ such that

$$N_i \cong \mathbb{C}He_i \qquad (1 \le i \le t).$$

In [18], Travis shows that M is a simple $\mathbb{C}[G]^H$-module if and only if M is of the form

$$M = e_i M_j, \quad \text{ where } \quad c_{ij} \neq 0,$$

for $1 \le i \le t$ and $1 \le j \le s$. Here, c_{ij}'s are as in (3.1). Moreover, the corresponding irreducible character is given by

$$\phi_{ij}(O(g)) := \frac{|H \cdot g|}{|H|} \sum_{h \in H} \chi_j(gh)\phi_i(h^{-1}).$$

Definition 3.4. For $1 \le i \le t$ and $1 \le j \le s$, the *spherical function associated with the pair* (ϕ_i, χ_j) is the function $Y_{ij} : G \to \mathbb{C}$ defined by

$$Y_{ij}(g) := \frac{1}{|H|} \sum_{h \in H} \chi_j(gh)\phi_i(h^{-1}) = \frac{1}{|H \cdot g|}\phi_{ij}(O(g)).$$

It is not difficult to check these:

$$\begin{aligned} Y_{ij}(g^{-1}) &= \overline{Y_{ij}(g)} \quad \text{ for all } g \in G, \\ Y_{ij}(hgh^{-1}) &= Y_{ij}(g) \quad \text{ for all } h \in H,\ g \in G. \end{aligned}$$

As a corollary of [18, Theorem 6], Travis obtains the following result:

Lemma 3.5. *The spherical functions Y_{ij} $(1 \le i \le t$ and $1 \le j \le s)$ generate the center of $\mathbb{C}[G]^H$.*

The following result is Proposition 3.2 of [2].

Proposition 3.6. *Let K and H be subgroups of G such that $K \trianglelefteq H \subseteq G$. For $\chi_i \in \operatorname{Irr}(G)$ and for $\psi_j \in \operatorname{Irr}(H)$ such that K is in the kernel of ψ_j, the set of restrictions of the spherical functions, Y_{ij}, to $\mathbb{C}[G]^H_K$ is a complete set of irreducible characters of $\mathbb{C}[G]^H_K$.*

Given a subgroup H of G, $\chi \in \operatorname{Irr}(G)$ and $\psi \in \operatorname{Irr}(H)$, let $c_{\chi\psi}$ denote the multiplicity of ψ in the restriction of χ to H. Let $\overline{H/K}$ denote the characters of H which have K in the kernel. We say H *is multiplicity-free on* K if for all $\chi \in \operatorname{Irr}(G)$ and for all $\psi \in \overline{H/K}$, $c_{\chi\psi}$ is zero or one. The following is Proposition 4.1 of [2].

Proposition 3.7. *The Schur algebra $\mathbb{C}[G]^H_K$ is commutative if and only if H is multiplicity-free on K.*

For $K = H$, the algebra $\mathbb{C}[G]_H^H$ is the double H-coset algebra. The Schur ring assertion was proved in Section 2; the Gelfand-pair hypothesis gives commutativity by the usual multiplicity-free criterion, equivalently by the preceding proposition with $K = H$.

Brender's main motivation for introducing the rings $\mathbb{C}[G]_K^H$ is the following result of Travis.

Proposition 3.8. [18, Corollary 1] *Let H be a subgroup of G. Then $\mathbb{C}[G]^H$ is a commutative Schur algebra if and only if H is a multiplicity-free subgroup in G.*

Recall that $\mathbb{C}[G]^H$ is the Schur ring generated by the H-conjugacy class sums in $\mathbb{C}[G]$. However, as explained above, a multiplicity-free subgroup (or, a strong Gelfand subgroup) is a subgroup $H \subset G$ such that the restriction of every irreducible character $\chi \in \mathrm{Irr}(G)$ to H is a multiplicity-free character of H. Notice that if H is a strong Gelfand subgroup, then it is a Gelfand subgroup. Hence, the Schur ring $\mathbb{C}[G]_K^H$ is commutative.

An important special case: the diagonal subgroup. As above, let G denote a finite group and let G^* denote the diagonal subgroup of the direct product $G \times G$. We have $(G \times G, G^*)$ is a Gelfand pair (see for example [13, Example 9, p. 396]).

We will describe the double coset algebra of the pair $(G \times G, G^*)$. Recall this algebra will be denoted by $\mathscr{D}(G)$ and the central algebra with linear basis the sums of the conjugacy class elements in G will be denoted by $\mathscr{C}(G)$.

For $g \in G$, we denote the conjugacy class containing g by $\mathbf{C}(g)$ and the sum of the elements in the conjugacy class $\mathbf{C}$ by

$$\widehat{\mathbf{C}} := \sum_{a \in \mathbf{C}} a.$$

As the following lemma is clear and well-known, the proof is omitted.

Lemma 3.9. *Let $\mathcal{C}(G)$ denote the set of all conjugacy classes of G. Then $\mathscr{C}(G)$ is generated by the elements $\{\widehat{\mathbf{C}} \mid \mathbf{C} \in \mathcal{C}(G)\}$.*

The following notation will be useful. For $g \in G$, we denote by $\mathbf{D}(g)$ the double G^*-coset of (g, e) in $G \times G$ and we use $\widehat{\mathbf{D}(g)}$ to denote the corresponding element

$$\widehat{\mathbf{D}(g)} := \sum_{(z,w) \in \mathbf{D}(g)} (z, w)$$

in $\mathbb{C}[G \times G]$. Note $\mathscr{D}(G)$ has a linear basis the distinct elements in the set $\{\widehat{\mathbf{D}(g)} \mid g \in G\}$.

We are now ready to prove Proposition 1.3. We recall its statement for convenience.

The double coset algebra, $\mathscr{D}(G)$, of the Gelfand pair $(G\times G, G^*)$ is isomorphic to $\mathscr{C}(G)$.

Proof of Proposition 1.3. First we describe a bijection between the conjugacy classes in G and the double cosets of G^* in $G\times G$.

We claim every double G^*-coset $G^*(x,y)G^*$, where $(x,y)\in G\times G$ can be written in the form $G^*(g,e)G^*$ for some $g\in G$. Indeed, clearly, we have $G^*(x,y)G^* = G^*(xy^{-1},e)(y,y)G^* = G^*(xy^{-1},e)G^*$. Letting $g:=xy^{-1}$ finishes the proof of our claim. Next, we observe that, for every g and a from G, we have

$$G^*(aga^{-1},e)G^* = G^*(a,a)(g,e)(a^{-1},a^{-1})G^* = G^*(g,e)G^*.$$

At the same time, if h and g are not conjugate to each other, but $G^*(g,e)G^* = G^*(h,e)G^*$ was true, then we would obtain the equality,

$$(x,x)(g,e)(y,y) = (h,e),$$

for some $x,y\in G$. But this implies that $x=y^{-1}$, and $xgx^{-1}=h$ which is absurd. These arguments show that two elements (x_1,x_2) and (y_1,y_2) lie in the same double G^*-coset if and only if $x_1x_2^{-1}$ and $y_1y_2^{-1}$ are conjugate in G. Let $\mathbf{D}$ be a double G^*-coset in $G\times G$. Then there exists $g\in G$ such that $\mathbf{D}=\mathbf{D}(g)$. We conclude from these observations that the following map is surjective:

$$\begin{aligned}\varphi_g : \mathbf{D}(g) &\longrightarrow \mathbf{C}(g)\\ (x,y) &\longmapsto xy^{-1}.\end{aligned} \tag{3.2}$$

We claim that the preimage of every element $aga^{-1}\in\mathbf{C}(g)$ under φ_g has exactly $|G|$ elements. Indeed, for every $b\in G$, we have (agb,ab) is mapped onto aga^{-1}. To show that there are no other elements in the fiber we apply a counting argument. Let $g_1,\dots,g_m$ be a list of mutually distinct representatives of conjugacy classes of G. Then our previous arguments show that we have a disjoint union $\mathbf{D}(g_1)\sqcup\cdots\sqcup\mathbf{D}(g_m)$. Furthermore, we have

$$G\times G = \mathbf{D}(g_1)\sqcup\cdots\sqcup\mathbf{D}(g_m),$$

and the maps $\varphi_{g_i} : \mathbf{D}(g_i)\to\mathbf{C}(g_i)$, for $i=1,\dots,m$, can be combined to give a global surjective map, $\varphi : G\times G\to G,\ (x,y)\mapsto xy^{-1}$. It follows that

$$|G\times G| = \sum_{i=1}^m |\varphi^{-1}(\mathbf{C}(g_i))| = \sum_{i=1}^m |\varphi_g^{-1}(\mathbf{C}(g_i))| \geq \sum_{i=1}^m |G||\mathbf{C}(g_i)|.$$

Since $\sum_{i=1}^m |\mathbf{C}(g_i)| = |G|$, the last inequality must be an equality. In particular, for every $i\in\{1,\dots,m\}$, the fibers of φ_{g_i} have exactly $|G|$ elements. This finishes the proof of our claim.

By Lemma 3.9, the center of $\mathbb{C}[G]$ is generated by the sums $\widehat{\mathbf{C}(g_i)}$, $i = 1, \ldots, m$. We proceed to show that the following map, extended by linearity, is an algebra isomorphism:

$$\begin{aligned} \Phi : Z(\mathbb{C}[G]) &\longrightarrow \mathbb{C}[G \times G]_{G^*}^{G^*} \\ \widehat{\mathbf{C}(g)} &\longmapsto \frac{1}{|G|}\widehat{\mathbf{D}(g)}. \end{aligned} \tag{3.3}$$

It is evident from our previous discussion that Φ is already a vector space isomorphism. Hence, it suffices to show that it is an algebra homomorphism.

We note that, for $g \in G$, the image of $\widehat{\mathbf{C}(g)}$ is given by

$$\begin{aligned}
\Phi(\widehat{\mathbf{C}(g)}) &= \frac{1}{|G|}\widehat{\mathbf{D}(g)} \\
&= \frac{1}{|G|} \sum_{\substack{b \in G, \\ aga^{-1} \in \mathbf{C}(g)}} (aga^{-1}b, b) \\
&= \frac{1}{|G|} \sum_{\substack{b \in G, \\ aga^{-1} \in \mathbf{C}(g)}} (aga^{-1}, e)(b, b) \\
&= \frac{1}{|G|} \left(\sum_{aga^{-1} \in \mathbf{C}(g)} (aga^{-1}, e) \right) \left(\sum_{b \in G} (b, b) \right) \\
&= \frac{1}{|G|} \sum_{aga^{-1} \in \mathbf{C}(g)} (aga^{-1}, e)\widehat{G}^* \\
&= \sum_{aga^{-1} \in \mathbf{C}(g)} (aga^{-1}, e) \qquad \text{(since the inner sum is independent of } b \in G.) \\
&= (\sum_{aga^{-1} \in \mathbf{C}(g)} aga^{-1}, e) \\
&= (\widehat{\mathbf{C}(g)}, e),
\end{aligned}$$

where $\widehat{G}^* = \frac{1}{|G|} \sum_{g \in G}$ is an idempotent which commutes with each $\mathbf{C}(g)$.

Now let g and h be two elements from G. Then we have

$$\widehat{\mathbf{C}(g)}\widehat{\mathbf{C}(h)} = \sum_{\substack{aga^{-1} \in \mathbf{C}(g), \\ chc^{-1} \in \mathbf{C}(h)}} aga^{-1}chc^{-1} = \sum_{i=1}^{m} a_i \widehat{\mathbf{C}(g_i)}$$

where $a_i \in \mathbb{C}$. Hence, by linearity, we have

$$\Phi(\widehat{\mathbf{C}(g)}\widehat{\mathbf{C}(h)}) = \sum_{i=1}^{m} a_i \Phi(\widehat{\mathbf{C}(g_i)}).$$

Next, we look closely at the corresponding product $\Phi(\widehat{\mathbf{D}(g)})\Phi(\widehat{\mathbf{D}(h)})$ in $\mathbb{C}[G \times G]_{G^*}^{G^*}$. Previously, we observed that $\widehat{\mathbf{D}(g)}$ is given by

$$\widehat{\mathbf{D}(g)} = |G| \sum_{aga^{-1}\in \mathbf{C}(g)} (aga^{-1}, e).$$

Likewise, we have $\widehat{\mathbf{D}(h)} = |G| \sum_{chc^{-1}\in \mathbf{C}(g)} (chc^{-1}, e)$. Hence, the product we are interested in is given by

$$\begin{aligned}
\Phi(\widehat{\mathbf{C}(g)})\Phi(\widehat{\mathbf{C}(h)}) &= \left(\sum_{aga^{-1}\in \mathbf{C}(g)} (aga^{-1}, e) \right) \left(\sum_{chc^{-1}\in \mathbf{C}(h)} (chc^{-1}, e) \right) \\
&= \sum_{\substack{aga^{-1}\in \mathbf{C}(g),\\ chc^{-1}\in \mathbf{C}(h)}} (aga^{-1}chc^{-1}, e) \\
&= (\sum_{\substack{aga^{-1}\in \mathbf{C}(g),\\ chc^{-1}\in \mathbf{C}(h)}} aga^{-1}chc^{-1}, e) \\
&= (\widehat{\mathbf{C}(g)}\widehat{\mathbf{C}(h)}, e) \\
&= (\sum_{i=1}^{m} a_i \widehat{\mathbf{C}(g_i)}, e) \\
&= \Phi \left(\sum_{i=1} a_i \widehat{\mathbf{C}(g_i)} \right) \qquad \text{(by the linearity of } \Phi) \\
&= \Phi(\widehat{\mathbf{C}(g)}\widehat{\mathbf{C}(h)}).
\end{aligned}$$

This finishes the proof. □

In the context of diagonal subgroups, we find the following Schur rings quite interesting: Let M be a subgroup of $G \times G$ such that

$$G^* \lhd M \subset G \times G \qquad \text{and} \qquad M \notin \{G^*, G \times G\}.$$

By Theorem 3.2, we see that

$$\mathbb{C}[G \times G]^{M}_{G^*} \subseteq \mathbb{C}[G \times G]^{G^*}_{G^*}.$$

Since the bigger ring is isomorphic to the center of the group ring of G, we see that $\mathbb{C}[G \times G]^{M}_{G^*}$ is isomorphic to a central Schur ring.

Example 3.10. Let G be a non-abelian group with nontrivial center, $Z(G)$. Let M denote the subgroup of $G \times G$ defined by

$$M := \langle (g, g), (z, e) \mid g \in G,\ z \in Z(G) \rangle.$$

When M is defined in this way we denote the algebra $\mathbb{C}[G \times G]^{M}_{G^*}$ by $\mathscr{E}(G)$. Evidently, M is properly contained between G^* and $G \times G$, and G^* is normal in M. (In Sections 4 and 6 we will fix a particular such subgroup M and provide details of the above and below results for specific families of groups.)

We proceed with the more general assumption that M is a proper subgroup of $G \times G$ such that $G^* \lhd M$. For $g \in G$, we define

$$\mathbf{E}(g) := M(g,1)G^*. \tag{3.4}$$

We will denote by $\widehat{\mathbf{E}(g)}$ the corresponding element

$$\widehat{\mathbf{E}(g)} := \sum_{(a,b)\in\mathbf{E}(g)} (a,b)$$

in $\mathbb{C}[G \times G]$. Since M is a disjoint union of (G^*, G^*) double cosets, we know that every $\mathbf{E}(g)$ is a union of (G^*, G^*) double cosets. For example, if M is obtained from the center of G as in Example 3.10, then we see that

$$\mathbf{E}(g) = M(g,1)G^* = \bigcup_{i=1}^{r}(z,1)G^*(g,1)G^* = \bigcup_{z\in Z(G)}(z,1)\mathbf{D}(g). \tag{3.5}$$

In particular, if $\mathbf{D}(g_1), \mathbf{D}(g_2), \ldots, \mathbf{D}(g_r)$ are the distinct (G^*, G^*) double cosets in $G\times G$, then the $\mathbf{E}(g_i) = \bigcup_{z\in Z(G)}(z,1)\mathbf{D}(g_i)$, for $i = 1, \ldots, r$ are the (not necessarily distinct) (M, G^*) double cosets in $G \times G$. However, notice that it is possible that $\mathbf{D}(z_1 g) = \mathbf{D}(z_2 g)$ for some $g \in G$ and $z_1, z_2 \in Z(G)$ with $z_1 \neq z_2$. Thus this union, in the description of $\mathbf{E}(g)$ in (3.5), is not necessarily a disjoint union. However the distinct sums $\widehat{\mathbf{E}(g)}$ for $g \in G$ form a linear basis of $\mathscr{E}(G)$.

4. Specific example where $G = GL(2, \mathbb{F}_q)$

In this section, M will be defined as in Example 3.10. Let q be a power of a prime and let $G = GL(2, \mathbb{F}_q)$. In this example, we will exhibit the bases of our Schur algebras $\mathscr{C}(G), \mathscr{D}(G)$, and $\mathscr{E}(G)$ for $G = GL(2, \mathbb{F}_q)$. We begin with a brief review of the conjugacy classes in G. There is a simple description of these classes that follows immediately from [4, Sec. 12.2, Theorem 17] of rational canonical forms. Indeed, an element $A \in G$, viewed as a linear transformation on $\mathbb{F}_q^2$, can have at most two invariant factors as in [4, Sec. 12.2, Theorem 14]. The invariant factor with the highest degree is the minimal polynomial $m_A(x)$ of A. It is a polynomial in $\mathbb{F}_q[x]$. Since the degree of $m_A(x)$ is either 1 or 2, its splitting into factors over $\mathbb{F}_q$ or $\mathbb{F}_{q^2}$ completely determines the conjugacy class of A. We will list all possibilities for a generic matrix A from G.

For shorthand notation denote

$$[a,b] := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } [a,a]' := \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

Let s denote a fixed element in $\mathbb{F}_{q^2}$ that generates the multiplicative group $\mathbb{F}_{q^2}^*$. Let $r = s^{q+1}$, so that r generates the multiplicative group $\mathbb{F}_q^*$. For $j \in \mathbb{Z}$, we will abuse notation and write $\mathbf{C}([s^j, s^{jq}])$ for the conjugacy class in G consisting of those elements in G conjugate to $[s^j, s^{jq}]$ over $\mathbb{F}_{q^2}$. We list the conjugacy classes of $GL(2, \mathbb{F}_q)$ (see, for example, [9]) using this notation.

(1) The $q-1$ conjugacy classes $\mathbf{C}([r^i, r^i])$, for $0 \leq i \leq q-2$. Each of these conjugacy classes are of size 1.
(2) The $q-1$ conjugacy classes $\mathbf{C}([r^i, r^i]')$, for $0 \leq i \leq q-2$. Each of these conjugacy classes are of size $q^2 - 1$.
(3) The $\frac{(q-1)(q-2)}{2}$ conjugacy classes $\mathbf{C}([r^i, r^j]) = \mathbf{C}([r^j, r^i])$, for $0 \leq i \neq j \leq q-2$. Each of these conjugacy classes are of size $q(q+1)$.
(4) The $\frac{q^2-q}{2}$ conjugacy classes $\mathbf{C}([s^i, s^{iq}]) = \mathbf{C}([s^{iq}, s^i])$, for $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Each of these conjugacy classes are of size $q(q-1)$.

We will refer to these conjugacy classes as Types 1, 2, 3 and 4.

We now look closely at the (M, G^*) double coset algebra $\mathbb{C}[G \times G]^M_{G^*}$ where M is defined (as in Example 3.10) by

$$M = \langle G^*, ([r^i, r^i], 1) \mid 0 \leq i \leq q-2 \rangle.$$

Using the bijection between the set of conjugacy classes of G and the set of (G^*, G^*) double cosets, we have the following (G^*, G^*) double cosets:

(1) The $q-1$ double cosets $\mathbf{D}([r^i, r^i])$, for $0 \leq i \leq q-2$. We will refer to these double cosets as *type 1* double cosets.
(2) The $q-1$ double cosets $\mathbf{D}([r^i, r^i]')$, for $0 \leq i \leq q-2$. We will refer to these double cosets as *type 2* double cosets.
(3) The $\frac{(q-1)(q-2)}{2}$ double cosets $\mathbf{D}([r^i, r^j]) = \mathbf{D}([r^j, r^i])$, for $0 \leq i \neq j \leq q-2$. We will refer to these double cosets as *type 3* double cosets.
(4) The $\frac{q^2-q}{2}$ double cosets $\mathbf{D}([s^i, s^{iq}]) = \mathbf{D}([s^{iq}, s^i])$, for $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$, where $[s^i, s^{iq}]$ represents an element in G conjugate over $\mathbb{F}_{q^2}$ to the diagonal matrix $[s^i, s^{iq}]$. We will refer to these double cosets as *type 4* double cosets.

We proceed to analyze the (M, G^*) double cosets.

We begin with the union of type 1 double cosets. First, we recall our notation that for every $g \in G$, there is an (M, G^*) double coset $\mathbf{E}(g) = \bigcup_{z \in Z(G)} (z, 1)\mathbf{D}(g)$. Hence, for a type 1 double coset, we have

$$\mathbf{E}(g) = \bigcup_{i=0}^{q-2} ([r^i, r^i], 1)\mathbf{D}(g).$$

Since $[r^i, r^i]$ is not conjugate to $[r^j, r^j]$ for $0 \leq i \neq j \leq q-2$, we get the double coset $\mathbf{E}([1, 1])$ is the following disjoint union:

$$\mathbf{E}([1, 1]) = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i]).$$

Thus the union of all the (G^*, G^*) double cosets of type 1 gives a single (M, G^*) double coset.

Similarly, since $[r^i, r^i][1,1]'$ is conjugate to $[r^i, r^i]'$, and since $[r^i, r^i]'$ is not conjugate to $[r^j, r^j]'$ for $i \neq j$, we get the double coset $\mathbf{E}([1,1]')$ as the disjoint union:

$$\mathbf{E}([1,1]') = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i]').$$

Thus the union of all the (G^*, G^*) double cosets of type 2 also form a single (M, G^*) double coset.

To determine the remaining (M, G^*) double cosets, we will temporarily assume q is odd.

Fixing an i such that $1 \leq i \leq q-2$, we have the double cosets

$$\mathbf{E}([1, r^i]) = \bigcup_{j=0}^{q-2} ([r^j, r^j], 1)\mathbf{D}([1, r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}]).$$

However, these double cosets are not distinct. Indeed, we have $\mathbf{C}([1, r^k]) = \mathbf{C}([r^k, 1])$ for $k = 0, \ldots, q-2$. Thus when j is such that $r^{i+j} = 1$, we will have

$$\mathbf{E}([1, r^i]) = \mathbf{E}([r^j, r^{i+j}]) = \mathbf{E}([1, r^j]).$$

Since $r^{i+j} = 1$ holds when $i + j = q - 1$ holds, we have

$$\mathbf{E}([1, r^i]) = \mathbf{E}([1, r^{q-(i+1)}]).$$

Thus, since q is odd, restricting i to the range $1 \leq i \leq \frac{q-1}{2}$ we have the distinct double cosets:

$$\mathbf{E}([1, r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}]).$$

We will now consider whether or not this union is disjoint.

Let X be the set of all type 3 (G^*, G^*) double cosets:

$$X = \{\mathbf{D}([r^a, r^b]) \mid 0 \leq a \neq b \leq q-2\}.$$

We have the center $Z(G)$ of G acts on X by $[r^j, r^j] \cdot \mathbf{D}([r^a, r^b]) = \mathbf{D}([r^{a+j}, r^{b+j}])$. Note that the $Z(G)$-orbit of $\mathbf{D}([1, r^i]) \in X$ is the (M, G^*) double coset $\mathbf{E}([1, r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}])$. Let

$$Stab_{Z(G)}(\mathbf{D}([r^a, r^b])) := \{[r^j, r^j] \in Z(G) \mid \mathbf{D}([r^{a+j}, r^{b+j}]) = \mathbf{D}([r^a, r^b])\}.$$

Fix an index i such that $1 \leq i \leq \frac{q-1}{2}$. Then we have

$$[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([1, r^i])) \iff \mathbf{D}([r^j, r^{i+j}]) = \mathbf{D}([1, r^i]) = \mathbf{D}([r^i, 1]).$$

In other words, $[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([1, r^i]))$ if and only if either $j = 0$ holds or both equalities, $r^{i+j} = 1$ and $r^i = r^j$, hold. Thus, the order of $Stab_{Z(G)}(\mathbf{D}([1, r^i]))$ is one except in the case where $r^i = r^{-i}$. Since $1 \leq i \leq \frac{q-1}{2}$, the equality $r^i = r^{-i}$ implies that $i = \frac{q-1}{2}$. In this case, the order of $Stab_{Z(G)}(\mathbf{D}([1, r^i]))$ is 2. Thus, by Orbit-Stabilizer Theorem, the $\mathbf{D}([r^j, r^{i+j}])$'s in the union $\displaystyle\bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}])$ will all be distinct unless $i = \frac{q-1}{2}$. In conclusion, for $i = 1, \ldots, \frac{q-1}{2} - 1$, we have the disjoint union:

$$\mathbf{E}([1, r^i]) = \mathbf{E}([r^i, 1]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([r^i, r^{i+j}]).$$

For $i = \frac{q-1}{2}$, each $\mathbf{D}([r^j, r^{i+j}]) = \mathbf{D}([r^j, -r^j])$ in the union

$$\mathbf{E}([1, r^i]) = \mathbf{E}([1, -1]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, -r^j])$$

appears twice since $r^{j+\frac{q-1}{2}} = -r^j$ and $\mathbf{D}([r^j, -r^j]) = \mathbf{D}([-r^i, r^j])$. Thus representing $\mathbf{E}([1, -1])$ as a disjoint union we have:

$$\mathbf{E}([1, -1]) = \mathbf{E}([-1, 1]) = \bigsqcup_{j=0}^{\frac{q-3}{2}} \mathbf{D}([r^j, -r^j]).$$

Note that the union of all $\mathbf{E}([1, r^i])$ double cosets for $i = 1, \ldots, \frac{q-1}{2} - 1$ contains $(\frac{q-1}{2} - 1)(q-1)$ type 3 (G^*, G^*) double cosets and $\mathbf{E}([1, -1])$ contains $\frac{q-1}{2}$ type 3 (G^*, G^*) double cosets. Thus, in summary, since

$$\left(\frac{q-1}{2} - 1\right)(q-1) + \frac{q-1}{2} = \frac{(q-1)(q-2)}{2},$$

every type 3 (G^*, G^*) double coset is a subset of one (and only one) of the $\mathbf{E}([1, r^i])$ double cosets $i = 1, \ldots, \frac{q-1}{2} - 1$, or a subset of $\mathbf{E}([1, -1])$.

It remains to consider the double cosets of the form $\mathbf{E}([s^i, s^{iq}])$, for $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We have

$$\mathbf{E}([s^i, s^{iq}]) = \bigcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j]).$$

We define Y as follows:

$$Y := \{\mathbf{D}([s^a, s^{aq}]) \mid s^a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q\}.$$

We have the center of G acts on Y by $[r^j, r^j] \cdot \mathbf{D}([s^a, s^{aq}]) = \mathbf{D}([s^a r^j, s^{aq} r^j])$. (Note as $r \in \mathbb{F}_q$, we have $r^q = r$. Thus $\mathbf{D}([s^a r^j, s^{aq} r^j]) = \mathbf{D}([s^a r^j, s^{aq} r^{jq}]) = \mathbf{D}([s^b, s^{bq}])$ where $b = a + j(q+1)$.)

Note that the orbit of this action of $Z(G)$ on $\mathbf{D}([s^i, s^{iq}]) \in Y$ is the (M, G^*) double coset $\mathbf{E}([s^i, s^{iq}]) = \bigcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j])$.

Fix an i such that $1 \leq i \leq \frac{q+1}{2}$. Note that for such an i, we have $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. We have $[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ if and only if $\mathbf{D}([s^i r^j, s^{iq} r^j]) = \mathbf{D}([s^i, s^{iq}])$. Thus $[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ if and only if $j = 0$ or both $s^i r^j = s^{iq}$ and $s^{iq} r^j = s^i$.

Focusing on the case where both $s^i r^j = s^{iq}$ and $s^{iq} r^j = s^i$ and solving both of these equations for r^j we get $s^{iq} s^{-i} = s^i s^{-iq}$. That is, $(s^{i(q-1)})^2 = 1$, so that $s^{i(q-1)} = \pm 1$. However, we have $i \leq \frac{q+1}{2}$, so $s^{i(q-1)} = -1$. Thus $i(q-1) = \frac{q^2-1}{2}$, so that $i = \frac{q+1}{2}$.

Thus for $i \neq \frac{q+1}{2}$, the order of $Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ is one and for $i = \frac{q+1}{2}$, the order of $Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ is two.

Thus by the Orbit-Stabilizer Theorem and for i restricted to the range $i = 1, \ldots, \frac{q-1}{2}$ we get the distinct double cosets

$$\mathbf{E}([s^i, s^{qi}]) = \mathbf{E}([s^{qi}, s^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i s^{j(q+1)}, s^{iq} s^{j(q+1)}]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j]).$$

When $i = \frac{q+1}{2}$ we get half as many (G^*, G^*) double cosets in this union, obtaining the (M, G^*) double coset

$$\mathbf{E}([s^{\frac{q+1}{2}}, s^{q(\frac{q+1}{2})}]) = \mathbf{E}([s^{q(\frac{q+1}{2})}, s^{\frac{q+1}{2}}]) = \mathbf{E}([s^{(\frac{q+1}{2})}, -s^{\frac{q+1}{2}}]) = \bigsqcup_{j=0}^{\frac{q-3}{2}} \mathbf{D}([s^{\frac{q+1}{2}} r^j, -s^{(\frac{q+1}{2})} r^j]).$$

Note that the union of all $\mathbf{E}([s^i, s^{iq}])$ double cosets for $i = 1, \ldots, \frac{q-1}{2}$ contains $(\frac{q-1}{2})(q-1)$ type 4 (G^*, G^*) double cosets and for $i = \frac{q+1}{2}$, $\mathbf{E}([s^i, -s^i])$ contains $\frac{q-1}{2}$ type 4 (G^*, G^*) double cosets. Thus, since

$$\left(\frac{q-1}{2}\right)(q-1) + \frac{q-1}{2} = \frac{q^2-q}{2},$$

every type 4 (G^*, G^*) double coset is a subset of one (and only one) of the $\mathbf{E}([s^i, s^{iq}])$ double cosets for $1 \leq i \leq \frac{q+1}{2}$. In summary, we obtained the following result in odd characteristic.

Proposition 4.1. *Let q be a power of an odd prime. Let s be a generator of $\mathbb{F}_{q^2}^*$ and let $r = s^{q+1}$. Let $G = GL(2, \mathbb{F}_q)$. We maintain the notation from the above for M and G^*. Then the distinct (M, G^*) double cosets in $G \times G$ are given by*

(1) $\mathbf{E}([1,1]) = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i])$.

(2) $\mathbf{E}([1,1]') = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i]')$.

(3) $\mathbf{E}([1, r^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([r^i, r^{i+j}])$, *for* $1 \leq i \leq \frac{q-3}{2}$.

(4) $\mathbf{E}([1,-1]) = \bigsqcup_{j=0}^{\frac{q-2}{2}} \mathbf{D}([r^j, -r^j])$.

(5) $\mathbf{E}([s^i, s^{qi}]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j])$, *for* $1 \leq i \leq \frac{q-1}{2}$.

(6) $\mathbf{E}([s^{\frac{q+1}{2}}, -s^{\frac{q+1}{2}}]) = \bigsqcup_{j=0}^{\frac{q-3}{2}} \mathbf{D}([s^{\frac{q+1}{2}} r^j, -s^{\frac{q+1}{2}} r^j])$.

As the corresponding Schur algebra has a linear basis with elements the sums of the elements in these double cosets, this Schur algebra has a linear basis whose number of elements is given by

$$1 + 1 + \frac{q-3}{2} + 1 + \frac{q-1}{2} + 1 = q + 2.$$

When instead q is a power of 2, we get the following result.

Proposition 4.2. *Let q be a power of 2 and let $G = GL(2, \mathbb{F}_q)$. The distinct (M, G^*) double cosets in $G \times G$ are:*

(1) $\mathbf{E}([1,1]) = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i])$.

(2) $\mathbf{E}([1,1]') = \bigsqcup_{i=0}^{q-2} \mathbf{D}([r^i, r^i]')$.

(3) $\mathbf{E}([1,r^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([r^i, r^{i+j}])$, *for* $1 \leq i \leq \frac{q-2}{2}$.

(4) $\mathbf{E}([s^i, s^{qi}]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j])$, *for* $1 \leq i \leq \frac{q}{2}$.

Thus, when q is a power of 2, the corresponding Schur algebra has a linear basis with $1 + 1 + \frac{q-2}{2} + \frac{q}{2} = q + 1$ elements.

Proof. The proof of this proposition follows from making a few adjustments to the proof of Proposition 4.1.

The (M, G^*) double cosets $\mathbf{E}([1,1])$ and $\mathbf{E}([1,1]')$ were described in the proof of Proposition 4.1 without assuming q was odd, so these double cosets also appear in the case when q is even.

Fixing an i such that $1 \leq i \leq q-2$, we again have the double cosets

$$\mathbf{E}([1,r^i]) = \bigcup_{j=0}^{q-2} ([r^j, r^j], 1)\mathbf{D}([1,r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}]).$$

As in the case when q was odd, we have $\mathbf{E}([1,r^i]) = \mathbf{E}([1,r^j])$ if and only if $j = q-(i+1)$. Thus, since q is now even, restricting i to the range $1 \leq i \leq \frac{q-2}{2}$ we have the distinct double cosets:

$$\mathbf{E}([1, r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}]).$$

We will now consider whether or not this union is disjoint. As in the case when q was odd, we have $[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([1, r^i]))$ if and only if $j = 0$ or $r^i = r^{-i}$. We will show that the latter case is not possible. Indeed, $r^i = r^{-i}$ if and only if r^i has order one or two. Since i is from $\{1, \ldots, q-2\}$, the element $r^i \in \mathbb{F}_q^*$ cannot be the identity element. Also, since the order of r is $q-1$ (which is odd), r^i cannot have order two for any i. In summary, we have $j = 0$ and the order of $Stab_{Z(G)}(\mathbf{D}([1, r^i]))$ is 1 for all i. Thus, by the Orbit-Stabilizer Theorem, the $\mathbf{D}([r^j, r^{i+j}])$'s in the union $\bigcup_{j=0}^{q-2} \mathbf{D}([r^j, r^{i+j}])$ will all be distinct. Thus for $i = 1, \ldots, \frac{q-2}{2}$ we have the disjoint union:

$$\mathbf{E}([1, r^i]) = \mathbf{E}([r^i, 1]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([r^i, r^{i+j}]).$$

Note that the union of all the $\mathbf{E}([1, r^i])$ double cosets for $i = 1, \ldots, \frac{q-2}{2}$ contain $(\frac{q-2}{2})(q-1)$ type 3 (G^*, G^*) double cosets. Therefore, every type 3 (G^*, G^*) double coset is a subset of one (and only one) of the $\mathbf{E}([1, r^i])$ double cosets.

It remains to consider the double cosets of the form $\mathbf{E}([s^i, s^{iq}])$, for $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Fix an i such that $1 \leq i \leq \frac{q}{2}$ and note that for such an i, we have $s^i \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. As in the case when q was odd and for $0 \leq j \leq q-2$, we have $[r^j, r^j] \in Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ if and only if $j = 0$ or $s^{i(q-1)} = \pm 1$. However, since $|s| = q^2 - 1$ (which is odd), $s^{i(q-1)} \neq \pm 1$ for every $i = 1, \ldots, \frac{q}{2}$. Thus the order of $Stab_{Z(G)}(\mathbf{D}([s^i, s^{iq}]))$ is one. It follows from the Orbit-Stabilizer Theorem that

$$\mathbf{E}([s^i, s^{qi}]) = \mathbf{E}([s^{qi}, s^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i s^{j(q+1)}, s^{iq} s^{j(q+1)}]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}([s^i r^j, s^{iq} r^j]),$$

for $i = 1, \ldots, \frac{q}{2}$. Notice that the union of all $\mathbf{E}([s^i, s^{iq}])$ double cosets, for $i = 1, \ldots, \frac{q}{2}$, contains $\frac{q^2-q}{2}$ type 4 (G^*, G^*) double cosets. This means that every type 4 (G^*, G^*) double coset is a subset of one (and only one) of the sets $\mathbf{E}([s^i, s^{iq}])$ for $1 \leq i \leq \frac{q}{2}$. This finishes the proof of our proposition. $\square$

5. Proof of the main theorem

In this section, we prove our Theorem 1.1 and discuss some consequences. Let us recall its statement for convenience.

Let M be a proper subgroup of $G \times G$ such that G^* is a normal subgroup of M. Then $\mathbb{C}[G \times G]_{G^*}^M$ is isomorphic to a unital central subalgebra of $\mathbb{C}[G]$.

Proof of Theorem 1.1. If $M = G^*$, then there is nothing to prove. We proceed with the assumption that G^* is a proper subgroup of M. In this case, we already pointed out before Example 3.10 that $\mathbb{C}[G \times G]^M_{G^*}$ is isomorphic to a subalgebra of the center of $\mathbb{C}[G]$. Hence, it remains to show that $\mathbb{C}[G \times G]^M_{G^*}$ is unital. To this end, let $(g_1, 1), \dots, (g_r, 1)$ be a complete list of representatives of the (M, G^*) double cosets in $G \times G$. Without loss of generality we assume that g_1 is the identity element of G. Evidently, $\mathbb{C}[G \times G]^M_{G^*}$ is spanned by the normalized elements

$$\mathbf{F}(g_i) := \frac{1}{|M|}\widehat{\mathbf{E}(g_i)}, \qquad i = 1, \dots, r,$$

where $\mathbf{E}(g)$ is defined as in (3.4). We note in passing that, for $g_1 = 1$, we have

$$\mathbf{F}(g_1) = \frac{1}{|M|}\widehat{M(g_1,1)G^*} = \frac{1}{|M|}\widehat{MG^*} = \frac{1}{|M||G|}\widehat{M}\widehat{G}^* = \frac{|G|}{|M||G|}\widehat{M} = \frac{1}{|M|}\widehat{M}.$$

Now, for each $j \in \{1, \dots, r\}$, we have

$$\begin{aligned}\mathbf{F}(g_1)\mathbf{F}(g_j) &= \frac{1}{|M|^2}\widehat{M}\widehat{M(g_j,1)G^*} \\ &= \frac{1}{|M|^2}\left(\sum_{m \in M} m\right)\widehat{M(g_j,1)G^*}.\end{aligned} \tag{5.1}$$

Let $x_{j_1}, \dots, x_{j_t}$ be the full list of elements of the double coset $M(g_j, 1)G^*$. Then M acts on the set $\{x_{j_1}, \dots, x_{j_t}\}$ as a permutation group. Hence, (5.1) is given by

$$\mathbf{F}(g_1)\mathbf{F}(g_j) = \frac{1}{|M|^2}|M|\widehat{M(g_j,1)G^*} = \frac{1}{|M|}\widehat{M(g_j,1)G^*} = \mathbf{F}(g_j).$$

In other words, $\mathbf{F}(g_1)$ is the identity element of the double coset algebra $\mathbb{C}[G\times G]^M_{G^*}$. This finishes the proof of our theorem. □

It is worth noticing that although $\mathbb{C}[G\times G]^M_{G^*}$ is isomorphic to a unital subalgebra of the center of the group ring $\mathbb{C}[G]$, the unit element of this subalgebra is not given by the identity element $e \in G$.

We continue with the assumption that K is a Gelfand subgroup of G, and H is a subgroup of G such that K is normal in H. By the transitivity of the induction of representations, it follows that

(1) K is a Gelfand subgroup of H,
(2) H is a Gelfand subgroup of G.

Assuming K is both a normal subgroup and a Gelfand subgroup of H puts severe restrictions on the quotient group H/K.

Lemma 5.1. *Let K be a normal subgroup of a finite group H. If K is a Gelfand subgroup of H, then H/K is an abelian group.*

Proof. Let χ denote the character of the induced trivial representation $\mathrm{ind}_K^H(\mathbf{1}_K)$. We decompose χ into irreducible representations of G:

$$\chi = \sum_{i=1}^{r} e_i \chi_i,$$

where $e_i \in \mathbb{N}$. Since $\mathrm{res}_K^H(\chi_i) = e_i \mathbf{1}_K$ by the Frobenius Reciprocity Theorem, we see that

$$\chi_i(1) = e_i \qquad \text{for every } i \in \{1, \dots, r\}.$$

In particular, we have

$$[H : K] = \chi(1) = \sum_{i=1}^{r} e_i \chi_i(1) = \sum_{i=1}^{r} e_i^2.$$

But (H, K) being a Gelfand pair implies that $e_i = 1$ for every $i \in \{1, \dots, r\}$. Therefore, $r = [H : K]$ showing that χ has $[H : K]$ mutually distinct constituents, that are, $\chi_1, \dots, \chi_{[H:K]}$. In particular, the dimensions of the corresponding representations are all 1. Note that if an irreducible representation χ_i of H is a constituent of the induced trivial representation $\mathrm{ind}_K^H(\mathbf{1}_K)$, then K is contained in the kernel of χ_i. Thus, after composing them with the canonical group homomorphism $H \to H/K$, the characters $\chi_1, \dots, \chi_{[H:K]}$ give the complete list of inequivalent characters $\overline{\chi}_1, \dots, \overline{\chi}_{[H:K]}$ of H/K. Since these are all degree 1 characters, it follows that H/K is an abelian group. □

We mention two consequences of our previous observation.

Corollary 5.2. *If (G, K) is a Gelfand pair, then the double coset algebra $\mathbb{C}[G]_H^H$ is a two sided ideal in the commutative double coset algebra $\mathbb{C}[G]_K^H$.*

Proof. Since K is normal in H, by Lemma 5.1 we know that H/K is abelian. Therefore every character ν in $\overline{H/K}$ is of degree one, and the restriction of ν to K is the trivial character of K. In other words, when we induce $\mathbf{1}_K$ to H, we obtain exactly the sum $\sum_{\nu \in \overline{H/K}} \nu$. By the transitivity property of the induction, we obtain

$$\begin{aligned} \mathrm{ind}_K^G(\mathbf{1}_K) &= \mathrm{ind}_H^G(\mathrm{ind}_K^H(\mathbf{1}_K)) \\ &= \sum_{\nu \in \overline{H/K}} \mathrm{ind}_H^G(\nu). \end{aligned} \tag{5.2}$$

Since (5.2) is a multiplicity-free representation of G, for $\chi \in \mathrm{Irr}(G)$, we have

$$(\chi, \mathrm{ind}_H^G(\nu)) \in \{0, 1\}.$$

Then, by the Frobenius Reciprocity, we see that $(\mathrm{res}_H^G(\chi), \nu) \in \{0, 1\}$. Therefore, H is multiplicity-free on K. It follows that $\mathbb{C}[G]_K^H$ is a commutative Schur algebra. □

We are now ready to prove Corollary 1.2. Let us recall its statement for convenience.

If M is a subgroup of $G \times G$ such that the diagonal subgroup G^* of $G \times G$ is a normal subgroup of M, then M/G^* is an abelian group.

Proof of Corollary 1.2. We already mentioned that G^* is a Gelfand subgroup of $G \times G$. The rest of the proof readily follows from Lemma 5.1. $\square$

Remark 5.3. We note that Corollary 1.2 can also be observed via a direct group-theoretic argument. The normalizer of the diagonal subgroup in $G \times G$ is given by $N_{G\times G}(G^*) = \{(h, zh) \mid h \in G, z \in Z(G)\}$. Setting $Z^* = \{(1, z) \mid z \in Z(G)\}$, one can readily verify that $N_{G\times G}(G^*) = G^* \times Z^*$. Since $M \leq N_{G\times G}(G^*)$ by normality, the quotient M/G^* embeds into the abelian group $Z^* \cong Z(G)$. Our representation-theoretic approach via Gelfand pairs, however, is intentional since it contextualizes this structural fact within the broader, overarching framework of multiplicity-free induced representations.

6. An illustration

In this section, G will be $GL(2, \mathbb{F}_q)$ and B denotes the Borel subgroup of G consisting of upper triangular matrices in G. Let T denote the maximal torus consisting of diagonal matrices in B. Let U denote the upper triangular subgroup of B which is a maximal unipotent subgroup normalized by T in B. Then we know that U is normal in B, and we have

$$B = U \rtimes T.$$

Let a and b be two elements from $\mathbb{F}_q^*$, and let $c \in \mathbb{F}_q$. We will describe the conjugacy class in B of the following element:

$$x := \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in B.$$

First of all, we notice that any conjugate of x in B has the same diagonal entries as x. This reduces our task to choosing a pair of diagonal entries $(a, b) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$. Then we consider the following cases:

(1) $a \neq b$,
(2) $a = b$.

We proceed with (1). In this case, we claim that, no matter what c is, the conjugacy class of x consists of all matrices of the form $\begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}$ where $c' \in \mathbb{F}_q$. To show this, we need to show that there exists an element $u \in B$ such that $uxu^{-1} = \begin{pmatrix} a & c' \\ 0 & b \end{pmatrix}$.

Indeed, if we let $u := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$, then we have

$$\begin{aligned} uxu^{-1} &= \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & c+bs \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a & c+bs-as \\ 0 & b \end{pmatrix}. \end{aligned}$$

But since $a \neq b$, by an appropriate choice of $s \in \mathbb{F}_q$, we see that the equation $c' = c + s(b-a)$ has a solution. This finishes the proof of our claim.

Next, we consider (2). In this case, we claim that the conjugacy class of x consists of all matrices of the form $\begin{pmatrix} a & cc' \\ 0 & a \end{pmatrix}$ where $c' \in \mathbb{F}_q^*$. To show this, we choose an element $m := \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in B$ such that $c' := xz^{-1}$. Then we have

$$\begin{aligned} mxm^{-1} &= \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \begin{pmatrix} x^{-1} & -y(xz)^{-1} \\ 0 & z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} xa & xc+ya \\ 0 & za \end{pmatrix} \begin{pmatrix} x^{-1} & -y(xz)^{-1} \\ 0 & z^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a & cc' \\ 0 & a \end{pmatrix}. \end{aligned}$$

This finishes the proof of our claim.

We note in passing that central Schur rings over related linear groups, specifically the projective special linear groups, have been previously investigated by Humphries and Wagner in [11]. Furthermore, see [14] for Marrow's related work on strong Gelfand pairs and linear groups. Our focus here on the Borel subgroup of $GL(2, \mathbb{F}_q)$ provides a complementary perspective, particularly regarding the fusion of double cosets.

Recall our earlier notation that $[a, b]$ stands for the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, and $[a, a]'$ stands for the matrix $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. Using this notation, our observations in the previous paragraphs readily imply the following lemma whose proof is omitted.

Lemma 6.1. *Each conjugacy class in B is given by one of the following type of conjugacy classes:*

Type 3 conjugacy class. For $(a,b) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$ such that $a \neq b$, the conjugacy class defined by

$$\mathbf{C}_B([a,b]) := \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid c \in \mathbb{F}_q \right\}.$$

Type 2 conjugacy class. For $(a,a) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$, the conjugacy class defined by

$$\mathbf{C}_B([a,a]') := \left\{ \begin{pmatrix} a & c \\ 0 & a \end{pmatrix} \mid c \in \mathbb{F}_q^* \right\}.$$

Type 1 conjugacy class. For $(a,a) \in \mathbb{F}_q^* \times \mathbb{F}_q^*$, the conjugacy class defined by

$$\mathbf{C}_B([a,a]) := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \right\}.$$

Remark 6.2. The number of type 3 conjugacy classes of B is $(q-1)(q-2)$. Note that each of these conjugacy classes contains exactly q elements. The number of type 2 conjugacy classes is $q-1$. Each of the type 2 conjugacy classes has exactly $q-1$ elements. Finally, the number of type 1 conjugacy classes is $q-1$ and each such conjugacy class has only one element. Therefore, the total number of conjugacy classes is given by

$$(q-1)(q-2) + (q-1) + (q-1) = q(q-1) = q^2 - q.$$

Since the number of conjugacy classes in $G := GL(2, \mathbb{F}_q)$ is $q^2 - 1$, we see that there are $(q-1)$ fewer conjugacy classes in B than in G.

As above, we denote by s a fixed element of $\mathbb{F}_{q^2}$ that generates the multiplicative group $\mathbb{F}_{q^2}^*$ and let $r = s^{q+1}$ and note r generates the multiplicative group $\mathbb{F}_q^*$. We now proceed to analyze the (M, B^*) double coset algebra $\mathscr{E}(B) := \mathbb{C}[B \times B]_{B^*}^M$ where, as above, $M = \langle B^*, (z,1) \mid z \in Z(B) \rangle$. Since the centers of G and B are equal, we have $M = \langle B^*, ([r^i, r^i], 1) \mid 0 \leq i \leq q-2 \rangle$.

Using the bijection between the set of conjugacy classes of B and the set of (B^*, B^*) double cosets, we have the following (B^*, B^*) double cosets:

(1) The $(q-1)(q-2)$ double cosets $\mathbf{D}_B([r^i, r^j])$, for $0 \leq i \neq j \leq q-2$. We will refer to these double cosets as *type 3* double cosets.
(2) The $q-1$ double cosets $\mathbf{D}_B([r^i, r^i]')$, for $0 \leq i \leq q-2$. We will refer to these double cosets as *type 2* double cosets.
(3) The $q-1$ double cosets $\mathbf{D}_B([r^i, r^i])$, for $0 \leq i \leq q-2$. We will refer to these double cosets as *type 1* double cosets.

We proceed to describe the construction of the (M, B^*) double cosets. For $g \in B$, we have the (M, B^*) double coset

$$\mathbf{E}_B(g) = \bigcup_{z \in Z(B)} (z,1)\mathbf{D}_B(g) = \bigcup_{i=0}^{q-2} ([r^i, r^i], 1)\mathbf{D}_B(g).$$

We begin with the (M, B^*) double cosets that are obtained from type 3 double cosets $\mathbf{D}_B(g)$. For $i \in \{0, 1, \dots, q-2\}$ we have

$$\mathbf{E}_B([1, r^i]) = \bigcup_{j=0}^{q-2} ([r^j, r^j], 1)\mathbf{D}_B([1, r^i]) = \bigcup_{j=0}^{q-2} \mathbf{D}_B([r^j, r^{i+j}]).$$

Since the element $w_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is not contained in B, the diagonal entries of an element of the form $[r^a, r^b]$ cannot be interchanged. Consequently, all double cosets $\mathbf{D}_B([r^j, r^{i+j}])$ for $j \in \{0, 1, \dots, q-2\}$ must be disjoint. In particular, for each $i \in \{0, 1 \dots, q-2\}$, we obtain a unique (M, B^*) double coset,

$$\mathbf{E}_B([1, r^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}_B([r^j, r^{i+j}]).$$

We now consider the (M, B^*) double cosets of type 2 (B^*, B^*) double cosets. It is easy to check that the center $Z(B)$ acts transitively on type 2 double cosets $\mathbf{D}_B([r^i, r^i]')$, $1 \le i \le q-2$. Therefore,

$$\mathbf{E}_B([1, 1]') = \bigsqcup_{j=0}^{q-2} \mathbf{D}_B([r^j, r^j]') = \mathbf{E}_B([r^i, r^i]')$$

for every $i \in \{0, 1, \dots, q-2\}$.

Finally, we consider the (M, B^*) double cosets of type 1 (B^*, B^*) double cosets. Similarly, to the previous case, the center acts transitively on type 1 double cosets $\mathbf{D}_B([r^i, r^i])$, $1 \le i \le q-2$. Therefore, we obtain

$$\mathbf{E}_B([1, 1]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}_B([r^j, r^j]) = \mathbf{E}_B([r^i, r^i])$$

for every $i \in \{0, 1, \dots, q-2\}$.

In summary, we obtained the following result.

Proposition 6.3. *Let s be a generator of $\mathbb{F}_{q^2}^*$ and let $r = s^{q+1}$. Let B denote the Borel subgroup consisting of upper triangular matrices in $GL(2, \mathbb{F}_q)$. Using the notation from the above for M and B^*, we have the following distinct (M, B^*) double cosets in $B \times B$:*

(1) $\mathbf{E}_B([1, 1]) = \bigsqcup_{i=0}^{q-2} \mathbf{D}_B([r^i, r^i])$.

(2) $\mathbf{E}_B([1,1]') = \bigsqcup_{i=0}^{q-2} \mathbf{D}_B([r^i, r^i]')$.

(3) $\mathbf{E}_B([1,r^i]) = \bigsqcup_{j=0}^{q-2} \mathbf{D}_B([r^j, r^{i+j}])$, *for* $1 \le i \le q-2$.

In particular, there are q distinct (M, B^*) double cosets in $B \times B$.

Recall $\mathscr{C}(B)$ denotes the Schur algebra with linear basis the distinct elements in $\{\widehat{\mathbf{C}_B(g)} \mid g \in B\}$, $\mathscr{D}(B)$ denotes the Schur algebra with linear basis the distinct elements in $\{\widehat{\mathbf{D}_B(g)} \mid g \in B\}$ and $\mathscr{E}(B)$ denotes the Schur algebra with linear basis the distinct elements in $\{\widehat{\mathbf{E}_B(g)} \mid g \in B\}$.

In order to better understand the structure of $\mathscr{E}(B)$, we will provide the structure constants of this algebra. First, however, we need the structure constants of $\mathscr{C}(B)$ and $\mathscr{D}(B)$.

Lemma 6.4. *Using the above notation, the structure constants of $\mathscr{C}(B)$ are given by the following. Let $a, b, c, d \in \mathbb{F}_q^*$.*

(1) $\widehat{\mathbf{C}_B([a,a])}\widehat{\mathbf{C}_B([b,b])} = \widehat{\mathbf{C}_B([ab,ab])}$
(2) $\widehat{\mathbf{C}_B([a,a])}\widehat{\mathbf{C}_B([b,b]')} = \widehat{\mathbf{C}_B([ab,ab]')}$
(3) $\widehat{\mathbf{C}_B([a,a])}\widehat{\mathbf{C}_B([b,c])} = \widehat{\mathbf{C}_B([ab,ac])}$
(4) $\widehat{\mathbf{C}_B([a,a]')}\widehat{\mathbf{C}_B([b,b]')} = (q-2)\widehat{\mathbf{C}_B([ab,ab]')} + (q-1)\widehat{\mathbf{C}_B([ab,ab])}$
(5) $\widehat{\mathbf{C}_B([a,a]')}\widehat{\mathbf{C}_B([b,c])} = (q-1)\widehat{\mathbf{C}_B([ab,ac])}$
(6) *If $ad \neq bc$, then* $\widehat{\mathbf{C}_B([a,b])}\widehat{\mathbf{C}_B([c,d])} = q\widehat{\mathbf{C}_B([ac,bd])}$
(7) *If $ad = bc$, then* $\widehat{\mathbf{C}_B([a,b])}\widehat{\mathbf{C}_B([c,d])} = q\widehat{\mathbf{C}_B([ac,bd]')} + q\widehat{\mathbf{C}_B([ac,bd])}$.

Proof. As $\widehat{\mathbf{C}_B([a,a])}$ is just the single central matrix $[a,a]$, formulas (1) - (3) are clear.

For a fixed a, there are $q-1$ matrices in $\mathbf{C}_B([a,a]')$ each of the form $g = \begin{pmatrix} a & e \\ 0 & a \end{pmatrix}$ for $e \in \mathbb{F}_q^*$. And for each such g, the set $g\mathbf{C}_B([b,b]')$ will contain matrices of types 1 and 2. For $e, e' \neq 0$ note that

$$\begin{pmatrix} a & e \\ 0 & a \end{pmatrix}\begin{pmatrix} b & e' \\ 0 & b \end{pmatrix} = \begin{pmatrix} ab & ae'+be \\ 0 & ab \end{pmatrix},$$

will be of type 1 if and only if $e' = -a^{-1}eb$. Thus for each of the $q-1$, g's in $\mathbf{C}_B([a,a]')$ the set $g\mathbf{C}_B([b,b]')$ will contain one copy of the type 1 conjugacy class $\mathbf{C}_B([ab,ab])$ and the remaining $q-2$ elements in the set $g\mathbf{C}_B([b,b]')$ will be distinct type 2 elements in $\mathbf{C}_B([ab,ab]')$. Thus formula (4) follows.

Again, fix a and consider the $q-1$ matrices in $\mathbf{C}_B([a,a]')$ each of the form $g = \begin{pmatrix} a & e \\ 0 & a \end{pmatrix}$ for $e \in \mathbb{F}_q^*$. Note for each such g, the set $g\mathbf{C}_B([b,c])$ equals the set $\mathbf{C}_B([ab,ac])$. Thus formula (5) follows.

For fixed a and b, there are q matrices in $\mathbf{C}_B([a,b])$ each of the form $g = \begin{pmatrix} a & e \\ 0 & b \end{pmatrix}$ for $e \in \mathbb{F}_q$. Then for fixed c and d such that $ac \neq bd$, the set $g\mathbf{C}_B([c,d])$ equals the set $\mathbf{C}_B([ac,bd])$ of type 3. Thus formula (6) follows. However if $ac = bd$, the set $g\mathbf{C}_B([c,d])$ will instead consist of matrices of types 1 and 2. Note

$$\begin{pmatrix} a & e \\ 0 & b \end{pmatrix}\begin{pmatrix} c & e' \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & ae'+de \\ 0 & bd \end{pmatrix},$$

which is of type 1 if and only if $e' = -a^{-1}ed$. Thus for each $g \in \mathbf{C}_B([c,d])$, the set $g\mathbf{C}_B([c,d])$ will contain only one copy of $\mathbf{C}_B([ac,bd])$ and the remaining $q-1$ elements in $g\mathbf{C}_B([c,d])$ are the set $\mathbf{C}_B([ac,bd]')$. Thus formula (7) follows. $\square$

The following lemma then follows immediately using the isomorphism Φ (given in Equation 3.3) between the algebras $\mathscr{D}(B)$ and the algebra $\mathscr{C}(B)$. Note we have $|B| = q(q-1)^2$.

Lemma 6.5. *Using the above notation, the structure constants of $\mathscr{D}(B)$ are given by the following. Let $a,b,c,d \in \mathbb{F}_q^*$.*

(1) $\widehat{\mathbf{D}_B([a,a])}\widehat{\mathbf{D}_B([b,b])} = |B|\widehat{\mathbf{D}_B([ab,ab])}$
(2) $\widehat{\mathbf{D}_B([a,a])}\widehat{\mathbf{D}_B([b,b]')} = |B|\widehat{\mathbf{D}_B([ab,ab]')}$
(3) $\widehat{\mathbf{D}_B([a,a])}\widehat{\mathbf{D}_B([b,c])} = |B|\widehat{\mathbf{D}_B([ab,ac])}$
(4) $\widehat{\mathbf{D}_B([a,a]')}\widehat{\mathbf{D}_B([b,b]')} = (q-2)|B|\widehat{\mathbf{D}_B([ab,ab]')} + (q-1)|B|\widehat{\mathbf{D}_B([ab,ab])}$
(5) $\widehat{\mathbf{D}_B([a,a]')}\widehat{\mathbf{D}_B([b,c])} = (q-1)|B|\widehat{\mathbf{D}_B([ab,ac])}$
(6) *If $ac \neq bd$, then* $\widehat{\mathbf{D}_B([a,b])}\widehat{\mathbf{D}_B([c,d])} = q|B|\widehat{\mathbf{D}_B([ac,bd])}$
(7) *If $ac = bd$, then* $\widehat{\mathbf{D}_B([a,b])}\widehat{\mathbf{D}_B([c,d])} = q|B|\widehat{\mathbf{D}_B([ac,bd]')} + q|B|\widehat{\mathbf{D}_B([ac,bd])}$.

We can now state and prove the structure constants of $\mathscr{E}(B)$.

Proposition 6.6. *Fix i,j such that $1 \leq i,j \leq q-2$. Using the above notation, the structure constants of $\mathscr{E}(B)$ are given by the following.*

(1) $\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,1])} = (q-1)|B|\widehat{\mathbf{E}_B([1,1])}$
(2) $\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,1]')} = (q-1)|B|\widehat{\mathbf{E}_B([1,1]')}$
(3) $\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,r^i])} = (q-1)|B|\widehat{\mathbf{E}_B([1,r^i])}$
(4) $\widehat{\mathbf{E}_B([1,1]')}\widehat{\mathbf{E}_B([1,1]')} = (q-1)(q-2)|B|\widehat{\mathbf{E}_B([1,1]')} + (q-1)^2|B|\widehat{\mathbf{E}_B([1,1])}$
(5) $\widehat{\mathbf{E}_B([1,1]')}\widehat{\mathbf{E}_B([1,r^i])} = (q-1)^2|B|\widehat{\mathbf{E}_B([1,r^i])}$

(6) *If $i+j=q-1$, then*
$$\widehat{\mathbf{E}_B([1,r^i])}\widehat{\mathbf{E}_B([1,r^j])} = q(q-1)|B|\Big(\widehat{\mathbf{E}_B([1,1]')} + \widehat{\mathbf{E}_B([1,1])}\Big)$$
(7) *If $i+j\neq q-1$, then* $\widehat{\mathbf{E}_B([1,r^i])}\widehat{\mathbf{E}_B([1,r^j])} = q(q-1)|B|\widehat{\mathbf{E}_B([1,r^{i+j}])}$.

Proof. As $\mathbf{E}_B([1,1]) = \bigsqcup_{k=0}^{q-2} \mathbf{D}_B([r^k,r^k])$, we have

$$\begin{aligned}
\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,1])} &= \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^l])}\big)\\
&= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\cdot\widehat{\mathbf{D}_B([r^l,r^l])}\\
&= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}])} \text{ (by the previous lemma)}\\
&= \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^m,r^m])} \text{ (by a change of variable)}\\
&= |B|\sum_{k=0}^{q-2}\widehat{\mathbf{E}_B([1,1])} = (q-1)|B|\widehat{\mathbf{E}_B([1,1])}.
\end{aligned}$$

Similarly, as $\mathbf{E}_B([1,1]') = \bigsqcup_{l=0}^{q-2} \mathbf{D}_B([r^l,r^l]')$, we have

$$\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,1]')} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^l]')}\big)$$
$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\cdot\widehat{\mathbf{D}_B([r^l,r^l]')} = \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}]')}$$
$$= \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^m,r^m]')} = |B|\sum_{k=0}^{q-2}\widehat{\mathbf{E}_B([1,1]')} = (q-1)|B|\widehat{\mathbf{E}_B([1,1]')}.$$

For a fixed i such that $1\leq i\leq q-2$, we have $\mathbf{E}_B([1,r^i]) = \bigsqcup_{l=0}^{q-2}\mathbf{D}_B([r^l,r^{i+l}])$ and thus

$$\widehat{\mathbf{E}_B([1,1])}\widehat{\mathbf{E}_B([1,r^i])} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^{i+l}])}\big)$$
$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k])}\cdot\widehat{\mathbf{D}_B([r^l,r^{i+l}])} = \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^{k+l},r^{i+k+l}])}$$
$$= \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}|B|\widehat{\mathbf{D}_B([r^m,r^{i+m}])} = |B|\sum_{k=0}^{q-2}\widehat{\mathbf{E}_B([1,r^i])} = (q-1)|B|\widehat{\mathbf{E}_B([1,r^i])}.$$

Furthermore,

$$\widehat{\mathbf{E}_B([1,1]')}\widehat{\mathbf{E}_B([1,1]')} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k]')}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^l]')}\big)$$

$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k]')}\cdot\widehat{\mathbf{D}_B([r^l,r^l]')}$$

$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}(q-2)|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}]')} + (q-1)|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}])}$$

$$= \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}(q-2)|B|\widehat{\mathbf{D}_B([r^m,r^m]')} + (q-1)|B|\widehat{\mathbf{D}_B([r^m,r^m])}$$

$$= \sum_{k=0}^{q-2}(q-2)|B|\widehat{\mathbf{E}_B([1,1]')} + (q-1)|B|\widehat{\mathbf{E}_B([1,1])}$$

$$= (q-1)(q-2)|B|\widehat{\mathbf{E}_B([1,1]')} + (q-1)^2|B|\widehat{\mathbf{E}_B([1,1])}.$$

In addition,

$$\widehat{\mathbf{E}_B([1,1]')}\widehat{\mathbf{E}_B([1,r^i])} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^k]')}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^{i+l}])}\big)$$

$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}(q-1)|B|\widehat{\mathbf{D}_B([r^{k+l},r^{i+k+l}])}$$

$$= \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}(q-1)|B|\widehat{\mathbf{D}_B([r^m,r^{i+m}])} = |B|(q-1)\sum_{k=0}^{q-2}\widehat{\mathbf{E}_B([1,r^i])} = (q-1)^2|B|\widehat{\mathbf{E}_B([1,r^i])}.$$

Note that for any $1\le k,l\le q-2$ we have $r^kr^l = r^{i+k}r^{j+l}$ if and only if $r^{i+j}=1$. That is, if and only if $i+j=q-1$. Thus if $i+j=q-1$, then

$$\widehat{\mathbf{E}_B([1,r^i])}\widehat{\mathbf{E}_B([1,r^j])} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^{i+k}])}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^{j+l}])}\big)$$

$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}q|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}]')} + q|B|\widehat{\mathbf{D}_B([r^{k+l},r^{k+l}])}$$

$$= \sum_{k=0}^{q-2}q|B|\widehat{\mathbf{E}_B([1,1]')} + q|B|\widehat{\mathbf{E}_B([1,1])} = q(q-1)|B|\widehat{\mathbf{E}_B([1,1]')} + q(q-1)|B|\widehat{\mathbf{E}_B([1,1])}.$$

However, if $i+j\neq q-1$, then

$$\widehat{\mathbf{E}_B([1,r^i])}\widehat{\mathbf{E}_B([1,r^j])} = \big(\sum_{k=0}^{q-2}\widehat{\mathbf{D}_B([r^k,r^{i+k}])}\big)\cdot\big(\sum_{l=0}^{q-2}\widehat{\mathbf{D}_B([r^l,r^{j+l}])}\big)$$

$$= \sum_{k=0}^{q-2}\sum_{l=0}^{q-2}q|B|\widehat{\mathbf{D}_B([r^{k+l},r^{i+j+k+l}])} = \sum_{k=0}^{q-2}\sum_{m=0}^{q-2}q|B|\widehat{\mathbf{D}_B([r^m,r^{i+j+m}])}$$

$$= \sum_{k=0}^{q-2} q|B|\widehat{\mathbf{E}_B([1, r^{i+j}])} = q(q-1)|B|\widehat{\mathbf{E}_B([1, r^{i+j}])}.$$ □

7. Association schemes attached to diagonal double cosets and Schur rings

We begin this section with a brief review of the connection between association schemes and Schur rings. For an explanation of the history and development of association schemes, see Bannai and Ito [1]. For further foundational details connecting these schemes to algebraic graph theory and group symmetries, we refer the reader to [8].

7.1. Association schemes and Schur rings. Let X be a finite set. An *association scheme* on X is a collection of relations $\{R_0, R_1, \dots, R_d\}$ on X such that

(1) $\bigsqcup_{i=0}^{d} R_i = X \times X$,
(2) $R_0 = \{(x, x) \mid x \in X\}$,
(3) for each i there exists j such that $R_i^\top = R_j$, where $R_i^\top = \{(y, x) \mid (x, y) \in R_i\}$,
(4) for each (i, j, k) there exists an integer $p_{ij}^k \in \mathbb{N}$ such that for every $(x, y) \in R_k$, the number of $z \in X$ with $(x, z) \in R_i$ and $(z, y) \in R_j$ is equal to p_{ij}^k (independent of $(x, y) \in R_k$).

The integers p_{ij}^k are the *intersection numbers* of the scheme. Equivalently, we can encode the scheme by its adjacency matrices $A_i \in Mat_X(\mathbb{C})$, where $(A_i)_{x,y} = 1$ if $(x, y) \in R_i$ and 0 otherwise. Then $A_0 = I$, $\sum_{i=0}^{d} A_i = J$ (the all-one matrix), and

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A_k.$$

When the *adjacency algebra* $\langle A_0, \dots, A_d \rangle$ is commutative, it is called the *Bose-Mesner algebra* of the scheme.

For the standard construction of an association scheme from a Schur ring, the identity must be a singleton basic set, because the scheme requires $R_0 = \{(x, x) : x \in X\}$. Since our convention for Schur rings allows the identity-containing basic set to be larger than $\{e\}$, we first pass to the identity refinement. If the Schur ring partition is $T_1, T_2, \dots, T_t$ with $e \in T_1$, then replace it by

$$\{e\}, \quad T_1 \setminus \{e\}, \quad T_2, \dots, T_t,$$

omitting $T_1 \setminus \{e\}$ if it is empty. This refinement preserves the inverse-closed property and the relevant commutativity and centrality needed here, and it is the partition used to define the relations R_i.

Let G be a finite group and let $\Pi = \{X_0, X_1, \ldots, X_d\}$ be either a Schur ring partition satisfying $X_0 = \{e\}$ or the identity-refined partition obtained as above. Thus $X_0 = \{e\}$ and $X_i^{-1} = X_j$ for some j (for each i), where $X_i^{-1} = \{g^{-1} \mid g \in X_i\}$. Let $S_{G,\Pi} \subseteq \mathbb{C}[G]$ be the corresponding Schur ring with basis $\widehat{X_i} := \sum_{g \in X_i} g$, for $i \in \{0, 1, \ldots, d\}$. We will look closely at the right regular action ρ of $\mathbb{C}[G]$ on $End(\mathbb{C}[G])$. For each class X_i, set

$$D_i := \rho(\widehat{X_i}) \in Mat_G(\mathbb{C}).$$

Note each D_i is a matrix whose entries consist only of zeros and ones. Furthermore, $D_0 = I_{|G|}$ and $\sum_i D_i = J_{|G|}$. Moreover,

$$\rho(S_{G,\Pi}) = \mathrm{span}_{\mathbb{C}}\{D_0, \ldots, D_d\}.$$

The elements of the partition Π give us relations on $X := G$ by

$$R_i := \{(x, y) \in G \times G \mid x^{-1}y \in X_i\}.$$

Furthermore, the adjacency matrix of R_i is precisely D_i, and the structure constants of the Schur ring multiplication

$$\widehat{X_i}\,\widehat{X_j} = \sum_{k=0}^{d} p_{ij}^k\,\widehat{X_k}$$

agree with the intersection numbers of the resulting association scheme. In this way, commutative Schur rings provide a uniform source of (commutative) association schemes, and conversely the adjacency algebra viewpoint can be used to study Schur rings.

7.2. Diagonal double cosets and the conjugacy class scheme. Recall from Proposition 1.3 that the double coset algebra $\mathscr{D}(G)$ of the Gelfand pair $(G \times G, G^*)$ is naturally isomorphic to the algebra $\mathscr{C}(G)$ of class functions (or equivalently, to $Z(\mathbb{C}[G])$). Concretely, the (G^*, G^*)-double cosets in $G \times G$ are parametrized by conjugacy classes of G. This has an immediate association scheme interpretation.

To explain, let Π be a partition of G into its conjugacy classes. The corresponding Schur ring is the center $Z(\mathbb{C}[G])$, and the associated relations

$$R_{\mathbf{C}} = \{(x, y) \in G \times G \mid x^{-1}y \in \mathbf{C}\}, \qquad \mathbf{C} \in \mathcal{C}(G),$$

form the *conjugacy class association scheme* on the vertex set G. Its Bose-Mesner algebra is canonically identified with $Z(\mathbb{C}[G]) \cong \mathscr{D}(G)$. We explain this in an explicit example.

Example 7.1. We will compute the conjugacy class scheme for the affine group over $\mathbb{F}_q$, which is directly related to the semidirect-product calculations appearing for the Borel subgroup $B = U \rtimes T$.

Let $\mathrm{Aff}(\mathbb{F}_q) = AGL(1,q)$ be the group of affine transformations $x \mapsto ax+b$ with $a \in \mathbb{F}_q^\times$ and $b \in \mathbb{F}_q$. We write elements as pairs (a,b) with multiplication

$$(a,b)(a',b') = (aa',\, b+ab').$$

Now it is not difficult to check that $AGL(1,q) \cong \mathbb{F}_q \rtimes \mathbb{F}_q^\times$.

Equivalently, we can represent $\mathrm{Aff}(\mathbb{F}_q)$ as a subgroup of the Borel subgroup of upper triangular matrices in $GL(2,\mathbb{F}_q)$ via the map

$$(a,b) \leftrightsquigarrow \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$

A direct computation shows that conjugation is given by

$$(c,d)(a,b)(c,d)^{-1} = \big(a,\ cb + (1-a)d\big).$$

From this, we list all of the conjugacy classes as follows:

- $\mathbf{C}_0 := \{(1,0)\}$ (the identity),
- $\mathbf{C}_T := \{(1,b) \mid b \in \mathbb{F}_q^\times\}$, a single class of size $q-1$ (all nontrivial translations),
- for each $a \in \mathbb{F}_q^\times \setminus \{1\}$,
$$\mathbf{C}_a := \{(a,b) \mid b \in \mathbb{F}_q\},$$
a class of size q.

Thus $AGL(1,q)$ has exactly q conjugacy classes, hence the associated scheme has q relations (including R_0).

We proceed to describe explicitly the associated association scheme. To ease our notation, we set

$$G := AGL(1,q) = \{(a,b) \mid a \in \mathbb{F}_q^\times,\ b \in \mathbb{F}_q\}, \qquad (a,b)(a',b') = (aa',\, b+ab').$$

The relevant partition, Π, of G has q blocks, namely $\mathbf{C}_0$, $\mathbf{C}_T$ and the $q-2$ blocks $\mathbf{C}_a$ such that $a \in \mathbb{F}_q^\times \setminus \{1\}$.

We consider the relations on $X = G$ defined by

$$R_\bullet = \{(x,y) \in G \times G \mid x^{-1}y \in \mathbf{C}_\bullet\}.$$

Let v_i be the number of elements $y \in G$ such that $(x,y) \in R_i$ for a fixed $x \in G$. (The v_i's are called the *valencies* of the association scheme.) The valencies in this example are

$$v_0 = |\mathbf{C}_0| = 1, \qquad v_T = |\mathbf{C}_T| = q-1, \qquad v_a = |\mathbf{C}_a| = q \quad (a \neq 1).$$

Moreover $R_T^\top = R_T$, and for $a \neq 1$ one has $R_a^\top = R_{a^{-1}}$ since $\mathbf{C}_a^{-1} = \mathbf{C}_{a^{-1}}$.

We proceed to compute the intersection numbers via class multiplication. For $A \subseteq G$, we use the notation $\widehat{A}$, as above, to denote the sum $\widehat{A} := \sum_{x\in A} x \in \mathbb{Z}[G]$. Now, if $A, B \subseteq G$, then in the group algebra we have

$$\widehat{A}\,\widehat{B} = \sum_{g\in G} N_{A,B}(g)\, g, \qquad N_{A,B}(g) := \#\{(x,y) \in A \times B \mid xy = g\}. \tag{7.1}$$

In particular, for conjugacy classes $\mathbf{C}_i, \mathbf{C}_j$ the element $\widehat{\mathbf{C}_i}\,\widehat{\mathbf{C}_j}$ lies in the algebra $Z(\mathbb{Z}[G])$, hence the coefficient $N_{\mathbf{C}_i,\mathbf{C}_j}(g)$ is constant on conjugacy classes. We will determine the coefficients in the product

$$\widehat{\mathbf{C}_i}\,\widehat{\mathbf{C}_j} = \sum_k p_{ij}^k\, \widehat{\mathbf{C}_k}.$$

To this end we consider three different cases:
(i) Translation class squared. Elements of $\mathbf{C}_T$ have the form $(1, b)$ with $b \neq 0$, and

$$(1, b)(1, b') = (1, b + b').$$

Fix $c \in \mathbb{F}_q$ and consider $g = (1, c)$. By (7.1), the coefficient of $(1, c)$ in $\widehat{\mathbf{C}_T}^2$ equals the number of ordered pairs $(b, b') \in (\mathbb{F}_q^\times)^2$ with $b + b' = c$. Since $b' = c - b$, we are counting the number of $b \in \mathbb{F}_q^\times$ such that also $c - b \neq 0$.

- If $c = 0$, then $b' = -b \neq 0$ for every $b \neq 0$, so there are $q - 1$ solutions.
- If $c \neq 0$, then among the $q-1$ choices of $b \neq 0$, the unique forbidden choice is $b = c$ (which forces $b' = 0$), so there are $q - 2$ solutions.

Thus every element of $\mathbf{C}_0$ occurs with coefficient $q-1$, every element of $\mathbf{C}_T$ occurs with coefficient $q-2$, and no other conjugacy class occurs. Hence, we conclude that

$$\widehat{\mathbf{C}_T}\,\widehat{\mathbf{C}_T} = (q-1)\widehat{\mathbf{C}_0} + (q-2)\widehat{\mathbf{C}_T}.$$

Equivalently,

$$p_{TT}^0 = q - 1, \qquad p_{TT}^T = q - 2, \qquad p_{TT}^a = 0 \ (a \neq 1).$$

(ii) Translation with a dilation class. Fix $a \neq 1$. For $(1, b) \in \mathbf{C}_T$ and $(a, t) \in \mathbf{C}_a$, we have

$$(1, b)(a, t) = (a, b + t).$$

Fix a target element $g = (a, u) \in \mathbf{C}_a$. By (7.1), the coefficient of (a, u) in $\widehat{\mathbf{C}_T}\,\widehat{\mathbf{C}_a}$ equals the number of ordered pairs (b, t) with $b \in \mathbb{F}_q^\times$, $t \in \mathbb{F}_q$ satisfying $b + t = u$. For each choice of $b \neq 0$ there is a unique solution $t = u - b \in \mathbb{F}_q$. Therefore each (a, u) occurs with coefficient $q - 1$, and no other class occurs:

$$\widehat{\mathbf{C}_T}\,\widehat{\mathbf{C}_a} = (q-1)\widehat{\mathbf{C}_a}.$$

Hence, in this case, we find

$$p_{T,a}^a = q - 1, \qquad p_{T,a}^k = 0 \text{ for } k \neq a.$$

(Equivalently, we have $\widehat{\mathbf{C}_a}\,\widehat{\mathbf{C}_T} = (q-1)\widehat{\mathbf{C}_a}$.)
(iii) Two dilation classes. Let $a, a' \neq 1$. For $(a,s) \in \mathbf{C}_a$ and $(a',t) \in \mathbf{C}_{a'}$, we have

$$(a,s)(a',t) = (aa',\ s + at).$$

We fix a target element $g = (aa', u)$. By (7.1), the coefficient of (aa', u) in $\widehat{\mathbf{C}_a}\,\widehat{\mathbf{C}_{a'}}$ equals the number of ordered pairs $(s,t) \in \mathbb{F}_q^2$ satisfying $s + at = u$. We can choose $t \in \mathbb{F}_q$ freely (there are q choices). Then $s = u - at$ is uniquely determined and lies in $\mathbb{F}_q$. Hence, each element of $\mathbf{C}_{aa'}$ occurs with coefficient q, and no other class occurs. In other words, we find that

$$\widehat{\mathbf{C}_a}\,\widehat{\mathbf{C}_{a'}} = q\,\widehat{\mathbf{C}_{aa'}}.$$

If $aa' \neq 1$, then $\mathbf{C}_{aa'}$ is one of our basis classes. On the other hand, if $aa' = 1$, then $a' = a^{-1}$ and we see that

$$\widehat{\mathbf{C}_1} = \sum_{b \in \mathbb{F}_q} (1,b) = \widehat{\mathbf{C}_0} + \widehat{\mathbf{C}_T},$$

so

$$\widehat{\mathbf{C}_a}\,\widehat{\mathbf{C}_{a^{-1}}} = q(\widehat{\mathbf{C}_0} + \widehat{\mathbf{C}_T}), \qquad \text{implying} \qquad p^0_{a,a^{-1}} = q, \quad p^T_{a,a^{-1}} = q.$$

These identities completely determine the intersection numbers of the conjugacy class association scheme for $AGL(1,q)$. In particular, this yields an explicit, uniform family of Bose-Mesner algebras of dimension q arising from the same class-algebra/diagonal double coset mechanism used throughout this article.

7.3. Final remarks. We close our paper by bridging the explicit double coset computations of Section 6 with the association schemes that we introduced above. While the standard conjugacy class scheme explored in Example 7.1 arises naturally from the (G^*, G^*) double cosets (when $M = G^*$), the primary novelty of our work lies in the coarser partitions generated by the intermediate subgroups $G^* \triangleleft M \subset G \times G$. We describe an instance of this by explicitly constructing the association scheme corresponding to the (M, B^*) double cosets of the Borel subgroup B.

Recall from Section 6 that the algebra $\mathscr{E}(B) = \mathbb{C}[B \times B]^M_{B^*}$ has a linear basis given by the sums over the double cosets $\mathbf{E}_B(g)$. Under the inverse of the isomorphism Φ defined in Equation (3.3), each basis element $\frac{1}{|B|}\widehat{\mathbf{E}_B(g)}$ maps to a central element in $\mathbb{C}[B]$. Since M acts to fuse the standard (B^*, B^*) double cosets, the inverse image of each $\mathbf{E}_B(g)$ corresponds to a union of conjugacy classes in B. Let us denote the underlying subset of B corresponding to this union as $\mathcal{K}_B(g)$. That is, we have

$$\Phi(\widehat{\mathcal{K}_B(g)}) = \frac{1}{|B|}\widehat{\mathbf{E}_B(g)}.$$

Now, since G^* is normal in M and $(B \times B, B^*)$ forms a Gelfand pair, the subsets $\mathcal{K}_B(g)$ form an inverse-closed partition of B, and the part containing e is refined, if necessary, by separating off the singleton $\{e\}$ as above. Let Π_B denote this identity-refined partition. Consequently, we can define a novel, coarser association scheme on the vertex set $X = B$. The relations are given by

$$R_Y = \{(x, y) \in B \times B \mid x^{-1}y \in Y\}, \qquad Y \in \Pi_B.$$

The adjacency matrices of these relations span a commutative Bose-Mesner algebra. Crucially, the structure constants of the Schur ring $\mathscr{E}(B)$, scaled by the group order as above, give the intersection numbers for this identity-refined *Diagonal M-Double Coset Association Scheme*, with the only additional bookkeeping coming from the possible split of the identity-containing part. Therefore, Proposition 6.6 is not merely an algebraic bookkeeping exercise. It provides the exact and explicit intersection numbers for this novel association scheme. For example, before the possible identity refinement, item (4) of Proposition 6.6 shows that composing the part corresponding to $\mathcal{K}_B([1,1]')$ with itself contributes to $\mathcal{K}_B([1,1]')$ with multiplicity $(q-1)(q-2)$, and to the identity-containing part $\mathcal{K}_B([1,1])$ with multiplicity $(q-1)^2$; after refinement, the latter contribution is separated between the singleton relation and its complement by the same coefficient count. The construction of new association schemes, particularly those with a small number of classes, remains an active area of research in algebraic combinatorics (see, for example, [19]). By varying the choice of the subgroup M strictly between G^* and $G \times G$, our mechanism provides a uniform algebraic machine for generating and computing the Bose-Mesner algebras of these new, coarse association schemes directly from the representation theory of finite groups.

Finally, we note that Hendrickson [10] found a bijection between the central Schur rings of a group G and the supercharacter theories of G (as defined by Diaconis and Isaacs [3]). In that context, a Schur ring is strictly defined with the added property that the identity element forms its own basic set (Property (3), discussed in Section 2). Since the double coset algebras explored in this article utilize a less restrictive definition (conceptually bridging them with Hecke algebras), it would be a fruitful direction for future research to investigate whether these diagonal subgroup constructions can provide deeper, generalized connections between Hecke algebras and supercharacter theories.

Acknowledgement. This research was initiated during the authors' visit to the Okinawa Institute of Science and Technology (OIST), with the first author participating through the Theoretical Sciences Visiting Program (TSVP). The authors are grateful to OIST for its support, and especially to Liron Speyer for facilitating this collaboration. The first author gratefully acknowledges funding from the Louisiana Board of Regents, award LEQSF(2023-25)-RD-A-21. The authors would also like to thank the reviewer for the insightful comments and suggestions which improved the clarity of this article.

Disclosure statement. The authors report that there are no competing interests to declare.

References

[1] E. Bannai and T. Ito, Algebraic Combinatorics I: Association Schemes, The Benjamin/Cummings Publishing Co., Inc., Menlo Park, CA, 1984.

[2] M. Brender, *A class of Schur algebras*, Trans. Amer. Math. Soc., 248(2) (1979), 435-444.

[3] P. Diaconis and I. M. Isaacs, *Supercharacters and superclasses for algebra groups*, Trans. Amer. Math. Soc., 360(5) (2008), 2359-2392.

[4] D. S. Dummit and R. M. Foote, Abstract Algebra, Third edition, John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[5] S. Evdokimov and I. Ponomarenko, *Permutation group approach to association schemes*, European J. Combin., 30(6) (2009), 1456-1476.

[6] I. M. Gelfand and M. L. Cetlin, *Finite-dimensional representations of the group of unimodular matrices*, Doklady Akad. Nauk SSSR (N.S.), 71 (1950), 825-828.

[7] I. M. Gelfand and M. L. Cetlin, *Finite-dimensional representations of groups of orthogonal matrices*, Doklady Akad. Nauk SSSR (N.S.), 71 (1950), 1017-1020.

[8] C. D. Godsil, Algebraic Combinatorics, Chapman and Hall Mathematics Series, Chapman & Hall, New York, 1993.

[9] Groupprops, Element structure of general linear group of degree two over a finite field, `https://groupprops.subwiki.org/wiki/Element_structure_of_general_linear_group_of_degree_two_over_a_finite_field`, 2019.

[10] A. O. F. Hendrickson, *Supercharacter theory constructions corresponding to Schur ring products*, Comm. Algebra, 40(12) (2012), 4420-4438.

[11] S. P. Humphries and D. R. Wagner, *Central Schur rings over the projective special linear groups*, Comm. Algebra, 45(12) (2017), 5325-5337.

[12] J. Karlof, *The subclass algebra associated with a finite group and subgroup*, Trans. Amer. Math. Soc., 207 (1975), 329-341.

[13] I. G. Macdonald, Symmetric Functions and Hall Polynomials, Second edition, Oxford Classic Texts in the Physical Sciences, The Clarendon Press, Oxford University Press, New York, 2015.

[14] J. E. Marrow, Strong Gelfand Pairs of Some Finite Groups, Ph.D. Thesis, Brigham Young University, 2024.

[15] M. Muzychuk and I. Ponomarenko, *Schur rings*, European J. Combin., 30(6) (2009), 1526-1539.

[16] I. Schur, *Zur theorie der einfach transitiven permutationsgruppen*, Sitzungsberichte der Preussischen Akademie der Wissenschaften, Physikalisch-Mathematische Klasse, (1933), 598-623.

[17] O. Tamaschke, *S-rings and the irreducible representations of finite groups*, J. Algebra, 1 (1964), 215-232.

[18] D. Travis, *Spherical functions on finite groups*, J. Algebra, 29 (1974), 65-76.

[19] E. R. van Dam, *Three-class association schemes*, J. Algebraic Combin., 10(1) (1999), 69-107.

[20] H. Wielandt, *Finite permutation groups and finite sums of group rings*, Proceedings of the International Colloquium on Algebra, Oxford University Press, (1969), 387-393.

Mahir Bilen Can (Corresponding Author)
Tulane University
New Orleans, Louisiana USA
e-mail: mahirbilencan@gmail.com

Julianne Rainbolt
Saint Louis University
Saint Louis, Missouri USA
e-mail: julianne.rainbolt@slu.edu