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LIE POLYNOMIALS IN A q-DEFORMED UNIVERSAL ENVELOPING ALGEBRA OF THE TWO-DIMENSIONAL NON-ABELIAN LIE ALGEBRA

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Abstract. The nonabelian two-dimensional Lie algebra over a field $\mathbb{F}$ has a presentation by generators A, B and relation $[A, B] = A$, with the universal enveloping algebra having a presentation by generators A, B and relation $AB - BA = A$. A solution to the Lie polynomial characterization problem in the corresponding class of q-deformed universal enveloping algebras, specifically of the algebra with relation $AB - qBA = A$ is presented.

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1. Introduction

Let $\mathbb{F}$ be a field. Up to isomorphism, there are only two (non-isomorphic) two-dimensional Lie algebras over $\mathbb{F}$: an abelian one, which we denote here by $\mathfrak{g}_0$, and a "solvable" one, which we shall denote by $\mathfrak{g}_1$. If $\mathfrak{g} \in \{\mathfrak{g}_0, \mathfrak{g}_1\}$, then, following [11, Theorem 3.1], the standard approach is to consider the derived (Lie) algebra $\mathfrak{g}'$ of $\mathfrak{g}$, or the $\mathbb{F}$-linear span of all possible Lie brackets of elements of $\mathfrak{g}$. Since $\mathfrak{g}$ is two-dimensional, we may fix a basis of $\mathfrak{g}$ consisting of two elements A and B. Since the Lie bracket is alternating (meaning the Lie bracket of anything with itself is zero), and is bilinear, any element of $\mathfrak{g}'$ is a scalar multiple of $[A, B]$ (the Lie bracket of A with B). Hence, the derived algebra $\mathfrak{g}'$ is at most one-dimensional. This gives the two classifications: $\mathfrak{g} = \mathfrak{g}_0$ (abelian) if $\mathfrak{g}'$ is the zero Lie algebra, or else $\mathfrak{g} = \mathfrak{g}_1$ (solvable). Routine arguments that make use of basis-to-basis linear maps such as $\phi : A \mapsto B, B \mapsto -A$, $\phi_t : A \mapsto A, B \mapsto tB$, and $\psi_t : A \mapsto tA, B \mapsto B$ (where $t \in \mathbb{F}\backslash\{0\}$ is a parameter) lead to proofs that the nonabelian two-dimensional Lie algebra $\mathfrak{g}_1$ is unique up to isomorphism. We refer the reader to [11, Chapters 3–4]

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for further details when we go to dimensions that are higher than two. This paper is about some bigger algebraic structures related to $\mathfrak{g}_1$.

Among the several isomorphic copies of $\mathfrak{g}_1$, we choose that which has a basis consisting of A and B that satisfy the "commutation relation" $[A,B]=A$. The universal enveloping algebra of $\mathfrak{g}_1$ is the (associative) algebra $\mathcal{U}(\mathfrak{g}_1)$ that has a presentation by generators A, B and relation $AB-BA=A$. Let $q\in\mathbb{F}$. In this work, we shall be interested in what is called the "q-analog" or "q-deformation" of the Lie bracket operation that was done on A and B, which results to the expression $AB-qBA$. The literature on q-analogs or q-deformations is extensive. With reference to the scope of this paper, what shall suffice is to mention here two of the important achievements made using q-analogs. The study of q-analogs of notions from ordinary calculus led to the discovery of many important notions and results in combinatorics, number theory, and other fields of mathematics [16, p. vii], and the q-analogs of commutation relations of important Hilbert space operators have been successfully applied to, for instance, particle physics, knot theory and general relativity [12, Chapter 12].

Since the q-deformation of the Lie bracket shall be considered later, we now mention some of the isomorphic forms of $\mathfrak{g}_1$ so that there shall be more clarity as to which of the isomorphisms are carried over, or not, after the q-deformation. If $\widetilde{\mathfrak{g}}_1$ is the Lie algebra over $\mathbb{F}$ with a basis consisting of A and B, subject to the relation $[A,B]=B$, then there exists a Lie algebra isomorphism $\mathfrak{g}_1\longrightarrow\widetilde{\mathfrak{g}}_1$ such that $A\mapsto B$ and $B\mapsto -A$. Given a nonzero $r\in\mathbb{F}$, if $\mathfrak{g}_r$ is the Lie algebra over $\mathbb{F}$ with a basis consisting of A, B that satisfy the commutation relation $[A,B]=rA$, then there exists a Lie algebra isomorphism $\mathfrak{g}_r\longrightarrow\mathfrak{g}_1$ that sends $A\mapsto A$ and $B\mapsto rB$. Also of interest here is the Lie algebra $\widetilde{\mathfrak{g}}_s$ over $\mathbb{F}$ with a basis consisting of A and B that obey the relation $[A,B]=sB$, given a nonzero $s\in\mathbb{F}$. There exists a Lie algebra isomorphism $\widetilde{\mathfrak{g}}_s\longrightarrow\widetilde{\mathfrak{g}}_1$ with the property that $A\mapsto sA$ and $B\mapsto B$. The universal enveloping algebras $\mathcal{U}(\mathfrak{g}_r)$, $\mathcal{U}(\widetilde{\mathfrak{g}}_s)$, for all $r,s\in\mathbb{F}\backslash\{0\}$, of the aforementioned Lie algebras are isomorphic. This may be proven routinely by the universal property of these universal enveloping algebras, together with the fact that, for the said isomorphic forms of the nonabelian two-dimensional Lie algebra, the inclusion maps $\mathfrak{g}_r\hookrightarrow\mathcal{U}(\mathfrak{g}_r)$ and $\widetilde{\mathfrak{g}}_s\hookrightarrow\mathcal{U}(\widetilde{\mathfrak{g}}_s)$ are injective [15, Exercise 17.2]. The isomorphism previously mentioned for the Lie algebras have their natural extension to algebra isomorphisms of the universal enveloping algebras. As a recapitulation, these algebra isomorphisms have the properties

$$\mathcal{U}(\mathfrak{g}_1) \longrightarrow \mathcal{U}(\widetilde{\mathfrak{g}}_1) \quad : \quad A \mapsto B, \quad B \mapsto -A, \tag{1}$$

$$\mathcal{U}(\mathfrak{g}_r) \longrightarrow \mathcal{U}(\mathfrak{g}_1) \quad : \quad A \mapsto A, \quad B \mapsto rB, \tag{2}$$

$$\mathcal{U}(\widetilde{\mathfrak{g}}_s) \longrightarrow \mathcal{U}(\widetilde{\mathfrak{g}}_1) \quad : \quad A \mapsto sA, \quad B \mapsto B. \tag{3}$$

Given a nonzero $q \in \mathbb{F}$, we now consider the q-deformation of the Lie bracket in the above algebras. Given a nonzero $r \in \mathbb{F}$, the algebra $\mathcal{U}(\mathfrak{g}_r)$ has a presentation by generators A, B and relation $AB - BA = rA$. The corresponding q-deformed algebra is what we shall denote by $\mathcal{U}_q(r, 0)$ that has a presentation by generators A, B and relation $AB - qBA = rA$. Analogously, the q-deformation $\mathcal{U}_q(0, s)$ for $\mathcal{U}(\widetilde{\mathfrak{g}}_s)$ has a presentation by generators A, B and relation $AB - qBA = sB$, where $s \in \mathbb{F}$ is nonzero.

The algebra $\mathcal{U}(\mathfrak{g}_1)$ has a natural Lie algebra stucture induced by the operation that sends any $X, Y \in \mathcal{U}(\mathfrak{g}_1)$ to $XY - YX$. The Lie subalgebra of $\mathcal{U}(\mathfrak{g}_1)$ generated by A and B is simply $\mathfrak{g}_1$, because of the relation $AB - BA = A$, and this is at the heart of the theory of universal enveloping algebras. This reduction of the Lie subalgebra to a smaller substructure is not necessarily true anymore for the algebra $\mathcal{U}_q(1, 0)$. In $\mathcal{U}_q(1, 0)$, we may still compute for Lie algebra expressions generated by A and B, but the new relation $AB - qBA = A$ does not imply that the Lie subalgebra of $\mathcal{U}_q(1, 0)$ generated by A and B, or the set of all "Lie polynomials" in $A, B \in \mathcal{U}_q(1, 0)$, shall be reduced into a small substructure. This is the main goal of this paper: to characterize all the Lie polynomials in A and B under a relation like $AB - qBA = A$, or what can be called the "Lie polynomial characterization problem" for the given presentation of $\mathcal{U}_q(1, 0)$ by generators and relations up to isomorphism.

Lie polynomial characterization problems were first studied in [2], which was about the universal Askey-Wilson algebra, an important mathematical object in algebraic combinatorics, which arose from mathematical physics. The Lie polynomial characterization problem was completely solved for the q-deformed Heisenberg algebra and some extensions of this algebraic structure [3,4,5,6,8,9,10]. The q-deformed Heisenberg algebra [13,14] is a q-analog of the Heisenberg-Weyl algebra [7], which is an algebraic structure important in quantum theory.

One can easily verify that when $q \neq 1$, there is no algebra homomorphism that corresponds to (1) for the q-deformed algebras $\mathcal{U}_q(s, 0)$ and $\mathcal{U}_q(0, s)$ that performs $A \mapsto \beta B$ and $B \mapsto \alpha A$ (for some nonzero $\alpha, \beta \in \mathbb{F}$), or the traditional map that "switches and scales" the two generators from $\mathcal{U}_q(s, 0)$ to $\mathcal{U}_q(0, s)$. If we assume that there is such homomorphism, then we would obtain equation $(q^2 - 1)BA + \frac{s}{\alpha} \cdot$

$(\alpha q+1)B = 0$ which would lead to a contradiction, since BA and B are linearly independent elements in $\mathcal{U}_q(0,s)$. We emphasize that this includes the case with $s = 1$, $\alpha = -1$ and $\beta = 1$, which serves as our proof that the algebra homomorphism $\mathcal{U}(\mathfrak{g}_1) \longrightarrow \mathcal{U}(\widetilde{\mathfrak{g}}_1)$ in (1), that sends $A \mapsto B$ and $B \mapsto -A$, has no q-analog. However, this is not sufficient reason to conclude that $\mathcal{U}_q(s,0)$ and $\mathcal{U}_q(0,s)$ are not isomorphic. In fact, in Section 5, we settle all these issues concerning isomorphisms leading to the conclusion that solving Lie polynomial characterization problem in $\mathcal{U}_q(1,0)$ is sufficient.

The aforementioned results were obtained with the aid of the Diamond Lemma for Ring Theory [1, Theorem 1.2], which is an ingenious and indispensable tool in the determination of a basis for an algebra given a certain kind of presentation. The proofs and computations for the algebras $\mathcal{U}_q(r,0)$ and $\mathcal{U}_q(0,s)$ that are based on the Diamond Lemma are analogous when done separately. For a better presentation of these proofs and computations, we decided to generalize the algebras $\mathcal{U}_q(r,0)$ and $\mathcal{U}_q(0,s)$ into an algebra $\mathcal{U}_q(r,s)$ which has a presentation by generators A, B and relation $AB - qBA = rA + sB$. The basis theorem, Theorem 4.3, is valid not only for the aforementioned restrictions on r and s for the relevant Lie algebras, but also for any choice of r and s in the field $\mathbb{F}$.

2. Preliminaries

Given a field $\mathbb{F}$, any $\mathbb{F}$-algebra shall be assumed to be associative and unital. Since only one field $\mathbb{F}$ will be used, we further drop the prefix "$\mathbb{F}$-" and simply use the term algebra. A Lie algebra structure is induced on an algebra $\mathcal{A}$ by the operation $[X,Y] := XY - YX$ for all $X, Y \in \mathcal{A}$. If $A_1, A_2, \ldots, A_n \in \mathcal{A}$, then the Lie subalgebra $\mathcal{K}$ of $\mathcal{A}$ generated by $A_1, A_2, \ldots, A_n$ is the smallest Lie subalgebra which contains $A_1, A_2, \ldots, A_n$. That is, if $\mathcal{S}$ is a Lie subalgebra of $\mathcal{A}$, if $\{A_1, A_2, \ldots, A_n\} \subseteq \mathcal{S}$, and if $\mathcal{S} \subseteq \mathcal{K}$, then $\mathcal{S} = \mathcal{K}$. In such a case, we refer to the elements of $\mathcal{K}$ as *Lie polynomials in* $A_1, A_2, \ldots, A_n$.

We denote the set of all nonnegative integers by $\mathbb{N}$, and the set of all positive integers by $\mathbb{Z}^+$. We fix $\nu \in \mathbb{Z}^+$, and let $\mathcal{X} = \mathcal{X}_\nu$ be a set with ν elements. The free monoid on $\mathcal{X}$ shall be denoted by $\langle \mathcal{X} \rangle$, while the free algebra generated by $\mathcal{X}$ shall be denoted by $\mathbb{F}\langle \mathcal{X} \rangle$. Most of the fundamental notions and properties of the aforementioned free monoid and free algebra may be seen, for instance, from [17, Chapter 1] or [18, Section 1.1], and we proceed with only the terminology and notation necessary. Any basis element of $\mathbb{F}\langle \mathcal{X} \rangle$ from the basis $\langle \mathcal{X} \rangle$ is called a *word* on $\mathcal{X}$. The *length* of a word $W \in \langle \mathcal{X} \rangle$ shall be denoted by $|W|$. Words of length 1 are precisely the elements of $\mathcal{X}$, and are called *letters*. The word of length 0 is called the *empty word* in $\langle \mathcal{X} \rangle$ and shall denoted by I, which is also the identity element under the concatenation multiplication in $\langle \mathcal{X} \rangle$. If $|W| \neq 0$, then W is said to be a *nonempty word*, but we further define $W^0 := I$. Multiplication in $\mathbb{F}\langle \mathcal{X} \rangle$ is determined by the concatenation product in $\langle \mathcal{X} \rangle$. Given a word $W \in \langle \mathcal{X} \rangle$, a word W' is said to be a *subword* of W if there exist words $L, R \in \langle \mathcal{X} \rangle$ such that $W = LW'R$.

If $\mathcal{X} = \{X_1, X_2, \ldots, X_\nu\}$, given $L_1, R_1, L_2, R_2, \ldots, L_m, R_m \in \mathbb{F}\langle \mathcal{X} \rangle$, let $\mathcal{I}$ be the (two-sided) ideal of $\mathbb{F}\langle \mathcal{X} \rangle$ generated by $L_1 - R_1, L_2 - R_2, \ldots, L_m - R_m$. The *algebra with generators* $X_1, X_2, \ldots, X_n$ *and relations* $L_1 = R_1$, $L_2 = R_2$, $\ldots$, $L_m = R_m$ is the quotient algebra $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$. With respect to the natural embedding $\mathcal{X} \hookrightarrow \mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$, if $\mathcal{K}$ is the Lie subalgebra of $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$ generated by $X_1, X_2, \ldots, X_\nu$, then a characterization of the elements of $\mathcal{K}$ is said to be a solution to the *Lie polynomial characterization problem* with respect to the aforementioned presentation of $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$.

We recall *Bergman's Diamond Lemma* or the *Diamond Lemma for Ring Theory* [1, Theorem 1.2], together with some related notions taken from [1, Section 1], which are crucial in determining a basis for $\mathbb{F}\langle \mathcal{X} \rangle / \mathcal{I}$. A set of ordered pairs of the form $\lambda = (W_\lambda, f_\lambda)$ where $W_\lambda \in \langle \mathcal{X} \rangle$ and $f_\lambda \in \mathbb{F}\langle \mathcal{X} \rangle$ is called a *reduction system*. Let S be a reduction system. Given $\lambda \in S$ and $L, R \in \langle \mathcal{X} \rangle$, by the *reduction* $\mathfrak{r}_{L\lambda R}$ we mean the linear mapping $\mathbb{F}\langle \mathcal{X} \rangle \longrightarrow \mathbb{F}\langle \mathcal{X} \rangle$ that fixes all elements of $\langle \mathcal{X} \rangle$ other than $LW_\lambda R$, and instead sends this basis element of $\mathbb{F}\langle \mathcal{X} \rangle$ to the element $Lf_\lambda R$. A reduction $\mathfrak{r}_{L\lambda R}$ acts *trivially* on an element K of $\mathbb{F}\langle \mathcal{X} \rangle$ if the coefficient of the basis element $LW_\lambda R$ in K is zero. If every reduction acts trivially on an element K, then K is *irreducible* (under S). We say that $K \in \mathbb{F}\langle \mathcal{X} \rangle$ is *reduction-finite* if for every infinite sequence $\mathfrak{r}_1, \mathfrak{r}_2, \ldots$ of reductions, there exists $N \in \mathbb{N}$ such that $\mathfrak{r}_i$ acts trivially on $(\mathfrak{r}_{i-1} \circ \mathfrak{r}_{i-2} \circ \cdots \circ \mathfrak{r}_1)(K)$ for all $i \geq N$. If K is reduction-finite, then a *final sequence* is any maximal finite sequence of reductions $\mathfrak{r}_i$, such that each $\mathfrak{r}_i$ acts nontrivially on $(\mathfrak{r}_{i-1} \circ \mathfrak{r}_{i-2} \circ \cdots \circ \mathfrak{r}_1)(K)$. Additionally, if K is reduction-finite and if its images under final sequences of reductions are the same, then we say that K is *reduction-unique*.

A 5-tuple $(\lambda, \tau, W_1, W_2, W_3)$ where $\lambda, \tau \in S$ and $W_1, W_2, W_3 \in \langle \mathcal{X} \rangle \backslash \{I\}$ is an *overlap ambiguity* if $W_\lambda = W_1 W_2$ and $W_\tau = W_2 W_3$. This ambiguity is said to be *resolvable* if there exist compositions of reductions $\mathfrak{r}$ and $\mathfrak{r}'$ such that $\mathfrak{r}(f_\lambda W_3) = \mathfrak{r}'(W_1 f_\tau)$. Also, a 5-tuple $(\lambda, \tau, W_1, W_2, W_3)$ where $\lambda \neq \tau \in S$ and $W_1, W_2, W_3 \in \langle \mathcal{X} \rangle$ is an *inclusion ambiguity* if $W_\lambda = W_2$ and $W_\tau = W_1 W_2 W_3$. This ambiguity is said to be *resolvable* if there exist compositions of reductions $\mathfrak{r}$

and $\mathfrak{r}'$ such that $\mathfrak{r}(f_\tau) = \mathfrak{r}'(W_1 f_\lambda W_3)$. By an *ambiguity* of S, we mean either an overlap ambiguity or an inclusion ambiguity.

Theorem 2.1. [Diamond Lemma] *Let S be a reduction system on $\mathbb{F}\langle\mathcal{X}\rangle$. Let $\mathcal{A}$ be an algebra with generators in $\mathcal{X}$ and relations $W_\lambda = f_\lambda$ for all $\lambda \in S$. The following conditions are equivalent.*

(i) *All ambiguities of S are resolvable.*
(ii) *All elements of $\mathbb{F}\langle\mathcal{X}\rangle$ are reduction-unique under S.*
(iii) *The set of all irreducible words on $\mathcal{X}$ with respect to S form a basis for $\mathcal{A}$.*

3. The algebra $\mathcal{U}_q(r,s)$

We now consider the case when $\mathcal{X}$ has only two elements A and B. Given $q, r, s \in \mathbb{F}$, let $\mathcal{I}_1 = \mathcal{I}_1(q,r,s)$ be the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by $AB - qBA - rA - sB$, and let $\mathcal{U}_q(r,s) := \mathbb{F}\langle\mathcal{X}\rangle/\mathcal{I}_1$. Succeeding computations will involve division by a power of q or by a field element of the form $1 - q^m$ for some nonzero integer m. Thus, we assume that $\mathbb{F}$ has characteristic zero, and the scalar q is nonzero, and is not a root of unity.

Proposition 3.1. *For any $n \in \mathbb{Z}^+$, the identities*

$$A^n B = B\sum_{t=0}^{n}\binom{n}{t}s^t(qA)^{n-t} + r\sum_{i=0}^{n-1}(qA+s)^{n-1-i}A^{i+1}, \tag{4}$$

$$AB^n = \sum_{t=0}^{n}\binom{n}{t}r^t(qB)^{n-t}A + s\sum_{i=0}^{n-1}(qB+r)^{n-1-i}B^{i+1} \tag{5}$$

hold in $\mathcal{U}_q(r,s)$.

Proof. Both identities simply reduce to the defining relation $AB - qBA = rA + sB$ of $\mathcal{U}_q(r,s)$ when $n = 1$. If the given identities hold for some $n \in \mathbb{Z}^+$, then, with the goal of performing induction on n, at $n+1$, the desired left-hand sides may be obtained by multiplying A from the left or by B from the right. In the resulting right-hand sides, the identity $AB = qBA + rA + sB$ may be used such that, after a finite number of steps, the desired linear combinations of words will appear in the new right-hand sides. By induction on n, the desired identities are indeed true in $\mathcal{U}_q(r,s)$. □

By introducing a new letter $C := [A,B] = AB - BA$, the algebra $\mathcal{U}_q(r,s)$ would consequently have the following presentation.

Lemma 3.2. *The algebra $\mathcal{U}_q(r,s)$ has a presentation by generators A, B, C and relations*

$$AB - qBA = rA + sB, \tag{6}$$

$$C = AB - BA. \tag{7}$$

Proof. Given $\mathcal{X}_3 = \{A, B, C\}$, let $\mathcal{I}_2$ be the ideal of $\mathbb{F}\langle\mathcal{X}_3\rangle$ generated by $AB - qBA - rA - sB$ and $C - AB + BA$. Since the generators in the respective presentations for $\mathbb{F}\langle\mathcal{X}\rangle/\mathcal{I}_1$ and $\mathbb{F}\langle\mathcal{X}_3\rangle/\mathcal{I}_2$ satisfy the relations of the other, a routine argument may be used to show that there exists an algebra isomorphism $\mathbb{F}\langle\mathcal{X}\rangle/\mathcal{I}_1 \longrightarrow \mathbb{F}\langle\mathcal{X}_3\rangle/\mathcal{I}_2$ which maps $A \mapsto A$, $B \mapsto B$ and $[A, B] \mapsto C$. □

We will often refer to some q-special relations from [13, Appendix C] such as the following. For a given $n \in \mathbb{N}$ and $z \in \mathbb{F}$,

$$\{n\}_z := \sum_{t=0}^{n-1} z^t, \tag{8}$$

$$(1-z)\{n\}_z = 1 - z^n. \tag{9}$$

If $n \leq 0$, then we interpret (8) as the empty sum 0.

Lemma 3.3. *Let $\xi_2 = \xi_2(q,s) := AC - qCA - sC$. For any $h \in \mathbb{Z}^+$,*

$$\sum_{i=1}^{h} q^{i-1}C^{i-1}\xi_2 C^{h-i} = AC^h - q^h C^h A - \{h\}_q sC^h. \tag{10}$$

Proof. The case $h = 1$ is simply the definition of ξ_2. Suppose (10) is true for some $h \in \mathbb{N}$. Multiplying both sides by C from the right, the resulting right-hand side is a linear combination of the words AC^{h+1}, C^hAC and C^{h+1}, where the AC in C^hAC may be replaced using the relation $AC = \xi_2 + qCA + sC$, which is immediate from the definition of ξ_2. Adding $q^hC^h\xi_2$ to both sides, we find that the identity is true at $h+1$. The desired result follows by induction on h. □

Proposition 3.4. *The algebra $\mathcal{U}_q(r,s)$ has a presentation by generators A, B, C and relations*

$$AB = \frac{rA + sB - qC}{1-q}, \tag{11}$$

$$AC = qCA + sC, \tag{12}$$

$$BA = \frac{rA + sB - C}{1-q}, \tag{13}$$

$$CB = qBC + rC, \tag{14}$$

$$BC^kA = \frac{q^k rC^kA + q^k sBC^k + \{k\}_q rsC^k - C^{k+1}}{q^k(1-q)}, \quad k \in \mathbb{Z}^+. \tag{15}$$

Proof. We view the left-hand side and right-hand side expressions of equations (11) to (15) as elements of $\mathbb{F}\langle\mathcal{X}_3\rangle$ with $\mathcal{X}_3 = \{A, B, C\}$ and define

$$\xi_1 := AB - \frac{rA + sB - qC}{1-q}, \tag{16}$$

$$\xi_2 := AC - qCA - sC, \tag{17}$$

$$\xi_3 := BA - \frac{rA + sB - C}{1-q}, \tag{18}$$

$$\xi_4 := CB - qBC - rC, \tag{19}$$

$$\xi_5(k) := BC^kA - \frac{q^k rC^kA + q^k sBC^k + \{k\}_q rsC^k - C^{k+1}}{q^k(1-q)}, \quad k \in \mathbb{Z}^+. \tag{20}$$

Also, we denote generators of $\mathcal{I}_2$ by

$$\zeta_1 = AB - qBA - rA - sB, \tag{21}$$

$$\zeta_2 = C - AB + BA. \tag{22}$$

Let $\mathcal{I}_3$ be the ideal of $\mathbb{F}\langle\mathcal{X}_3\rangle$ generated by

$$\{\xi_1, \xi_2, \xi_3, \xi_4\} \cup \{\xi_5(k)\} : k \in \{1, 2, 3, \ldots\}. \tag{23}$$

We claim that $\mathcal{I}_2 = \mathcal{I}_3$. The relations (16) and (18) may be used in some routine computations to obtain

$$\xi_3 - \xi_1 = C - AB + BA, \tag{24}$$

$$\xi_1 - q\xi_3 = AB - qBA - rA - sB, \tag{25}$$

provided $q \neq 1$. With the use of (24) and (25), we have $\zeta_1, \zeta_2 \in \mathcal{I}_3$. Thus we have $\mathcal{I}_2 \subseteq \mathcal{I}_3$. Next we show that $\mathcal{I}_3 \subseteq \mathcal{I}_2$. Observe that

$$\begin{aligned} \zeta_1 + q\zeta_2 &= AB - qBA - rA - sB + qC - qAB + qBA, \\ &= (1-q)AB - rA - sB + qC, \\ \frac{\zeta_1 + q\zeta_2}{1-q} &= AB - \frac{rA + sB - qC}{1-q}. \end{aligned} \tag{26}$$

From (26), we can easily derive

$$AB = \frac{\zeta_1 + q\zeta_2}{1-q} + \frac{rA + sB - qC}{1-q}. \tag{27}$$

By further routine computations,

$$A\zeta_1 - \zeta_1A + A\zeta_2 - q\zeta_2A = AC - qCA - sC, \tag{28}$$

$$\frac{\zeta_2 + \zeta_1}{1-q} = BA - \frac{rA + sB - C}{1-q}, \tag{29}$$

$$\zeta_1B - B\zeta_1 + \zeta_2B - qB\zeta_2 = CB - qBC - rC. \tag{30}$$

Equations (26) and (29) clearly show that ξ_1 and ξ_3 are linear combinations of ζ_1 and ζ_2, and so we have $\xi_1, \xi_3 \in \mathcal{I}_2$. Also, because of absorbing property of ideals, equations (28) and (30) suggest that $\xi_2, \xi_4 \in \mathcal{I}_2$.

From Lemma 3.3, we can easily obtain

$$AC^k = \sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} + q^k C^k A + \{k\}_q s C^k. \tag{31}$$

Since we have established $\xi_2, \xi_3 \in \mathcal{I}_2$, we have $\xi_3 C^n, B\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} \in \mathcal{I}_2$. Observe that

$$\begin{aligned}
(1-q)\xi_3 C^k &= (1-q)BAC^k - rAC^k - sBC^k + C^{k+1}, \\
&= (1-q)BAC^k - r\left(\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} + q^k C^k A + \{k\}_q s C^k\right) \\
&\quad -sBC^k + C^{k+1}, \\
&= (1-q)BAC^k - r\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} - q^k r C^k A - \{k\}_q rs C^k \\
&\quad -sBC^k + C^{k+1},
\end{aligned}$$

and transposing the summation to the left-hand side, we obtain

$$\begin{aligned}
(1-q)\xi_3 C^k + r\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} &= (1-q)BAC^k - q^k r C^k A - \{k\}_q rs C^k \\
&\quad -sBC^k + C^{k+1}.
\end{aligned}$$

Also with Lemma 3.3, we have

$$\begin{aligned}
(1-q)B\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} &= (1-q)BAC^k - (1-q)q^k BC^k A \\
&\quad -(1-q)\{k\}_q sBC^k.
\end{aligned}$$

The previous two identities, together with the earlier one with left-hand side AC^k, may be used in routine computations to show that

$$(1-q)\xi_3 C^k + r\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i} - (1-q)B\sum_{i=1}^{k} q^{i-1}C^{i-1}\xi_2 C^{k-i}, \tag{32}$$

is equal to the linear combination

$$(1-q)q^k BC^k A - q^k r C^k A - q^k sBC^k - \{k\}_q rs C^k + C^{k+1},$$

and this implies that $\xi_5(k)$ for any k is a linear combination of elements in $\mathcal{I}_2$ and that $\xi_5(k) \in \mathcal{I}_2$. We now have $\mathcal{I}_3 \subseteq \mathcal{I}_2$. Thus, we have $\mathcal{U}_q(r,s) = \mathbb{F}\langle \mathcal{X}_3 \rangle / \mathcal{I}_3$. $\square$

4. A basis for $\mathcal{U}_q(r,s)$

A basis of an algebra holds essential information for understanding its algebraic structure [19, p. 10]. So we choose a basis for $\mathcal{U}_q(r,s)$ based from its presentation on Proposition 3.4 using Bergman's Diamond Lemma.

We use relations of $\mathcal{U}_q(r,s)$ given in Proposition 3.4 to construct a reduction system in $\mathbb{F}\langle\mathcal{X}_3\rangle$. Let

$$\sigma_1 = \left(AB, \frac{rA+sB-qC}{1-q}\right), \tag{33}$$

$$\sigma_2 = \left(AC, qCA+sC\right), \tag{34}$$

$$\sigma_3 = \left(BA, \frac{rA+sB-C}{1-q}\right), \tag{35}$$

$$\sigma_4 = \left(CB, qBC+rC\right), \tag{36}$$

$$\tau_k = \left(BC^kA, \frac{q^k rC^kA + q^k sBC^k + \{k\}_q rsC^k - C^{k+1}}{q^k(1-q)}\right), \quad k\in\mathbb{Z}^+. \tag{37}$$

Then $R := \{\sigma_i, \tau_k \ : \ i\in\{1,2,3,4\}, k\in\{1,2,3,\ldots\}\}$ is a reduction system in $\mathbb{F}\langle\mathcal{X}_3\rangle$ for $\mathcal{U}_q(r,s)$ in three generators. In order to use an implication in Bergman's Diamond Lemma, first we show that any ambiguity of R is resolvable. It is routine to show that there is no inclusion ambiguity given the reduction system R. In addition, all overlap ambiguities that do not involve an element $\tau_k\in R$ are

$$\Phi_1 = (\sigma_1,\sigma_3,A,B,A), \tag{38}$$

$$\Phi_2 = (\sigma_2,\sigma_4,A,C,B), \tag{39}$$

$$\Phi_3 = (\sigma_3,\sigma_1,B,A,B), \tag{40}$$

$$\Phi_4 = (\sigma_3,\sigma_2,B,A,C), \tag{41}$$

$$\Phi_5 = (\sigma_4,\sigma_3,C,B,A), \tag{42}$$

while all the overlap ambiguities that depend on an integer parameter (k) are

$$\Phi_6(k) = (\sigma_1,\tau_k,A,B,C^kA), \tag{43}$$

$$\Phi_7(k) = (\sigma_4,\tau_k,C,B,C^kA), \tag{44}$$

$$\Phi_8(k) = (\tau_k,\sigma_1,BC^k,A,B), \tag{45}$$

$$\Phi_9(k) = (\tau_k,\sigma_2,BC^k,A,C). \tag{46}$$

Proposition 4.1. *For any $n, m \in \mathbb{Z}^+$,*

$$A^nC^m = \sum_{i=0}^{n} \binom{n}{i} q^{mn-mi}(\{m\}_q s)^i C^m A^{n-i}, \text{ and} \tag{47}$$

$$C^mB^n = \sum_{i=0}^{n} \binom{n}{i} q^{mn-mi}(\{m\}_q r)^i B^{n-i} C^m. \tag{48}$$

Proof. The desired relations in the statement may be obtained by routine induction based on the relations (12) and (14) in Proposition 3.4. An argument similar to that done in Proposition 3.1 may be used in aid of the induction. □

Lemma 4.2. *All ambiguities of R are resolvable.*

Proof. We prove this lemma directly by determining compositions of reductions $\mathfrak{r}_i$ and $\mathfrak{r}_i'$ for each ambiguity $\Phi_1, \ldots, \Phi_5$ and $\Phi_6(k), \ldots, \Phi_9(k)$ that will satisfy condition for resolvable ambiguity. For any positive integer k and any $U, W \in \langle \mathcal{X}_3 \rangle$, we let

$$\begin{aligned} \mathfrak{a}_{(k,W)} &:= \mathfrak{r}_{C^{k-1}\sigma_2 W} \circ \mathfrak{r}_{C^{k-2}\sigma_2 CW} \circ \mathfrak{r}_{C^{k-3}\sigma_2 C^2 W} \circ \cdots \circ \mathfrak{r}_{C\sigma_2 C^{k-2} W} \circ \mathfrak{r}_{\sigma_2 C^{k-1} W}, \\ \mathfrak{b}_{(k,U)} &:= \mathfrak{r}_{UC^{k-1}\sigma_4} \circ \mathfrak{r}_{UC^{k-2}\sigma_4 C} \circ \mathfrak{r}_{UC^{k-3}\sigma_4 C^2} \circ \cdots \circ \mathfrak{r}_{UC\sigma_4 C^{k-2}} \circ \mathfrak{r}_{U\sigma_4 C^{k-1}}. \end{aligned}$$

With Proposition 4.1, we take note of the following simple results:

$$\mathfrak{a}_{(k,W)}(AC^kW) = q^kC^kAW + \{k\}_q sC^kW, \tag{49}$$

$$\mathfrak{b}_{(k,U)}(UC^kB) = q^kUBC^k + \{k\}_q rUC^k. \tag{50}$$

For simpler notation, given $\mu \in R$, we write $\mathfrak{r}_{I\mu I}, \mathfrak{r}_{U\mu I}, \mathfrak{r}_{I\mu V}$ as $\mathfrak{r}_\mu, \mathfrak{r}_{U\mu}, \mathfrak{r}_{\mu V}$, respectively.

We first consider $\Phi_1 = (\sigma_1, \sigma_3, A, B, A)$. Notice that ABA is precisely the nontrivial word involved with this overlap ambiguity. Let $\mathfrak{r}_1 = \mathfrak{r}_{\sigma_3}$ and $\mathfrak{r}_1' = \mathfrak{r}_{\sigma_2} \circ \mathfrak{r}_{\sigma_1}$. Observe that

$$\begin{aligned} \mathfrak{r}_1(f_{\sigma_1}A) &= \mathfrak{r}_{\sigma_3}\left(\frac{rA^2 + sBA - qCA}{1-q}\right), \\ &= \frac{rA^2 + s\left(\frac{rA+sB-C}{1-q}\right) - qCA}{1-q}, \\ &= \frac{(1-q)rA^2 + rsA + s^2B - sC - (1-q)qCA}{(1-q)^2}, \end{aligned}$$

and

$$\begin{aligned}
\mathfrak{r}_1'(Af_{\sigma_3}) &= (\mathfrak{r}_{\sigma_2}\circ\mathfrak{r}_{\sigma_1})\left(\frac{rA^2+sAB-AC}{1-q}\right),\\
&= \mathfrak{r}_{\sigma_2}\left(\frac{rA^2+s\left(\frac{rA+sB-qC}{1-q}\right)-AC}{1-q}\right),\\
&= \frac{rA^2+s\left(\frac{rA+sB-qC}{1-q}\right)-(qCA+sC)}{1-q},\\
&= \frac{(1-q)rA^2+rsA+s^2B-sC-(1-q)qCA}{(1-q)^2},\\
&= \mathfrak{r}_1(f_{\sigma_1}A).
\end{aligned}$$

Thus, for the ambiguity $\Phi_1=(\sigma_1,\sigma_3,A,B,A)$, if $\mathfrak{r}_1=\mathfrak{r}_{\sigma_3}$ and $\mathfrak{r}_1'=\mathfrak{r}_{\sigma_2}\circ\mathfrak{r}_{\sigma_1}$, then we have $\mathfrak{r}_1'(Af_{\sigma_3})=\mathfrak{r}_1(f_{\sigma_1}A)$, which implies resolvability of the ambiguity. To complete the proof, we check all other ambiguities. The process involves routine computations like those done above for Φ_1. We only summarize below what compositions of reductions are used for each ambiguity, which lead to the desired resolvability condition, like the equation $\mathfrak{r}_1'(Af_{\sigma_3})=\mathfrak{r}_1(f_{\sigma_1}A)$ for Φ_1. Again, such equations may be verified routinely for each of the remaining ambiguities.

(i) For $\Phi_2=(\sigma_2,\sigma_4,A,C,B)$, if $\mathfrak{r}_2=\mathfrak{r}_{\sigma_4}\circ\mathfrak{r}_{C\sigma_1}$ and $\mathfrak{r}_2'=\mathfrak{r}_{\sigma_2}\circ\mathfrak{r}_{\sigma_1C}$, then $\mathfrak{r}_2(f_{\sigma_2}B)=\mathfrak{r}_2'(Af_{\sigma_4})$.

(ii) For $\Phi_3=(\sigma_3,\sigma_1,B,A,B)$, if $\mathfrak{r}_3=\mathfrak{r}_{\sigma_4}\circ\mathfrak{r}_{\sigma_1}$ and $\mathfrak{r}_3'=\mathfrak{r}_{\sigma_3}$, then $\mathfrak{r}_3(f_{\sigma_3}B)=\mathfrak{r}_3'(Bf_{\sigma_1})$.

(iii) For $\Phi_4=(\sigma_3,\sigma_2,B,A,C)$, if $\mathfrak{r}_4=\mathfrak{r}_{\sigma_2}$ and $\mathfrak{r}_4'=\mathfrak{r}_{\tau_1}$, then $\mathfrak{r}_4(f_{\sigma_3}C)=\mathfrak{r}_4'(Bf_{\sigma_2})$.

(iv) For $\Phi_5=(\sigma_4,\sigma_3,C,B,A)$, if $\mathfrak{r}_5=\mathfrak{r}_{\tau_1}$ and $\mathfrak{r}_5'=\mathfrak{r}_{\sigma_4}$, then $\mathfrak{r}_5(f_{\sigma_4}A)=\mathfrak{r}_5'(Cf_{\sigma_3})$.

(v) For $\Phi_6(k)=(\sigma_1,\tau_k,A,B,C^kA)$, if $\mathfrak{r}_6=\mathfrak{r}_{\tau_k}\circ\mathfrak{a}_{(k,A)}$ and $\mathfrak{r}_6'=\mathfrak{a}_{(k+1,I)}\circ\mathfrak{a}_{(k,I)}\circ\mathfrak{r}_{\sigma_1C^k}\circ\mathfrak{a}_{(k,A)}$, then $\mathfrak{r}_6(f_{\sigma_1}C^kA)=\mathfrak{r}_6'(Af_{\tau_k})$.

(vi) For $\Phi_7(k)=(\sigma_4,\tau_k,C,B,C^kA)$, if $\mathfrak{r}_7=\mathfrak{r}_{\tau_{k+1}}$ and $\mathfrak{r}_7'=\mathfrak{r}_{\sigma_4}$, then $\mathfrak{r}_7(f_{\sigma_4}C^kA)=\mathfrak{r}_7'(Cf_{\tau_k})$.

(vii) For $\Phi_8(k)=(\tau_k,\sigma_1,BC^k,A,B)$, if $\mathfrak{r}_8=\mathfrak{b}_{(k+1,I)}\circ\mathfrak{b}_{(k,I)}\circ\mathfrak{b}_{(k,B)}\circ\mathfrak{r}_{C^k\sigma_1}$ and $\mathfrak{r}_8'=\mathfrak{b}_{(k,B)}\circ\mathfrak{r}_{\tau_k}$, then $\mathfrak{r}_8(f_{\tau_k}B)=\mathfrak{r}_8'(BC^kf_{\sigma_1})$.

(viii) For $\Phi_9(k)=(\tau_k,\sigma_2,BC^k,A,C)$, if $\mathfrak{r}_9=\mathfrak{r}_{C^k\sigma_2}$ and $\mathfrak{r}_9'=\mathfrak{r}_{\tau_{k+1}}$, then $\mathfrak{r}_9(f_{\tau_k}C)=\mathfrak{r}_9'(BC^kf_{\sigma_2})$.

These results suggest that with $\mathfrak{r}_i$ and $\mathfrak{r}_i'$ for $i\in\{1,2,\dots,9\}$, all ambiguities $\Phi_1,\dots,\Phi_5$ and $\Phi_6(k)$, … , $\Phi_9(k)$ are resolvable. This completes the proof. □

Theorem 4.3. *The elements*

$$B^l C^m, C^m A^t, \qquad (l, m \in \mathbb{N},\ t \in \mathbb{Z}^+), \tag{51}$$

form a basis for $\mathcal{U}_q(r, s)$*.*

Proof. We consider R whose elements are given by (33)-(37) based from the defining relations of $\mathcal{U}_q(r, s)$ as previously stated in Proposition 3.4. We first show that

$$\{B^h C^j, C^j A^k \ :\ h, j \in \mathbb{N}, k \in \mathbb{Z}^+\}, \tag{52}$$

is the set of all irreducible words under R. Notice that collection (52) is clearly a set of irreducible words with respect to the reduction system R since words AB, AC, BA, CB and BC^kA with $k \in \mathbb{Z}^+$ do not appear as a subword in any of its elements. Suppose W is not in (52). Then W must have a subword of the form $A^xC^yB^z$ or $B^uC^yA^w$ where $x, y, z \in \mathbb{N}, u, w \in \mathbb{Z}^+$ and at most one of the powers x, y, z for $A^xC^yB^z$ is equal to zero. It is clear that we cannot have two or three variables among x, y and z to be zero for $A^xC^yB^z$ because it will contradict our supposition. This means that we only have to consider cases when $x = 0$, $y = 0$, $z = 0$, and when $x, y, z \in \mathbb{Z}^+$ for $A^xC^yB^z$. Meanwhile, we have cases $y = 0$, and $y \neq 0$ for $B^uC^yA^w$. If $x = 0$, then a reduction which involves σ_4 would act nontrivially on $A^xC^yB^z = C^yB^z$. If $y = 0$, a reduction which involves σ_1 would act nontrivially on $A^xC^yB^z = A^xB^z$, while a reduction which involves σ_3 would act nontrivially on $B^uC^yA^w = B^uA^w$. And if $z = 0$, a reduction which involves σ_2 would act nontrivially on $A^xC^yB^z = A^xC^y$. For the case $x, y, z \in \mathbb{Z}^+$, reductions which involve σ_2 or σ_4 would act nontrivially on $A^xC^yB^z$, while reductions which involve τ_y would act nontrivially on $B^uC^y\alpha^w$ when $y \neq 0$.

It is clear that in any of the mentioned cases, W is not irreducible. Thus, any irreducible element with respect to the reduction system R are in (52). Now, we only need to show that elements in (52) form a basis for $\mathcal{U}_q(r, s)$. To do this, we invoke Bergman's Diamond Lemma. The only implication we need from the Diamond Lemma is that: if all the ambiguities of a reduction system S are resolvable and if $\mathcal{K}$ is the ideal of $\mathbb{F}\langle\mathcal{X}\rangle$ generated by all $W_\sigma - f_\sigma(\sigma \in S)$, then the images of all the S-irreducible words under the canonical map $\mathbb{F}\langle\mathcal{X}\rangle \longrightarrow \mathbb{F}\langle\mathcal{X}\rangle/\mathcal{K}$ form a basis for $\mathbb{F}\langle\mathcal{X}\rangle/\mathcal{K}$. If we take $S = R$, $\mathcal{K} = \mathcal{I}_3$ generated by expressions in (16)-(22), then with Lemma 4.2, the elements in (52) form a basis for $\mathcal{U}_q(r, s)$. $\square$

5. The isomorphism class of the algebra $\mathcal{U}_q(1, 0)$

In this section, we discuss our reason for choosing the algebra $\mathcal{U}_q(1, 0)$ for the Lie polynomial characterization in the class of q-deformed universal enveloping algebras for two-dimensional nonabelian Lie algebras. We start by considering two general cases for the algebra $\mathcal{U}_q(r,s)$. First, the case $r \neq 0$ and $s = 0$, that is, the algebra $\mathcal{U}_q(r,0)$ that has presentation by generators A, B and relation

$$AB - qBA = rA. \tag{53}$$

Second, we have the case $r = 0$ and $s \neq 0$, that is, the algebra $\mathcal{U}_q(0,s)$ with a presentation by generators A, B and relation

$$AB - qBA = sB. \tag{54}$$

To proceed, we again make use of the additional generator $C = [A,B] = AB - BA$ so that the algebra $\mathcal{U}_q(r,0)$ would have a presentation by generators A, B, C and relations

$$\begin{aligned} AB &= \frac{rA - qC}{1-q}, && (55)\\ AC &= qCA, && (56)\\ BA &= \frac{rA - C}{1-q}, && (57)\\ CB &= qBC + rC, && (58)\\ BC^kA &= \frac{q^k rC^kA - C^{k+1}}{q^k(1-q)}, \quad k \in \mathbb{Z}^+, && (59)\end{aligned}$$

which follows directly from Proposition 3.4. From the same Proposition 3.4, the algebra $\mathcal{U}_q(0,s)$ would have a presentation by generators A, B, C and relations

$$\begin{aligned} AB &= \frac{sB - qC}{1-q}, && (60)\\ AC &= qCA + sC, && (61)\\ BA &= \frac{sB - C}{1-q}, && (62)\\ CB &= qBC, && (63)\\ BC^kA &= \frac{q^k sBC^k - C^{k+1}}{q^k(1-q)}, \quad (k \in \mathbb{Z}^+). && (64)\end{aligned}$$

For some important proofs that shall come later, we will need a generalization of (61) and (63) which is in the following.

Proposition 5.1. *For any $n, m \in \mathbb{Z}^+$,*

$$\begin{aligned} A^nC^m &= q^{nm}C^mA^n, && (65)\\ C^mB^n &= \sum_{i=0}^{n} \binom{n}{i} q^{mn-mi}(\{m\}_q r)^i B^{n-i}C^m. && (66)\end{aligned}$$

Proof. Set $s = 0$ in Proposition 4.1. □

We now show that working on the algebra $\mathcal{U}_q(1,0)$ shall suffice for us to accomplish the Lie polynomial characterization intended for this paper. We do this by showing that the relevant algebras actually belong to the isomorphism class (under algebra isomorphisms) of the algebra $\mathcal{U}_q(1,0)$.

Proposition 5.2. *There is an algebra isomorphism* $\mathcal{U}_{\frac{1}{q}}(0,1) \longrightarrow \mathcal{U}_q(1,0)$ *that sends* $A \mapsto \beta B$ *and* $B \mapsto \alpha A$*, where* $\alpha, \beta \in \mathbb{F}$ *are nonzero.*

Proof. The algebra $\mathcal{U}_{\frac{1}{q}}(0,1)$ is generated by $F := -B$ and $G := \frac{-1}{q}A$, while $\mathcal{U}_q(1,0)$ is generated by $U := -qB$ and $V := -A$. Using the defining relations of these algebras, the relation $FG - qGF = F$ holds in $\mathcal{U}_{\frac{1}{q}}(0,1)$ (which has the defining relation $AB - \frac{1}{q}BA = B$), and the relation $UV - \frac{1}{q}VU = V$ in $\mathcal{U}_q(1,0)$ (which has the defining relation $AB - qBA = A$). We have thus shown that $\mathcal{U}_{\frac{1}{q}}(0,1)$ and $\mathcal{U}_q(1,0)$ are homomorphic images of each other. More precisely, there exist algebra homomorphisms $\Psi : \mathcal{U}_{\frac{1}{q}}(0,1) \longrightarrow \mathcal{U}_q(1,0)$ and $\Upsilon : \mathcal{U}_q(1,0) \longrightarrow \mathcal{U}_{\frac{1}{q}}(0,1)$ such that

$$\Psi : A \mapsto -qB, \quad B \mapsto -A,$$
$$\Upsilon : A \mapsto -B, \quad B \mapsto \frac{-1}{q}A,$$

which, by routine computations, satisfy the conditions $(\Upsilon \circ \Psi)(A) = A$ and $(\Upsilon \circ \Psi)(B) = B$. This completes the proof. $\square$

As we said in Section 1, the traditional algebra homomorphisms that perform $A \mapsto \beta B$ and $B \mapsto \alpha A$ (for some nonzero $\alpha, \beta \in \mathbb{F}$), or algebra homomorphisms that "switch and scale" the two generators from $\mathcal{U}_q(r,0)$ to $\mathcal{U}_q(0,s)$ consequently imply that $q = 1$ or $q = -1$, which is counter to the assumption that q is not a root of unity. Proposition 5.2, however, shows us that, for the case $r = 1 = s$, by making a change in parameter for in one of the algebras (that is, from q to $\frac{1}{q}$), a "switch and scale" isomorphism is obtained. This shall greatly simplify the Lie polynomial characterization later. The problem now shifts into a different direction: could all algebras $\mathcal{U}_q(r,0)$ be "represented" by the case $r = 1$, and similarly, for the algebras $\mathcal{U}_q(0,s)$ by the case $s = 1$? This is addressed by the following.

Proposition 5.3. (i) *For any nonzero* $r \in \mathbb{F}$*, there is an algebra isomorphism* $\mathcal{U}_q(r,0) \longrightarrow \mathcal{U}_q(1,0)$ *that sends* $A \mapsto A$ *and* $B \mapsto rB$.

(ii) *For any nonzero* $s \in \mathbb{F}$*, there is an algebra isomorphism* $\mathcal{U}_q(0,s) \longrightarrow \mathcal{U}_q(0,1)$ *that sends* $A \mapsto sA$ *and* $B \mapsto B$.

Proof. The proof is based on an argument similar to that done in the proof of Proposition 5.2. $\square$

Since we have now established an isomorphism between $\mathcal{U}_q(1,0)$ and $\mathcal{U}_q(r,0)$ for any nonzero $r \in \mathbb{F}$, we now have the justification to drop the parameter r and simply consider algebra the $\mathcal{U}_q(1,0)$ for our succeeding results and computations. This is from the first part of Proposition 5.3, and by the second part, we can do similarly for the algebras $\mathcal{U}_q(0,s)$ for all nonzero $s \in \mathbb{F}$. Ultimately, the two remaining algebras $\mathcal{U}_q(1,0)$ and $\mathcal{U}_q(0,1)$ have the same algebra structure as implied by Proposition 5.2. Finally, this gives us sufficient reason to work only on the algebra $\mathcal{U}_q(1,0)$ for the Lie polynomial characterization in the class of q-deformed universal enveloping algebras for two-dimensional nonabelian Lie algebras.

We now proceed with exhibiting elements of the algebra $\mathcal{U}_q(1,0)$ in terms of its basis elements based on Theorem 4.3. The results in this section are crucial and work as our initial step in constructing Lie polynomials in $\mathcal{U}_q(1,0)$. We reiterate for emphasis that the algebra $\mathcal{U}_q(1,0)$ has a presentation by generators A, B and relation

$$AB - qBA = A. \tag{67}$$

Let $n \in \mathbb{Z}^+$. Routine induction using (53) results to

$$A^nB = q^nBA^n + \{n\}_qA^n, \tag{68}$$

both sides of which, we multiply by B^{n-1} from the right. The resulting right-hand side is a linear combination of only two words. The first term is $(q^nBA^n)B^{n-1}$. To the expressions in parentheses, we substitute using $q^nBA^n = A^nB - \{n\}_qA^n$, which is just one equivalent form of (68). We now have

$$A^nB^n = (q^nBA + \{n\}_qA)A^{n-1}B^{n-1}, \tag{69}$$

which we shall use to prove

$$A^nB^n = \prod_{i=0}^{n-1}(q^{n-i}BA + \{n-i\}_qA). \tag{70}$$

The case $n = 1$ is simply the relation (67). Suppose (70) is true for some $n \in \mathbb{Z}^+$. We consider the case $A^{n+1}B^{n+1}$ of equation (69). The inductive hypothesis, with the aid of (67), may then be used on the resulting right-hand side, and (70) follows.

Using the relation (57) on (70), routine computations may be used to prove

$$A^nB^n = \prod_{i=0}^{n-1}\frac{A - q^{n-i}C}{(1-q)}. \tag{71}$$

We also have the following equivalent expressions under $\mathcal{U}_q(1,0)$.

Corollary 5.4. *For any $n, m \in \mathbb{Z}^+$,*

$$A^n C^m = q^{nm} C^m A^n, \quad (72)$$

$$C^m B^n = \sum_{i=0}^{n} \binom{n}{i} q^{mn-mi} (\{m\}_q)^i B^{n-i} C^m. \quad (73)$$

Proof. Take $r = 1$ of equations (65) and (66) in Proposition 5.1. □

Theorem 5.5. *If all the letters B, C and A appear (meaning with nonzero exponent) in the word $B^y C^k A^w$, then if $B^y C^k A^w$ is further written as a linear combination of the basis elements from Theorem 4.3, then the letter C appears (meaning has nonzero exponent) in all words in the said linear combination.*

Proof. The desired statement may more precisely be stated as: given $k, w, y \in \mathbb{Z}^+$, the word $B^y C^k A^w \in \mathcal{U}_q(1,0)$ is in $\mathcal{S} := \text{Span}\{B^l C^m, C^m A^l \ : \ l \in \mathbb{N}, \ m \in \mathbb{Z}^+\}$. We clarify that $\mathcal{S}$ is only a proper subset of the basis of $\mathcal{U}_q(1,0)$ from Theorem 4.3: this basis includes words where the letter C does not appear, while $\mathcal{S}$ is the set of all words in the basis in each of which, the letter C appears. Let $k \in \mathbb{Z}^+$. Elements of the form $BC^k A$ are in $\mathcal{S}$ because of the relation (59). In particular, $BC^k A \in \mathcal{S}_0 := \text{Span}\{C^m A^l \ : \ l \in \mathbb{N}, \ m \in \mathbb{Z}^+\}$. If for some $w \in \mathbb{Z}^+$, we have $BC^k A^w \in \mathcal{S}_0$, then we have an equation that expresses $BC^k A^w$ as a linear combination of basis elements of the form $C^m A^l$. We multiply both sides of this equation by A from the right, and so, $BC^k A^{w+1}$ is a linear combination of elements of the form $C^m A^{l+1}$, and this proves that $BC^k A^{w+1} \in \mathcal{S}_0$. By induction, $BC^k A^w \in \mathcal{S}_0$, for all $w \in \mathbb{Z}^+$. We perform another induction, with the statement $BC^k A \in \mathcal{S}_0$ as the basis step. Suppose that for some $w \in \mathbb{Z}^+$, we have $B^w C^k A^w \in \mathcal{S}_0$. Thus, $B^w C^k A^w$ is a linear combination of elements of the form $C^m A^l$, and consequently, $B^{w+1} C^k A^{w+1}$ is a linear combination of elements of the form $BC^m A^{l+1}$, which have been proven earlier to be elements of $\mathcal{S}_0$. By induction, $B^w C^k A^w \in \mathcal{S}_0$ for all $w \in \mathbb{Z}^+$. An analogous induction argument may be used to show that $B^w C^k A^{w+n} \in \mathcal{S}_0$ for all $n \in \mathbb{N}$. The remaining case is when the exponent of B is strictly greater than the exponent of A in the word $B^y C^k A^w$. We may write such a word as $B^{w+t} C^k A^w$, where $t \in \mathbb{Z}^+$. If $t = 1$, then using the fact that $B^w C^k A^w \in \mathcal{S}_0$, the element $B^w C^k A^w$ is a linear combination of elements of the form $C^m A^l$. Consequently, $B^{w+1} C^k A^w$ is a linear combination of elements of the form $BC^m A^l$, which have been proven earlier to be in $\mathcal{S}_0 \subseteq \mathcal{S}$. If for some $t \in \mathbb{Z}^+$, we have $B^{w+t} C^k A^w \in \mathcal{S}$, then this element is a linear combination of elements of the form $B^\lambda C^\mu$ and $C^m A^l$. Thus, $B^{w+t+1} C^k A^w$ is a linear combination of elements of the form $B^{\lambda+1} C^\mu$ and $BC^m A^l$. The latter have been proven earlier to be in $\mathcal{S}_0 \subseteq \mathcal{S}$, while the former are spanning set elements of $\mathcal{S}$. Therefore, $B^{w+t+1}C^kA^w \in \mathcal{S}$. This completes the proof. □

Proposition 5.6. *Given $k, w, y \in \mathbb{Z}^+$ and $t \in \mathbb{N}$,*

$$C^kA^wB^yC^t = \sum_{i=0}^{y} \binom{y}{i} q^{wy-wi}(\{w\}_q)^i \sum_{t=0}^{y-i} \binom{y-i}{t} q^{k(y-i-t)+tw}(\{k\}_q)^t B^{y-i-t}C^{k+t}A^w. \quad (74)$$

Proof. From (68), we find that $A^wB = q^wBA^w + \{w\}_qA^w$. Multiplying both sides by C^k, we find that the identity

$$C^kA^wB^y = \sum_{i=0}^{y} \binom{y}{i} q^{wy-wi}(\{w\}_q)^iC^kB^{y-i}A^w, \quad (75)$$

is true in $\mathcal{U}_q(1,0)$ for the case $y = 1$. The proof of (75) may be completed by induction, with some aid again from the equation $A^wB = q^wBA^w + \{w\}_qA^w$.

Using equation (73) in Corollary 5.4, we have

$$C^kB^{y-i} = \sum_{t=0}^{y-i} \binom{y-i}{t} q^{k(y-i)-kt}(\{k\}_q)^tB^{y-i-t}C^k,$$

which we subtitute into (75) to obtain

$$C^kA^wB^y = \sum_{i=0}^{y} \binom{y}{i} q^{wy-wi}(\{w\}_q)^i \sum_{t=0}^{y-i} \binom{y-i}{t} q^{k(y-i)-kt}(\{k\}_q)^tB^{y-i-t}C^kA^w,$$

both sides of which, we multiply by C^t from the right. The resulting right-hand side is a linear combination of words of the form $B^{y-i-t}C^kA^wC^t$ where the subword A^wC^t may be replaced by $q^{wt}C^tA^w$ using (72). The result is (74), as desired. □

Remark 5.7. [Hilbert space representations in a q-oscillator] For interested readers, we mention here a realization, or more precisely, a representation of the algebras $\mathcal{U}_q(1,0)$ and $\mathcal{U}_q(0,1)$ in terms of Hilbert space operators. We refer the reader to [4, Section 3] for the details on how two operators a and a^+ act on a fixed complete orthonormal basis of an infinite-dimensional seperable Hilbert space such that the relation $aa^+ - qa^+a = 1$ holds, where 1 is the identity operator. If we now introduce the *number operator* $N := a^+a$, then the relations $aN - qNa = a$ and $Na^+ - qa^+N = a^+$ hold. Thus, we have a representation of $\mathcal{U}_q(1,0)$ given by $A \mapsto a$ and $B \mapsto N$, and a representation of $\mathcal{U}_q(0,1)$, for which, $A \mapsto N$ and $B \mapsto a^+$.

6. Constructing Lie polynomials in A, B of $\mathcal{U}_q(1,0)$

Let $\mathcal{L}_{(1,0)}$ denote the Lie subalgebra of $\mathcal{U}_q(1,0)$ generated by A and B. Fix an element $U \in \mathcal{L}_{(1,0)}$. We recall the linear map ad U that sends $V \mapsto [U,V]$ for any $V \in \mathcal{L}_{(1,0)}$. We exhibit some important elements of the Lie subalgebra $\mathcal{L}_{(1,0)}$ generated by A and B.

Proposition 6.1. *For any $m \in \mathbb{Z}^+$,*

$$(\text{ad } C)^m A \quad = \quad (1-q)^m C^m A, \tag{76}$$

$$(\text{ad } C)^m B \quad = \quad (q-1)^m BC^m + (q-1)^{m-1} C^m. \tag{77}$$

Proof. Use induction on m, with routine computations that involve the relations (56) and (58) with $r = 1$. □

Lemma 6.2. *For any $m \in \mathbb{Z}^+$, the following holds in $\mathcal{U}_q(1,0)$:*

$$q^m [C^m A, B] \quad = \quad \{m+1\}_q C^{m+1}. \tag{78}$$

Proof. Use (55) and (59) on $[C^m A, B] = C^m AB - BC^m A$ given $r = 1$. □

Proposition 6.3. *For any $m \in \mathbb{Z}^+$,*

$$C^m, BC^m, C^m A \in \mathcal{L}_{(1,0)}. \tag{79}$$

Proof. Equation (76) implies that $C^m A \in \mathcal{L}_{(1,0)}$. With Lemma 6.2, we consequently have all elements of the form C^m for any $m \in \mathbb{Z}^+$ are in $\mathcal{L}_{(1,0)}$. Isolating the term with BC^m in equation (77), we find that $BC^m \in \mathcal{L}_{(1,0)}$ since C^m, $(\text{ad } C)^m B \in \mathcal{L}_{(1,0)}$. □

Lemma 6.4. *Fix $m \in \mathbb{Z}^+$. For any $n \in \mathbb{Z}^+$,*

$$(-\text{ad } A)^n (C^m A) \quad = \quad (1-q^m)^n C^m A^{n+1}, \tag{80}$$

$$(\text{ad } B)^n (BC^m) \quad = \quad ((1-q^m)B - \{m\}_q)^n \, BC^m. \tag{81}$$

Proof. Use (72) and (73) and induction on n. □

Theorem 6.5. *The following elements*

$$A, B, B^n C^m, C^m A^k, \qquad (m, k \in \mathbb{Z}^+, n \in \mathbb{N}), \tag{82}$$

form a basis for Lie algebra $\mathcal{L}_{(1,0)}$.

Proof. Let $\mathcal{K}$ be the span of the elements in (82). Notice that elements in (82) are linearly independent as they are basis elements of $\mathcal{U}_q(1,0)$. To show that $\mathcal{K}$ is equal to $\mathcal{L}_{(1,0)}$, we only have to show that the following conditions are satisfied:

(i) $A, B \in \mathcal{K}$,
(ii) $\mathcal{K}$ is a Lie subalgebra of $\mathcal{U}_q(1,0)$,
(iii) $\mathcal{K} \subseteq \mathcal{L}_{(1,0)}$.

The condition (i) immediately follows from the definition of $\mathcal{K}$. For condition (ii), we show that $\mathcal{K}$ is closed under the Lie bracket operation. That is, we show that for any basis elements L, R of $\mathcal{K}$ from(82), $[L, R]$ is a linear combination of the elements in (82). Given $t,u,v,w\in\mathbb{Z}^+$ and $x,y\in\mathbb{N}$, routine use of Corollary 5.4 gives us

$$\begin{aligned}
\big[C^u,C^t\big] &= 0, && (83)\\
[C^u,A] &= (1-q^u)C^uA, && (84)\\
[C^uA^w,A] &= (1-q^u)C^uA^{w+1}, && (85)\\
\big[C^uA^w,C^t\big] &= (q^{tw}-1)C^{t+u}A^w, && (86)\\
\big[C^uA^w,C^tA^v\big] &= (q^{tw}-q^{uv})C^{t+u}A^{v+w}, && (87)\\
[C^u,B] &= (q^u-1)BC^u+\{u\}_qC^u, && (88)\\
\big[B^tC^u,B\big] &= (q^u-1)B^{t+1}C^u+\{u\}_qB^tC^u, && (89)\\
\big[C^u,B^wC^t\big] &= \sum_{i=0}^{w}\binom{w}{i}q^{uw-ui}(\{u\}_q)^iB^{w-i}C^{t+u}-B^wC^{t+u}, && (90)\\
\big[B^vC^u,B^wC^t\big] &= \sum_{i=0}^{w}\binom{w}{i}q^{uw-ui}(\{u\}_q)^iB^{v+w-i}C^{t+u}\\
&\quad -\sum_{i=0}^{v}\binom{v}{i}q^{tv-ti}(\{t\}_q)^iB^{w+v-i}C^{t+u}, && (91)\\
\big[B^tC^u,A\big] &= B^tC^uA-\sum_{i=0}^{t}\binom{t}{i}q^{t+u-i}B^{t-i}C^uA, && (92)\\
[C^uA^w,B] &= C^uA^wB-BC^uA^w, && (93)\\
\big[C^uA^w,B^yC^t\big] &= C^uA^wB^yC^t-B^yC^{t+u}A^w. && (94)
\end{aligned}$$

The commutation relations of the basis elements of $\mathcal{K}$ in (82) are summarized in the following table.

$[\cdot,\cdot]$	A	B	B^yC^t	C^tA^v
A	0			
B	$-C$	0		
B^xC^u	(92), (84)	(89), (88)	(91), (90), (83)	
C^uA^w	(85)	(93)	(94), (86)	(87)

For each of the right-hand sides of the relations (83)–(91), we find that the result of the Lie bracket is a linear combination of (82), and is hence in $\mathcal{K}$. For the last three relations (92)–(94), we use Theorem 5.5 and Proposition 5.6 to deduce that the right-hand sides are also linear combinations of (82). Hence, condition (ii) is satisfied.

To prove (iii), we show that every basis element of $\mathcal{K}$ is in $\mathcal{L}_{(1,0)}$. The basis elements A,B,C have this property by the definition of $\mathcal{L}_{(1,0)}$. The rest of the basis elements of $\mathcal{K}$ are in $\mathcal{L}_{(1,0)}$ because of Lemma 6.4. At this point, we have proven $\mathcal{K} \subseteq \mathcal{L}_{(1,0)}$. This completes the proof. $\square$

From the standpoint of this study, the introduction of a q-deformation from the Lie algebra relation $AB - BA = A$ to the "q-relation" $AB - qBA = A$ results to a rich Lie-algebraic structure, an infinite-dimensional space of Lie polynomials in A and B which contrasts the reduction to a low-dimensional Lie algebra for the non-deformed case. The interplay between the associative and nonassociative algebraic structures in the same space is one important mathematical perspective that comes from studying Lie polynomial characterization problems.

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