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SEPARABLE FUNCTORS AND FIRM MODULES

Patrik Lundström

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Abstract. We develop a theory of separable ring extensions and separable functors for nonunital rings in the setting of firm modules. We prove nonunital analogues of classical results on functorial separability and semisimplicity, and apply these results to obtain a locally unital version of Maschke's theorem for group rings.

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1. Introduction

Let A be a ring. By this we mean that A is associative, but not necessarily unital. Suppose that B is another ring and $f : B \to A$ is a ring homomorphism. In that case, we say that A is a ring extension of B and we indicate this by writing A/B.

Recall that A/B is said to be separable if the multiplication map $\mu : A \otimes_B A \to A$, defined by the additive extension of $\mu(a \otimes a') = aa'$, for $a, a' \in A$, has a section in the category of A-bimodules, that is, if there is an A-bimodule map $\sigma : A \to A \otimes_B A$ such that $\mu \circ \sigma = \mathrm{id}_A$. Here, the B-bimodule structure on A is defined via f. Separable ring extensions generalize the classical notion of separable algebras over fields, which in turn extends separability of field extensions (see, e.g., [29] and the references therein).

Separable ring extensions have been studied by numerous authors (see, e.g., [4,6,8,11,12,13,15,16,20,21,23,25,26]). One reason for the sustained interest in these extensions is that important properties of the ground ring, such as semisimplicity, are often inherited by the larger ring. Perhaps the most classical example of this is the following result.

Maschke's theorem. *Let G be a finite group of order $|G|$. Let B be a semisimple unital ring with $|G|$ invertible in B. Then the group ring $A = B[G]$ is semisimple.*

Classically, this result is proved by a direct semisimplicity argument (see, e.g., [14, Thm. (6.1)] or Maschke's original proof [19]). However, separable ring extensions provide a more conceptual explanation of Maschke's theorem. In [21] (see Definition 2.1), Năstăsescu, Van den Bergh and Van Oystaeyen introduce the notion of a separable functor. Such a functor reflects splittings: if a morphism or exact sequence splits after applying the functor, then it already splits beforehand. In the same article ([21, Prop. 1.3]), they prove the following result.

Theorem 1.1. *Let $f : B \to A$ be a homomorphism of unital rings. Then A/B is separable if and only if the restriction functor* $\mathrm{Res} : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$ *is separable.*

Specializing to group rings, by [8, p. 41], we have:

Theorem 1.2. *Let G be a finite group of order $|G|$. Let B be a unital ring such that $|G|$ is invertible in B. Let A denote the group ring $B[G]$. Then A/B is separable.*

By Theorem 1.1, separability of A/B is equivalent to separability of the restriction functor $\mathrm{Res} : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$. Consequently, every exact sequence of A-modules that splits as a sequence of B-modules already splits A-linearly. If B is semisimple, separability of A/B then implies that A is semisimple, thereby recovering Maschke's theorem as a consequence of separability and functorial splitting.

The rings and modules considered in the classical situation above are unital. However, many important algebraic objects are defined as infinite direct sums or arise from local constructions, and therefore do not in general possess a global unit. This naturally raises the following question:

> *Is it possible to extend the classical separable functorial machinery from the unital setting to appropriate categories of modules over nonunital rings?*

To our knowledge, there appear to be very few results on separability of nonunital ring extensions. The only references we have found are [4], where Brzeziński, Kadison, and Wisbauer study connections between separability of A-rings and A-corings, and [3], where Böhm and Gómez-Torrecillas study splittings of the comultiplication map in the context of coalgebras.

Apart from these works, we obtained in [6, Proposition 26] a nonunital version of Theorem 1.1 under the assumption that A and B are rings with enough idempotents. Recall from Fuller [10] that a ring A has enough idempotents if there exists a set $\{e_i\}_{i\in I}$ of pairwise orthogonal idempotents in A such that $A = \oplus_{i\in I} Ae_i =$

$\oplus_{i\in I} e_i A$. The module categories considered in [6] are the categories ${}_A\mathrm{UMod}$ of unitary left A-modules, that is, modules M satisfying $AM = M$, where A is assumed to be a ring with enough idempotents.

In the present article, we extend the separability result of [6] to the more general setting of firm modules introduced by Quillen [24] (see Theorem 1.3). Recall that if A is a ring and M is a left A-module, then M is said to be firm if the multiplication map $A \otimes_A M \to M$, induced by the additive extension of $A \otimes_A M \ni a \otimes m \mapsto am \in M$, for $a \in A$ and $m \in M$, is an isomorphism. We denote the category of such modules by ${}_A\mathrm{FMod}$. The ring A is said to be firm if it is firm as a left A-module. Following Marín and Laan [18], we say that a ring homomorphism $f : B \to A$ is left firm if A, considered as a left B-module via f, is firm. We prove the following firm version of Theorem 1.1:

Theorem 1.3. *Let A be a firm ring and suppose that $f : B \to A$ is a left firm ring homomorphism. Then A/B is separable if and only if the restriction functor* $\mathrm{Res}_f : {}_A\mathrm{FMod} \to {}_B\mathrm{FMod}$ *is separable.*

We then apply this result to the study of semisimple modules in the category ${}_A\mathrm{FMod}$ when A is a left s-unital ring (see Theorem 1.4). Recall from Tominaga [27] that a left A-module M is said to be s-unital, if for every $m \in M$, one has $m \in Am$. A ring is said to be left s-unital if it is s-unital as a left module over itself. We say that a ring homomorphism $f : B \to A$ is left s-unital if A, regarded as a left B-module via f, is s-unital. We prove the following s-unital semisimplicity result:

Theorem 1.4. *Let A and B be left s-unital rings with B left semisimple. Suppose that $f : B \to A$ is a left s-unital ring homomorphism such that A/B is separable. Then A is left semisimple.*

This allows us to prove a locally unital version of Maschke's theorem (see Theorem 1.6). Recall from Ánh and Márki [1] that A is called a ring with local units if for each finite subset $X \subseteq A$, there exists an idempotent $e \in A$ such that $ex = x = xe$ for all $x \in X$. We first establish the following locally unital analogue of Theorem 1.2:

Theorem 1.5. *Let G be a finite group of order $|G|$. Let B be a ring with local units satisfying $|G|B = B$. Let A denote the group ring $B[G]$. Then A/B is separable.*

Combining this with Theorem 1.4, we obtain:

Theorem 1.6. *Let G be a finite group of order $|G|$. Let B be a semisimple ring with local units satisfying $|G|B = B$. Then the group ring $A = B[G]$ is semisimple.*

Here is an outline of the article.

In Section 2, we fix our categorical conventions and recall the definition of a separable functor. We then develop a systematic account of the categorical properties reflected by separable functors. In particular, we show that separability reflects key classes of morphisms, such as monomorphisms, epimorphisms, sections, retractions, and isomorphisms, as well as structural properties of objects, including subobject and quotient simplicity and semisimplicity, and the existence of initial, terminal, and zero objects.

In Section 3, we introduce our conventions on nonunital rings and module categories, with particular emphasis on unitary, s-unital, and firm modules. We clarify the relationships between these notions and show that they coincide over left s-unital rings. We then study separability of functors between module categories associated to a ring homomorphism $f : B \to A$, focusing on the restriction functor Res_f and the induction functor $\mathrm{Ind}_f = A \otimes_B -$ on categories of firm modules.

In Section 4, we apply the preceding functorial results to the study of simplicity and semisimplicity in the category ${}_A\mathrm{FMod}$ of firm modules over a left s-unital ring A. We show that the usual module-theoretic notions of simple and semisimple modules coincide with the categorical notions of subobject and quotient simplicity and semisimplicity. These results are then used to prove the nonunital version of Maschke's theorem for group rings over locally unital rings mentioned above, as well as a corresponding hereditary result, thereby extending classical semisimplicity results to a broad nonunital setting.

2. Separable functors

Throughout, we use the following notation. For a set X, we let $|X|$ denote its cardinality, and we let $\mathbb{N}$ denote the set of positive integers. We let C denote a category. By this we mean that C consists of a class of objects C_0, a class of morphisms C_1, domain and codomain maps $d, c : \mathsf{C}_1 \to \mathsf{C}_0$, and a partially defined associative composition $\mathsf{C}_1 \times \mathsf{C}_1 \ni (g, f) \mapsto gf \in \mathsf{C}_1$, defined whenever $d(g) = c(f)$. For each $M \in \mathsf{C}_0$, there is an identity morphism $\mathrm{id}_M \in \mathsf{C}_1$ satisfying $d(\mathrm{id}_M) = c(\mathrm{id}_M) = M$, $f\mathrm{id}_M = f$ whenever $d(f) = M$, and $\mathrm{id}_N f = f$ whenever $c(f) = N$. If f is a morphism in C with $d(f) = M$ and $c(f) = N$, then we write $f : M \to N$, and we denote the class of all such morphisms by $\mathrm{Hom}_{\mathsf{C}}(M, N)$.

Definition 2.1. Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor, always assumed to be covariant. Following [21], we say that F is separable if for all $M, N \in \mathsf{C}_0$, there is a function

(SF0) $\mathsf{R}_{M,N} : \mathrm{Hom}_{\mathsf{D}}(\mathsf{F}(M), \mathsf{F}(N)) \to \mathrm{Hom}_{\mathsf{C}}(M, N)$ such that

(SF1) $\mathsf{R}_{M,N}(\mathsf{F}(f)) = f$, for $f : M \to N$ in C, and

(SF2) $\mathsf{F}(g)f' = g' \,\mathsf{F}(f) \Rightarrow g\,\mathsf{R}_{M,N}(f') = \mathsf{R}_{M',N'}(g')f$, for $f : M \to M'$, $g : N \to N'$ in C, and $f' : \mathsf{F}(M) \to \mathsf{F}(N)$, $g' : \mathsf{F}(M') \to \mathsf{F}(N')$ in D.

At times, we indicate the dependence on F by writing $\mathsf{R}^{\mathsf{F}}_{M,N}$ instead of $\mathsf{R}_{M,N}$.

Proposition 2.2. *Suppose that $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ is a separable functor. Let $M, N \in \mathsf{C}_0$.*

(a) $\mathsf{R}_{M,M}(\mathrm{id}_{\mathsf{F}(M)}) = \mathrm{id}_M$*;*

(b) $\mathrm{Hom}_{\mathsf{C}}(M, N) \neq \emptyset \iff \mathrm{Hom}_{\mathsf{D}}(\mathsf{F}(M), \mathsf{F}(N)) \neq \emptyset$*;*

(c) F *is faithful.*

Proof. For (a) see [21, p. 399]; (b) and (c) are immediate. □

The next result shows that the assignment R behaves almost like a functor, in the sense that the naturality condition (SF2) can be replaced by (SF3) below.

Proposition 2.3. *Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor. Then F is separable if and only if for all $M, N \in \mathsf{C}_0$, there is a map $\mathsf{R}_{M,N}$ satisfying* (SF0), (SF1)*, and*

(SF3) $\mathsf{R}_{M,P}(gf) = \mathsf{R}_{N,P}(g)\,\mathsf{R}_{M,N}(f)$ *for* $M, N, P \in \mathsf{C}_0$*,* $f : \mathsf{F}(M) \to \mathsf{F}(N)$*,* $g : \mathsf{F}(N) \to \mathsf{F}(P)$*, whenever* $f \in \mathsf{F}(\mathrm{Hom}_{\mathsf{C}}(M, N))$ *or* $g \in \mathsf{F}(\mathrm{Hom}_{\mathsf{C}}(N, P))$.

Proof. This is essentially [25, Lemma 1.1]. □

Proposition 2.4. *Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ and $\mathsf{G} : \mathsf{D} \to \mathsf{E}$ be functors. Put $\mathsf{H} := \mathsf{G}\,\mathsf{F}$.*

(a) F *and* G *separable* $\Longrightarrow$ H *separable;*

(b) H *separable* $\Longrightarrow$ F *separable.*

Proof. This is [21, Lemma 1.1]. □

The following result is probably well known. Since we were unable to find an appropriate reference, we include a proof.

Lemma 2.5. *Every separable functor reflects limits and colimits.*

Proof. Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a separable functor. Let $\mathsf{G} : \mathsf{J} \to \mathsf{C}$ be a diagram, and let $(\lambda_j : L \to C_j)_{j\in\mathsf{J}}$ be a cone in C such that $(\mathsf{F}(\lambda_j) : \mathsf{F}(L) \to \mathsf{F}(C_j))_{j\in\mathsf{J}}$ is a limit cone of $\mathsf{F} \circ \mathsf{G}$ in D. Given any cone $(\alpha_j : X \to C_j)_{j\in\mathsf{J}}$ in C, there exists a unique morphism $h : \mathsf{F}(X) \to \mathsf{F}(L)$ with $\mathsf{F}(\lambda_j)h = \mathsf{F}(\alpha_j)$ for all $j \in \mathsf{J}$. Put $g := \mathsf{R}_{X,L}(h)$. Then, for each $j \in \mathsf{J}$, (SF3) and (SF1) imply that $\lambda_j g = \lambda_j\,\mathsf{R}_{X,L}(h) = \mathsf{R}_{X,C_j}(\mathsf{F}(\lambda_j)h) = \mathsf{R}_{X,C_j}(\mathsf{F}(\alpha_j)) = \alpha_j$. Thus g is a mediating morphism. If $g' : X \to L$ is another such morphism, then $\mathsf{F}(\lambda_j)\,\mathsf{F}(g') = \mathsf{F}(\alpha_j) = \mathsf{F}(\lambda_j)\,\mathsf{F}(g)$ for all $j \in \mathsf{J}$, so by the uniqueness of h we get $\mathsf{F}(g') = \mathsf{F}(g)$. By Proposition 2.2(c), $g' = g$. Thus $(\lambda_j : L \to C_j)_{j\in\mathsf{J}}$ is a limit cone in C. Hence, F reflects limits. The proof for colimits is dual. □

Let $f : M \to N$ be a morphism in C. Then f is called a monomorphism if for all morphisms $g_1, g_2 : P \to M$, the equality $fg_1 = fg_2$ implies $g_1 = g_2$. Dually, f is called an epimorphism if for all morphisms $h_1, h_2 : N \to Q$, the equality $h_1 f = h_2 f$ implies $h_1 = h_2$.

Proposition 2.6. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor. Let* $f \in \mathsf{C}_1$*.*

(a) $\mathsf{F}(f)$ *monomorphism* $\Longrightarrow$ f *monomorphism;*
(b) $\mathsf{F}(f)$ *epimorphism* $\Longrightarrow$ f *epimorphism.*

Proof. This follows from Lemma 2.5. Indeed, a morphism is a monomorphism if and only if $(\mathrm{id}, \mathrm{id})$ is the pullback of the corresponding pair (f, f). Thus (a) holds. The statement in (b) is dual. For more details, see e.g. [17, p. 72, Exercise 4]. □

Let $f : M \to N$ and $g : N \to M$ be morphisms in C such that $fg = \mathrm{id}_N$. Then f is called a retraction of g, and g is called a section of f. If also $gf = \mathrm{id}_M$, then f is said to be an isomorphism with inverse g.

Proposition 2.7. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor and let* $f : M \to N$ *be a morphism in* C*.*

(a) $r : \mathsf{F}(N) \to \mathsf{F}(M)$ *a retraction of* $\mathsf{F}(f) \Longrightarrow \mathsf{R}_{N,M}(r)$ *a retraction of* f*;*
(b) $s : \mathsf{F}(N) \to \mathsf{F}(M)$ *a section of* $\mathsf{F}(f) \Longrightarrow \mathsf{R}_{N,M}(s)$ *a section of* f*;*
(c) $g : \mathsf{F}(N) \to \mathsf{F}(M)$ *an inverse of* $\mathsf{F}(f) \Longrightarrow \mathsf{R}_{N,M}(g)$ *an inverse of* f*.*

Proof. This is [21, Prop. 1.2.1]. □

Let $M \in \mathsf{C}_0$. Consider the class of all monomorphisms in C with codomain M. We define an equivalence relation $\sim_M$ on this class as follows. Given monomorphisms $f : P \to M$ and $g : Q \to M$, we write $f \sim_M g$ if there exists an isomorphism $h : P \to Q$ with $f = gh$. The equivalence class $[f]_{\sim_M}$ of such a monomorphism $f : P \to M$ is called a subobject of M. We denote the collection of subobjects of M by $\mathrm{Sub}_{\mathsf{C}}(M)$.

Lemma 2.8. *Suppose that* $f : P \to M$ *and* $g : Q \to M$ *are monomorphisms in* C*. Then* $[f]_{\sim_M} = [g]_{\sim_M}$ *if and only if there are morphisms* $p : P \to Q$ *and* $q : Q \to P$ *such that* $f = gp$ *and* $g = fq$*.*

Proof. See the discussion in [17, p. 122]. □

Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor that preserves monomorphisms. Take $M \in \mathsf{C}_0$. Define $\alpha_M : \mathrm{Sub}_{\mathsf{C}}(M) \to \mathrm{Sub}_{\mathsf{D}}(\mathsf{F}(M))$ by $\alpha_M([m]_{\sim_M}) = [\mathsf{F}(m)]_{\sim_{\mathsf{F}(M)}}$, for monomorphisms $m : P \to M$.

Proposition 2.9. *Let* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *be a functor that preserves monomorphisms. Let* $M \in \mathsf{C}_0$*. Then* α_M *is well-defined. If* F *is also separable, then* α_M *is injective.*

Proof. Let $f : P \to M$ and $g : Q \to M$ be monomorphisms in C such that $f = gh$ for some isomorphism $h : P \to Q$. Then $\mathsf{F}(f) = \mathsf{F}(g)\,\mathsf{F}(h)$. By the assumptions, $\mathsf{F}(f)$ and $\mathsf{F}(g)$ are monomorphisms. Hence $[\mathsf{F}(f)]_{\sim_{\mathsf{F}(M)}} = [\mathsf{F}(g)]_{\sim_{\mathsf{F}(M)}}$. Thus, α_M is well-defined.

Suppose now that F is separable and $[\mathsf{F}(f)]_{\sim_{\mathsf{F}(M)}} = [\mathsf{F}(g)]_{\sim_{\mathsf{F}(M)}}$. By Lemma 2.8, there exist morphisms $p : \mathsf{F}(P) \to \mathsf{F}(Q)$ and $q : \mathsf{F}(Q) \to \mathsf{F}(P)$ with $\mathsf{F}(f) = \mathsf{F}(g)p$ and $\mathsf{F}(g) = \mathsf{F}(f)q$. By (SF1) and (SF3), $f = \mathsf{R}_{P,M}(\mathsf{F}(f)) = \mathsf{R}_{P,M}(\mathsf{F}(g)p) = \mathsf{R}_{Q,M}(\mathsf{F}(g))\,\mathsf{R}_{P,Q}(p) = g\,\mathsf{R}_{P,Q}(p)$, and similarly, $g = \mathsf{R}_{Q,M}(\mathsf{F}(g)) = \mathsf{R}_{Q,M}(\mathsf{F}(f)q) = \mathsf{R}_{P,M}(\mathsf{F}(f))\,\mathsf{R}_{Q,P}(q) = f\,\mathsf{R}_{Q,P}(q)$. Hence, by Lemma 2.8, $[f]_{\sim_M} = [g]_{\sim_M}$. □

We introduce the following terminology. We say that $M \in \mathsf{C}_0$ is subobject trivial if $|\operatorname{Sub}_{\mathsf{C}}(M)| = 1$, subobject nontrivial if $|\operatorname{Sub}_{\mathsf{C}}(M)| \geq 2$, and subobject simple if $|\operatorname{Sub}_{\mathsf{C}}(M)| = 2$.

Proposition 2.10. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor which preserves monomorphisms. Let* $M \in \mathsf{C}_0$ *be subobject nontrivial and* $\mathsf{F}(M)$ *subobject simple in* D*. Then* M *is subobject simple in* C*.*

Proof. Since $\mathsf{F}(M)$ is subobject simple in D, $|\operatorname{Sub}_{\mathsf{D}}(\mathsf{F}(M))| = 2$. The map α_M is injective, by Proposition 2.9. Thus, $|\operatorname{Sub}_{\mathsf{C}}(M)| \leq 2$. On the other hand, since M is subobject nontrivial, $|\operatorname{Sub}_{\mathsf{C}}(M)| \geq 2$. Therefore $|\operatorname{Sub}_{\mathsf{C}}(M)| = 2$. □

Let $M \in \mathsf{C}_0$. We say that M is subobject semisimple if every monomorphism $m : P \to M$ has a retraction $r : M \to P$.

Proposition 2.11. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor which preserves monomorphisms. Let* $M \in \mathsf{C}_0$ *have the property that* $\mathsf{F}(M)$ *is subobject semisimple in* D*. Then* M *is subobject semisimple in* C*.*

Proof. Suppose that $f : P \to M$ is a monomorphism in C. Since F preserves monomorphisms, $\mathsf{F}(f) : \mathsf{F}(P) \to \mathsf{F}(M)$ is a monomorphism in D. Because $\mathsf{F}(M)$ is subobject semisimple, $\mathsf{F}(f)$ has a retraction $r : \mathsf{F}(M) \to \mathsf{F}(P)$. By Proposition 2.7(a), $\mathsf{R}_{M,P}(r)$ is a retraction of f. Hence, M is subobject semisimple in C. □

Example 2.12. Let **Set** denote the category of sets.

(a) In **Set**, the monomorphisms are precisely the injective maps. Using this, it is easy to see that all sets are subobject semisimple. All singleton sets $\{*\}$ are subobject simple since, up to equivalence, there are precisely two injections into

$\{*\}$ namely $\mathrm{id}_{\{*\}}$ and the empty function $\emptyset : \emptyset \to \{*\}$. It is not hard to see that the singleton sets in fact are all subobject simple sets. Therefore, in **Set**, every subobject simple object is trivially subobject semisimple.

(b) In an arbitrary category, a subobject simple object need not be subobject semisimple. Indeed, consider the category whose objects are M and N, and whose only morphisms are $\mathrm{id}_M, \mathrm{id}_N$ and $f : N \to M$. Subobjects of M correspond to the monomorphisms into M, that is id_M and f. Therefore, M is subobject simple. But f does not have a retraction. Hence, M is not subobject semisimple.

Let $M \in \mathsf{C}_0$. Consider the class of all epimorphisms in C with domain M. We define an equivalence relation $\approx_M$ on this class as follows. Given epimorphisms $f : M \to P$ and $g : M \to Q$, we write $f \approx_M g$ if there exists an isomorphism $h : Q \to P$ with $f = hg$. The equivalence class $[f]_{\approx_M}$ of such an epimorphism $f : M \to P$ is called a quotient object of M. We denote the collection of quotient objects of M by $\mathrm{Quot}_{\mathsf{C}}(M)$.

Lemma 2.13. *Suppose that $f : M \to P$ and $g : M \to Q$ are epimorphisms in C. Then $[f]_{\approx_M} = [g]_{\approx_M}$ if and only if there are morphisms $p : Q \to P$ and $q : P \to Q$ such that $f = pg$ and $g = qf$.*

Proof. Dual to Lemma 2.8. □

Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor that preserves epimorphisms. Take $M \in \mathsf{C}_0$. Define $\beta_M : \mathrm{Quot}_{\mathsf{C}}(M) \to \mathrm{Quot}_{\mathsf{D}}(\mathsf{F}(M))$ by $\beta_M([e]_{\approx_M}) = [\mathsf{F}(e)]_{\approx_{\mathsf{F}(M)}}$, for epimorphisms $e : M \to P$.

Proposition 2.14. *Let $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ be a functor that preserves epimorphisms. Then β_M is well-defined. If F is also separable, then β_M is injective.*

Proof. Dual to Proposition 2.9, using Proposition 2.7(b). □

We say that an object $M \in \mathsf{C}_0$ is quotient trivial if $|\mathrm{Quot}_{\mathsf{C}}(M)| = 1$, quotient nontrivial if $|\mathrm{Quot}_{\mathsf{C}}(M)| \geq 2$, and quotient simple if $|\mathrm{Quot}_{\mathsf{C}}(M)| = 2$.

Proposition 2.15. *Suppose that $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ is a separable functor which preserves epimorphisms. Let $M \in \mathsf{C}_0$ be quotient nontrivial and $\mathsf{F}(M)$ quotient simple in D. Then M is quotient simple in C.*

Proof. Dual to Proposition 2.10, using Proposition 2.14. □

Let $M \in \mathsf{C}_0$. We say that M is quotient semisimple if every epimorphism $e : M \to P$ has a section $s : P \to M$.

Proposition 2.16. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor which preserves epimorphisms. Let* $M \in \mathsf{C}_0$ *have the property that* $\mathsf{F}(M)$ *is quotient semisimple in* D*. Then* M *is quotient semisimple in* C*.*

Proof. Dual to Proposition 2.11, using Proposition 2.7(b). □

Example 2.17. (a) In **Set**, the epimorphisms are precisely the surjective maps. Using this, it is easy to see that all sets are quotient semisimple. All two-element sets are quotient simple since, up to equivalence, there are precisely two surjections from such a set: the identity map and the unique surjection onto a singleton. One can easily show that sets of other cardinalities are not quotient simple. Thus, in **Set**, every quotient simple object is trivially quotient semisimple.

(b) By taking the dual of the category in Example 2.12(b), we see that for arbitrary categories, a quotient simple object is not necessarily quotient semisimple.

Recall that $I \in \mathsf{C}_0$ is called initial if $|\mathrm{Hom}_{\mathsf{C}}(I, M)| = 1$ for all $M \in \mathsf{C}_0$. Dually, $T \in \mathsf{C}_0$ is called terminal if $|\mathrm{Hom}_{\mathsf{C}}(M, T)| = 1$, for all $M \in \mathsf{C}_0$. An object that is both initial and terminal is called a zero object. A category with a zero object is called pointed.

Proposition 2.18. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor. Let* $I, Z, T \in \mathsf{C}_0$*.*

(a) $\mathsf{F}(I)$ *initial in* $\mathsf{D} \Longrightarrow I$ *initial in* C*;*
(b) $\mathsf{F}(T)$ *terminal in* $\mathsf{D} \Longrightarrow T$ *terminal in* C*;*
(c) $\mathsf{F}(Z)$ *a zero object in* $\mathsf{D} \Longrightarrow Z$ *a zero object in* C*.*

Proof. This follows from Lemma 2.5, since terminal objects are limits of the empty diagram and initial objects are colimits of the empty diagram. □

Let C be a pointed category with a zero object $Z \in \mathsf{C}_0$. Take $M, N \in \mathsf{C}_0$. We denote the unique morphisms $Z \to M$ and $M \to Z$ by $0_{Z,M}$ and $0_{M,Z}$, respectively, and we set $0_{M,N} := 0_{Z,N} 0_{M,Z}$. Clearly, $0_{Z,N}$ is a monomorphism and $0_{M,Z}$ is an epimorphism.

Lemma 2.19. *Suppose that* C *is a pointed category with a zero object* Z*. Let* $M \in \mathsf{C}_0$*. The following statements are equivalent:*

(i) M *is a zero object in* C*;* (ii) *there is an isomorphism* $Z \to M$*;*
(iii) $0_{Z,M} \sim_M \mathrm{id}_M$*;* (iv) M *is subobject trivial;*
(v) $0_{M,Z} \approx_M \mathrm{id}_M$*;* (vi) M *is quotient trivial.*

Proof. It suffices to prove (i)⇒(iii)⇒(iv)⇒(i), because the equivalence (i)⇔(ii) is well known, and the circle (i)⇒(v)⇒(vi)⇒(i) is dual.

(i)⇒(iii): Suppose that M is a zero object. Then it follows that $0_{Z,M} : Z \to M$ is an isomorphism, so $0_{Z,M} = \mathrm{id}_M\, 0_{Z,M}$. Hence $0_{Z,M} \sim_M \mathrm{id}_M$.

(iii)⇒(iv): Suppose that $0_{Z,M} \sim_M \mathrm{id}_M$. Let $h : P \to M$ be a monomorphism. Choose an isomorphism $g : M \to Z$ such that $\mathrm{id}_M = 0_{Z,M} g$. Since Z is a zero object, $g = 0_{M,Z}$. Thus $h = \mathrm{id}_M h = 0_{Z,M} 0_{M,Z} h$. Now $0_{M,Z} h : P \to Z$, so by uniqueness of morphisms into Z, $0_{M,Z} h = 0_{P,Z}$. Hence $h = 0_{Z,M} 0_{P,Z}$. Also, $h 0_{Z,P} = 0_{Z,M} 0_{P,Z} 0_{Z,P} = 0_{Z,M}$, since $0_{P,Z} 0_{Z,P} = \mathrm{id}_Z$. Therefore, by Lemma 2.8, $h \sim_M 0_{Z,M}$. Thus every subobject of M equals $[0_{Z,M}]_{\sim_M}$, so M is subobject trivial.

(iv)⇒(i): Suppose that M is subobject trivial. $0_{Z,M} \sim_M \mathrm{id}_M$. Hence there exists an isomorphism $f : M \to Z$ such that $\mathrm{id}_M = 0_{Z,M} f$. Therefore $0_{Z,M}$ is an isomorphism. Since Z is a zero object, it follows that M is a zero object. □

Proposition 2.20. *Let C be a pointed category with a zero object Z. Let $M \in \mathsf{C}_0$.*

(a) *M subobject simple $\Longrightarrow$ M subobject semisimple;*

(b) *M quotient simple $\Longrightarrow$ M quotient semisimple.*

Proof. We prove (a). The proof of (b) is dual, using Lemma 2.19(v) in place of Lemma 2.19(iii).

Suppose that M is subobject simple. By Lemma 2.19(iii), $\mathrm{Sub}_{\mathsf{C}}(M)$ consists of the two distinct classes $[0_{Z,M}]_{\sim_M}$ and $[\mathrm{id}_M]_{\sim_M}$. Let $m : P \to M$ be a monomorphism. We now show that m has a retraction. To this end, we consider two cases.

Case 1: $[m]_{\sim_M} = [0_{Z,M}]_{\sim_M}$. Then there is an isomorphism $f : P \to Z$ with $m = 0_{Z,M} f$. But since $0_{P,Z}$ is the unique morphism $P \to Z$ it follows that $f = 0_{P,Z}$ is an isomorphism and $m = 0_{Z,M} 0_{P,Z} = 0_{P,M}$. Put $r := 0_{M,P}$. Then $rm : P \to P$. By Lemma 2.19(i), P is a zero object of C. In particular, id_P is the unique morphism $P \to P$, so $rm = \mathrm{id}_P$.

Case 2: $[m]_{\sim_M} = [\mathrm{id}_M]_{\sim_M}$. Then there is an isomorphism $f : P \to M$ with $m = \mathrm{id}_M f = f$. But then $rm = \mathrm{id}_P$ for $r := f^{-1}$. □

Example 2.21. Let **Grp** denote the category of groups. It is well known that this category is pointed with the trivial group as a zero object. Let G denote a group.

(a) In **Grp**, monomorphisms are exactly injective homomorphisms, and two monomorphisms into G represent the same subobject of G if and only if they have the same image in G. Therefore, the elements of $\mathrm{Sub}_{\mathbf{Grp}}(G)$ correspond to the set of subgroups of G. By Cauchy's theorem (see [9, Thm. 11, p. 93]), G is subobject simple precisely when G is a cyclic group of prime order. By a result by Baer [2, Thm. 3], G is subobject semisimple if and only if G is an abelian group all of whose elements have finite square-free order. Note that Baer [2] does not use our categorical terminology.

(b) In **Grp**, epimorphisms are precisely surjective homomorphisms, and the quotient object of G represented by an epimorphism $q : G \to Q$ corresponds to the quotient group $G/\ker(q)$. So quotient objects of G correspond to isomorphism classes of quotient groups of G. A group G is therefore quotient simple in the categorical sense if and only if G has no nontrivial proper normal subgroups. Thus, quotient simple groups are exactly the simple groups in the classical sense. Furthermore, G is quotient semisimple if and only if every normal subgroup of G is complemented. Indeed, an epimorphism $G \to Q$ splits precisely when its kernel N has a complement H in G with $G = NH$ and $N \cap H = 1$, so that $G \cong N \rtimes H$. Thus quotient semisimple groups are exactly the nC-groups studied in the group-theoretic literature (see, e.g., [7]). To the author's knowledge, no complete classification of nC-groups is known.

Recall that $P \in \mathsf{C}_0$ is called projective if for every epimorphism $e : M \to N$ and every $f : P \to N$, there is a morphism $g : P \to M$ with $eg = f$. Dually, $Q \in \mathsf{C}_0$ is called injective if for every monomorphism $m : M \to N$ and every $f : M \to Q$, there is a morphism $g : N \to Q$ with $gm = f$.

Proposition 2.22. *Suppose that* $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ *is a separable functor. Let* $P, Q \in \mathsf{C}_0$.

(a) $\mathsf{F}(P)$ *projective and* F *preserves epimorphisms* $\Longrightarrow$ P *projective;*

(b) $\mathsf{F}(Q)$ *injective and* F *preserves monomorphisms* $\Longrightarrow$ Q *injective.*

Proof. This is [21, Prop. 1.2.2 and Prop. 1.2.3]. □

3. Module categories

Let A be a ring. By this we mean that A is associative, but not necessarily unital. Let M be a left A-module. By this we mean that M is an additive group equipped with a biadditive map $A \times M \ni (a, m) \mapsto am \in M$ satisfying $(aa')m = a(a'm)$ for $a, a' \in A$ and $m \in M$. We let ${}_A\mathrm{Mod}$ denote the category having left A-modules as objects and A-linear maps as morphisms. Similarly, the category Mod_A of right A-modules is defined. Let B be another ring. If M is a left A-module and a right B-module, then M is said to be an A-B-bimodule if $(am)b = a(mb)$ for $a \in A$, $m \in M$ and $b \in B$. We let ${}_A\mathrm{Mod}_B$ denote the category having A-B-bimodules as objects and as morphisms maps that are simultaneously left A-linear and right B-linear. If M is an A-B-bimodule where $A = B$, then M is said to be an A-bimodule.

Suppose that M is a left A-module. Then M is said to be unital if there is $a \in A$ such that for every $m \in M$, the equality $am = m$ holds. We let AM denote the set of all finite sums of elements of the form am for $a \in A$ and $m \in M$.

Following Ánh and Márki [1], we say that M is unitary if $AM = M$. We let $\mu_{A,M}$ denote the multiplication map $A \otimes_A M \to M$ defined by the additive extension of $\mu_{A,M}(a \otimes m) = am$, for $a \in A$ and $m \in M$. Following Quillen [24], we say that M is firm if $\mu_{A,M}$ is an isomorphism in ${}_A\mathrm{Mod}$. Finally, following Tominaga [27], we say that M is s-unital if for every $m \in M$, we have $m \in Am$. The ring A is said to be left unital (s-unital, unitary, firm) if it is unital (s-unital, unitary, firm) as a left A-module. In the sequel, we will make use of the following result by Tominaga [27, Thm. 1] (see also [22]):

Proposition 3.1. *Suppose that A is a ring and M is a left A-module. Then M is s-unital if and only if for each finite subset $X \subseteq M$, there exists $a \in A$ such that $am = m$ for all $m \in X$.*

We introduce the following full subcategories of ${}_A\mathrm{Mod}$:

- ${}_A\mathrm{UMod}$, the category of unitary left A-modules;
- ${}_A\mathrm{FMod}$, the category of firm left A-modules;
- ${}_A\mathrm{SMod}$, the category of s-unital left A-modules;
- ${}_A\mathrm{Mod}^1$, the category of unital left A-modules.

Clearly, the following implications hold for these categories:

$$\text{unital} \Longrightarrow \text{s-unital} \Longrightarrow \text{unitary}, \quad \text{and} \quad \text{unital} \Longrightarrow \text{firm} \Longrightarrow \text{unitary}. \tag{1}$$

By the following examples, none of these implications is reversible.

Example 3.2. (a) s-unital $\not\Rightarrow$ unital: Let K be a field, and consider the ring $K^{(\mathbb{N})}$ consisting of all sequences $(k_n)_{n\in\mathbb{N}}$, where $k_n \in K$, and all but finitely many $k_n = 0$, with pointwise addition and multiplication. Then $K^{(\mathbb{N})}$ is left s-unital but not left unital.

(b) unitary $\not\Rightarrow$ s-unital: Consider the ring $C_0(\mathbb{N}, \mathbb{R})$ consisting of all real sequences $(a_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} a_n = 0$, with pointwise addition and multiplication. From $(a_n)_{n\in\mathbb{N}} = (\sqrt{|a_n|})_{n\in\mathbb{N}} \cdot (\mathrm{sgn}(a_n)\sqrt{|a_n|})_{n\in\mathbb{N}}$, it follows that $C_0(\mathbb{N}, \mathbb{R})$ is unitary. However, this ring is not left s-unital, which is easily seen by considering the sequence $b := (1/n)_{n\in\mathbb{N}}$; there is no element $a \in C_0(\mathbb{N}, \mathbb{R})$ such that $ab = b$.

(c) unitary $\not\Rightarrow$ firm: see [5, Example 1.2].

(d) firm $\not\Rightarrow$ unital: The ring $K^{(\mathbb{N})}$ from (a) is not unital. It is easy to see, either by a direct argument or by using Proposition 3.3 below, that it is firm as a left module over itself.

If we restrict to modules over left s-unital rings, then the classes of s-unital modules, firm modules, and unitary modules coincide.

Proposition 3.3. *Let A be a left s-unital ring. Then ${}_A\mathrm{UMod} = {}_A\mathrm{FMod} = {}_A\mathrm{SMod}$.*

Proof. Take $M \in {}_A\mathrm{Mod}$. By (1), it is enough to show M unitary $\Rightarrow M$ s-unital $\Rightarrow M$ firm. Suppose first that M is unitary. Take $m \in M$. Since $AM = M$, there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in M$ with $\sum_{i=1}^n a_i m_i = m$. Since A is left s-unital, there is, by Proposition 3.1, $b \in A$ with $ba_i = a_i$, for all $i \in \{1, \ldots, n\}$. Then $bm = \sum_{i=1}^n ba_i m_i = \sum_{i=1}^n a_i m_i = m$, showing that M is s-unital.

Suppose now that M is s-unital. Surjectivity of $\mu_{A,M}$ follows immediately from the fact that M is s-unital. Let us show that $\mu_{A,M}$ is injective. Suppose that $z \in A \otimes_A M$ satisfies $\mu_{A,M}(z) = 0$. Take $n \in \mathbb{N}$, $a_i \in A$ and $x_i \in M$, for $i = 1, \ldots, n$, with $z = \sum_{i=1}^n a_i \otimes x_i$. Then $\sum_{i=1}^n a_i x_i = \mu_{A,M}(z) = 0$. Since A is left s-unital, there exists $b \in A$ such that $ba_i = a_i$ for each i. Hence, $z = \sum_{i=1}^n ba_i \otimes x_i = \sum_{i=1}^n b \otimes a_i x_i = b \otimes \sum_{i=1}^n a_i x_i = b \otimes 0 = 0$. Therefore, M is firm. $\square$

Lemma 3.4. *Suppose that A is a ring and M is a left A-module. Then M is firm if and only if $\mu_{A,M} : A \otimes_A M \to M$ has a section in ${}_A\mathrm{Mod}$.*

Proof. The "only if" part is trivial. Now we show the "if" part. Suppose that $\mu_{A,M}$ has a section σ in ${}_A\mathrm{Mod}$. Then $\mu_{A,M}$ is surjective. It remains to show that $\mu_{A,M}$ is injective. Take $z \in \ker(\mu_{A,M})$. Write $z = \sum_{i=1}^n a_i \otimes x_i$ for some $n \in \mathbb{N}$, $a_i \in A$ and $x_i \in M$. For each i, take $n_i \in \mathbb{N}$, $b_{ij} \in A$ and $y_{ij} \in M$ with $\sigma(x_i) = \sum_{j=1}^{n_i} b_{ij} \otimes y_{ij}$. Then

$$\begin{aligned} z &= \textstyle\sum_{i=1}^n a_i \otimes x_i = \sum_{i=1}^n a_i \otimes \mu_{A,M}(\sigma(x_i)) = \sum_{i=1}^n \sum_{j=1}^{n_i} a_i \otimes b_{ij} y_{ij} \\ &= \textstyle\sum_{i=1}^n \sum_{j=1}^{n_i} a_i b_{ij} \otimes y_{ij} = \sum_{i=1}^n a_i \sigma(x_i) = \sigma\left(\sum_{i=1}^n a_i x_i\right) \\ &= \textstyle(\sigma \circ \mu_{A,M})\left(\sum_{i=1}^n a_i \otimes x_i\right) = \sigma(\mu_{A,M}(z)) = \sigma(0) = 0, \end{aligned}$$

showing that $\mu_{A,M}$ is injective. $\square$

Let $f : B \to A$ be a ring homomorphism, and M a left A-module. We view M as a left B-module by letting B act on M via f, that is, for $b \in B$ and $m \in M$, we define $b \cdot m := f(b)m$. This defines the restriction functor $\mathrm{Res}_f : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$. We say that f is left unital (respectively s-unital, firm, unitary) if the left B-module $\mathrm{Res}_f(A)$ is unital (respectively s-unital, firm, unitary).

Proposition 3.5. *Let $f : B \to A$ be a ring homomorphism.*

(a) *f left unital $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{Mod}^1$;*
(b) *f left s-unital $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{SMod}$;*
(c) *f left unitary $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{UMod} \to {}_B\mathrm{UMod}$;*
(d) *f left firm $\Longrightarrow \mathrm{Res}_f : {}_A\mathrm{FMod} \to {}_B\mathrm{FMod}$.*

Proof. Let M be a left A-module.

(a) Suppose that f is left unital and that M is unitary as a left A-module. Since A is unital as a left B-module, there is $b \in B$ with $f(b)a = a$, for all $a \in A$. Take $m \in M$. Since M is unitary as a left A-module, there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in M$ with $m = \sum_{i=1}^n a_i m_i$. Then $f(b)m = \sum_{i=1}^n f(b) a_i m_i = \sum_{i=1}^n a_i m_i = m$, showing that M is unital as a left B-module.

(b) Suppose that f is left s-unital and that M is unitary as a left A-module. Take $m \in M$. Then there are $n \in \mathbb{N}$, $a_1, \ldots, a_n \in A$ and $m_1, \ldots, m_n \in M$ such that $m = \sum_{i=1}^n a_i m_i$. Since A is s-unital as a left B-module, by Proposition 3.1, there is $b \in B$ such that $f(b)a_i = a_i$ for all $i \in \{1, \ldots, n\}$. Then $f(b)m = \sum_{i=1}^n f(b) a_i m_i = \sum_{i=1}^n a_i m_i = m$, showing that M is s-unital as a left B-module.

(c) Suppose that f is left unitary and that M is unitary as a left A-module. Then $f(B)A = A$ and $AM = M$. Hence, $f(B)M = f(B)AM = AM = M$, showing that M is unitary as a left B-module.

(d) Suppose that f is left firm and M is firm as a left A-module. By Lemma 3.4, we need to show that $\mu_{B,M}$ has a section in ${}_B\mathrm{Mod}$. To this end, we first note that, by the assumptions, $\mu_{B,A}$ has a section $\sigma_{B,A}$ in ${}_B\mathrm{Mod}$ and $\mu_{A,M}$ has a section $\sigma_{A,M}$ in ${}_A\mathrm{Mod}$. Define $\sigma : M \to B \otimes_B M$ by $\sigma := (\mathrm{id}_B \otimes \mu_{A,M}) \circ (\sigma_{B,A} \otimes \mathrm{id}_M) \circ \sigma_{A,M}$. Then σ, being a composition of morphisms in ${}_B\mathrm{Mod}$, is again a morphism in ${}_B\mathrm{Mod}$. We now show that $\mu_{B,M} \circ \sigma = \mathrm{id}_M$. Take $m \in M$, $p \in \mathbb{N}$, $m_1, \ldots, m_p \in M$ and $a_1, \ldots, a_p \in A$ with $\sigma_{A,M}(m) = \sum_{i=1}^p a_i \otimes m_i$. By the assumptions, $\sum_{i=1}^p a_i m_i = (\mu_{A,M} \circ \sigma_{A,M})(m) = m$. For each $i \in \{1, \ldots, p\}$, take $q_i \in \mathbb{N}$, $a_1^{(i)}, \ldots, a_{q_i}^{(i)} \in A$ and $b_1^{(i)}, \ldots, b_{q_i}^{(i)} \in B$ with $\sigma_{B,A}(a_i) = \sum_{j=1}^{q_i} b_j^{(i)} \otimes a_j^{(i)}$. By the assumptions, for each $i \in \{1, \ldots, p\}$, we get $\sum_{j=1}^{q_i} f(b_j^{(i)}) a_j^{(i)} = (\mu_{B,A} \circ \sigma_{B,A})(a_i) = a_i$. Therefore,

$$\begin{aligned}
(\mu_{B,M} \circ \sigma)(m) &= (\mu_{B,M} \circ (\mathrm{id}_B \otimes \mu_{A,M}) \circ (\sigma_{B,A} \otimes \mathrm{id}_M) \circ \sigma_{A,M})(m) \\
&= \textstyle\sum_{i=1}^p (\mu_{B,M} \circ (\mathrm{id}_B \otimes \mu_{A,M}) \circ (\sigma_{B,A} \otimes \mathrm{id}_M))(a_i \otimes m_i) \\
&= \textstyle\sum_{i=1}^p \sum_{j=1}^{q_i} (\mu_{B,M} \circ (\mathrm{id}_B \otimes \mu_{A,M}))(b_j^{(i)} \otimes a_j^{(i)} \otimes m_i) \\
&= \textstyle\sum_{i=1}^p \sum_{j=1}^{q_i} \mu_{B,M}(b_j^{(i)} \otimes a_j^{(i)} m_i) \\
&= \textstyle\sum_{i=1}^p \sum_{j=1}^{q_i} f(b_j^{(i)}) a_j^{(i)} m_i = \sum_{i=1}^p a_i m_i = m,
\end{aligned}$$

showing that $\mu_{B,M} \circ \sigma = \mathrm{id}_M$. □

Let $f : B \to A$ be a ring homomorphism. Consider A as a B-bimodule via f, that is $b \cdot a := f(b)a$ and $a \cdot b := af(b)$, for $a \in A$ and $b \in B$. Recall from Section 1 that A/B is called separable if the multiplication map $\mu_{A/B} : A \otimes_B A \to A$, defined by the additive extension of $\mu(a \otimes a') = aa'$, for $a, a' \in A$, has a section in ${}_A\mathrm{Mod}_A$, that is if there is an A-bimodule map $\sigma : A \to A \otimes_B A$ such that $\mu_{A/B} \circ \sigma = \mathrm{id}_A$.

Proof of Theorem 1.3. First we show the "only if" statement. Let Res_f be separable. We wish to show that A/B is separable. Put $\sigma_{B,A} := \mu_{B,A}^{-1}$. Let $\sigma' : A \to A \otimes_B A$ be the morphism in ${}_B\mathrm{Mod}_A$ defined by $\sigma' := (f \otimes \mathrm{id}_A) \circ \sigma_{B,A}$. Then

$$\mathrm{Res}_f(\mu_{A/B}) \circ \sigma' = \mathrm{id}_A \tag{2}$$

as morphisms in ${}_B\mathrm{FMod}$. Indeed, take $a \in A$. Then $\sigma_{B,A}(a) = \sum_{i=1}^n b_i \otimes a_i$ for some $n \in \mathbb{N}$, $b_i \in B$ and $a_i \in A$ with $\sum_{i=1}^n f(b_i)a_i = a$. Therefore

$$\begin{aligned}(\mathrm{Res}_f(\mu_{A/B}) \circ \sigma')(a) &= (\mathrm{Res}_f(\mu_{A/B}) \circ (f \otimes \mathrm{id}_A) \circ \sigma_{B,A})(a) \\ &= \textstyle\sum_{i=1}^n (\mathrm{Res}_f(\mu_{A/B}) \circ (f \otimes \mathrm{id}_A))(b_i \otimes a_i) \\ &= \textstyle\sum_{i=1}^n \mathrm{Res}_f(\mu_{A/B})(f(b_i) \otimes a_i) = \sum_{i=1}^n f(b_i)a_i = a.\end{aligned}$$

Since A is firm, both ${}_AA$ and ${}_A(A \otimes_B A)$ belong to ${}_A\mathrm{FMod}$, so the map $\mathsf{R}^{\mathrm{Res}_f}_{A,\, A\otimes_B A}$ is well defined. Let $\sigma : A \to A \otimes_B A$ be the morphism in ${}_A\mathrm{Mod}$ defined by $\sigma := \mathsf{R}^{\mathrm{Res}_f}_{A,\, A\otimes_B A}(\sigma')$. By (SF3) and (2), $\mu_{A/B} \circ \sigma = \mathrm{id}_A$. What remains to show is that σ is a morphism in Mod_A. To this end, take $a \in A$ and let $\alpha_a : A \to A$ and $\beta_a : A \otimes_B A \to A \otimes_B A$ denote the left B-linear maps given by right multiplication by a. Since σ' is a morphism in ${}_B\mathrm{Mod}_A$, it follows that $\sigma' \circ \alpha_a = \beta_a \circ \sigma'$ as morphisms in ${}_B\mathrm{Mod}$. By (SF3), $\sigma \circ \alpha_a = \beta_a \circ \sigma$ as left A-linear maps, since $\mathrm{Res}_f(\alpha_a) = \alpha_a$ and $\mathrm{Res}_f(\beta_a) = \beta_a$. Therefore, σ is right A-linear.

Now we show the "if" statement. Suppose that $\mu_{A/B}$ has a section σ in ${}_A\mathrm{Mod}_A$. Let M and N be firm left A-modules. Set $\sigma_{A,M} := \mu_{A,M}^{-1}$. Define

$$\mathsf{R}^{\mathrm{Res}_f}_{M,N} : \mathrm{Hom}_B(\mathrm{Res}_f(M), \mathrm{Res}_f(N)) \to \mathrm{Hom}_A(M, N)$$

in the following way. Let $i_N : A \otimes_B N \to A \otimes_A N$ be the map $a \otimes_B n \mapsto a \otimes_A n$. Take $g \in \mathrm{Hom}_B(\mathrm{Res}_f(M), \mathrm{Res}_f(N))$. Put

$$\mathsf{R}^{\mathrm{Res}_f}_{M,N}(g) := \mu_{A,N} \circ i_N \circ (\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes \mu_{A,M}) \circ (\sigma \otimes \mathrm{id}_M) \circ \sigma_{A,M}.$$

Then $\mathsf{R}^{\mathrm{Res}_f}_{M,N}(g)$, being defined as a composition of morphisms in ${}_A\mathrm{Mod}$, is again a morphism in ${}_A\mathrm{Mod}$. To facilitate the rest of the proof, we now describe $\mathsf{R}^{\mathrm{Res}_f}_{M,N}(g)$ elementwise. Take $m \in M$, $n \in \mathbb{N}$, $a_i \in A$ and $m_i \in M$ with $\sum_{i=1}^n a_i m_i = m$. For each i, take $n_i \in \mathbb{N}$ and $a_{ij}, a'_{ij} \in A$ such that $\sigma(a_i) = \sum_{j=1}^{n_i} a_{ij} \otimes a'_{ij}$. Then

$$\begin{aligned}&\mathsf{R}^{\mathrm{Res}_f}_{M,N}(g)(m) \\ =\;& (\mu_{A,N} \circ i_N \circ (\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes \mu_{A,M}) \circ (\sigma \otimes \mathrm{id}_M) \circ \sigma_{A,M})(m) \\ =\;& \textstyle\sum_{i=1}^n (\mu_{A,N} \circ i_N \circ (\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes \mu_{A,M}) \circ (\sigma \otimes \mathrm{id}_M))(a_i \otimes m_i) \\ =\;& \textstyle\sum_{i=1}^n \sum_{j=1}^{n_i} (\mu_{A,N} \circ i_N \circ (\mathrm{id}_A \otimes g) \circ (\mathrm{id}_A \otimes \mu_{A,M}))(a_{ij} \otimes a'_{ij} \otimes m_i) \\ =\;& \textstyle\sum_{i=1}^n \sum_{j=1}^{n_i} (\mu_{A,N} \circ i_N \circ (\mathrm{id}_A \otimes g))(a_{ij} \otimes a'_{ij} m_i) \\ =\;& \textstyle\sum_{i=1}^n \sum_{j=1}^{n_i} \mu_{A,N}(a_{ij} \otimes g(a'_{ij} m_i)) = \sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} g(a'_{ij} m_i).\end{aligned}$$

Now we show (SF1). Suppose that g is a morphism in ${}_A\mathrm{Mod}$. Then

$$\begin{aligned}
\mathsf{R}^{\mathrm{Res}_f}_{M,N}(g)(m) &= \sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} g(a'_{ij} m_i) = g\left(\sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} a'_{ij} m_i\right) \\
&= g\left(\sum_{i=1}^n \sum_{j=1}^{n_i} \mu_{A/B}(a_{ij} \otimes a'_{ij}) m_i\right) \\
&= g\left(\sum_{i=1}^n \mu_{A/B}\left(\sum_{j=1}^{n_i} a_{ij} \otimes a'_{ij}\right) m_i\right) \\
&= g\left(\sum_{i=1}^n \mu_{A/B}(\sigma(a_i)) m_i\right) = g\left(\sum_{i=1}^n a_i m_i\right) = g(m).
\end{aligned}$$

Finally, we show (SF3). Let M, N, P be modules in ${}_A\mathrm{FMod}$. Let $g \in \mathrm{Hom}_B(N, P)$ and $h \in \mathrm{Hom}_B(M, N)$. Take $m \in M$. We consider two cases.

Case 1: $g \in \mathrm{Res}_f(\mathrm{Hom}_A(N, P))$, that is g is A-linear. Then

$$\begin{aligned}
\mathsf{R}^{\mathrm{Res}_f}_{M,P}(g \circ h)(m) &= \sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} g(h(a'_{ij} m_i)) = g\left(\sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} h(a'_{ij} m_i)\right) \\
&= g\left(\mathsf{R}^{\mathrm{Res}_f}_{M,N}(h)(m)\right) = \left(\mathsf{R}^{\mathrm{Res}_f}_{N,P}(g) \circ \mathsf{R}^{\mathrm{Res}_f}_{M,N}(h)\right)(m).
\end{aligned}$$

Case 2: $h \in \mathrm{Res}_f(\mathrm{Hom}_A(M, N))$, that is h is A-linear. Since $\sum_{i=1}^n a_i m_i = m$, it follows that $\sum_{i=1}^n a_i h(m_i) = h(m)$. Hence

$$\begin{aligned}
\mathsf{R}^{\mathrm{Res}_f}_{M,P}(g \circ h)(m) &= \sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} g(h(a'_{ij} m_i)) = \sum_{i=1}^n \sum_{j=1}^{n_i} a_{ij} g(a'_{ij} h(m_i)) \\
&= \mathsf{R}^{\mathrm{Res}_f}_{N,P}(g)(h(m)) = \left(\mathsf{R}^{\mathrm{Res}_f}_{N,P}(g) \circ \mathsf{R}^{\mathrm{Res}_f}_{M,N}(h)\right)(m).
\end{aligned}$$

This establishes (SF3). By Proposition 2.3, Res_f is separable. □

Proposition 3.6. *Let $f : B \to A$ and $g : C \to B$ be left firm ring homomorphisms. Suppose that A and B are firm rings.*

(a) *A/B separable and B/C separable $\Longrightarrow$ A/C separable for $f \circ g : C \to A$.*
(b) *A/C separable for $f \circ g : C \to A$ $\Longrightarrow$ A/B separable.*

Proof. This follows immediately from Theorem 1.3 and Proposition 2.4. □

Suppose that $f : B \to A$ is a ring homomorphism. The induction functor $\mathrm{Ind}_f : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ assigns to any left B-module N the left A-module $A \otimes_B N$, where A acts on $A \otimes_B N$ by left multiplication and the right B-action on A is given via f.

Proposition 3.7. *Let $f : B \to A$ be a ring homomorphism.*

(a) *f left unital $\Longrightarrow$ $\mathrm{Ind}_f : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}^1$;*
(b) *f left s-unital $\Longrightarrow$ $\mathrm{Ind}_f : {}_B\mathrm{Mod} \to {}_A\mathrm{SMod}$;*
(c) *f left unitary $\Longrightarrow$ $\mathrm{Ind}_f : {}_B\mathrm{Mod} \to {}_A\mathrm{UMod}$;*
(d) *f right firm $\Longrightarrow$ $\mathrm{Ind}_f : {}_B\mathrm{FMod} \to {}_A\mathrm{FMod}$.*

Proof. Let N be a left B-module. First we show (a), (b) and (c) simultaneously. Let f be left unital (respectively s-unital, unitary). Then A, considered as a left B-module, is unital (respectively s-unital, unitary). Hence, the same conclusion holds for A viewed as a left module over itself. Therefore, the left A-module $A \otimes_B N$ is also unital (respectively s-unital, unitary).

(d) Suppose that f is right firm and N is firm in ${}_B\mathrm{FMod}$. Put $M := A \otimes_B N$. By Lemma 3.4, we need to show that $\mu_{A,M}$ has a section σ in ${}_A\mathrm{Mod}$. Set $\sigma_{B,N} := \mu_{B,N}^{-1}$. Let $i_M : A \otimes_B M \to A \otimes_A M$ be the map $a \otimes_B m \mapsto a \otimes_A m$. Define $\sigma : M \to A \otimes_A M$ by $\sigma := i_M \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_N) \circ (\mathrm{id}_A \otimes \sigma_{B,N})$. Then σ, being a composition of morphisms in ${}_A\mathrm{Mod}$, is again a morphism in ${}_A\mathrm{Mod}$. Now we show that $\mu_{A,M} \circ \sigma = \mathrm{id}_M$. Take $a \in A$ and $n \in N$. Take $k \in \mathbb{N}$, $b_i \in B$ and $n_i \in N$ with $\sigma_{B,N}(n) = \sum_{i=1}^k b_i \otimes n_i$, so that $\sum_{i=1}^k b_i n_i = (\mu_{B,N} \circ \sigma_{B,N})(n) = n$. Then

$$\begin{aligned}(\mu_{A,M} \circ \sigma)(a \otimes n) &= (\mu_{A,M} \circ i_M \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_N) \circ (\mathrm{id}_A \otimes \sigma_{B,N}))(a \otimes n)\\ &= \textstyle\sum_{i=1}^k (\mu_{A,M} \circ i_M \circ (\mathrm{id}_A \otimes f \otimes \mathrm{id}_N))(a \otimes b_i \otimes n_i)\\ &= \textstyle\sum_{i=1}^k (\mu_{A,M} \circ i_M)(a \otimes f(b_i) \otimes n_i) = \sum_{i=1}^k af(b_i) \otimes n_i\\ &= \textstyle\sum_{i=1}^k a \otimes b_i n_i = a \otimes \sum_{i=1}^k b_i n_i = a \otimes n,\end{aligned}$$

showing that $\mu_{A,M} \circ \sigma = \mathrm{id}_M$. □

A ring homomorphism $f : B \to A$ is called firm if it is both left and right firm.

Theorem 3.8. *Let B be a firm ring and suppose that $f : B \to A$ is a firm ring homomorphism. Then the functor $\mathrm{Ind}_f : {}_B\mathrm{FMod} \to {}_A\mathrm{FMod}$ is separable if and only if f is a split monomorphism in the category of B-bimodules.*

Proof. First we show the "only if" statement. Suppose that Ind_f is separable. We wish to show that f is a split monomorphism in the category of B-bimodules. Set $\sigma_{A,B} := \mu_{A,B}^{-1}$ and $\gamma := (\mu_{A/B} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes \sigma_{A,B})$. Then γ is a morphism in ${}_A\mathrm{Mod}_B$. Now we show that

$$\mu_{A,B} \circ \gamma \circ \mathrm{Ind}_f(f) = \mu_{A,B} \circ \mathrm{Ind}_f(\mathrm{id}_B). \tag{3}$$

Suppose that $a \in A$ and $b \in B$. Take $n \in \mathbb{N}$, $a_i \in A$ and $b_i \in B$ such that $\sigma_{A,B}(f(b)) = \sum_{i=1}^n a_i \otimes b_i$. Then $\sum_{i=1}^n a_i f(b_i) = \mu_{A,B}(\sigma_{A,B}(f(b))) = f(b)$. Thus,

$$\begin{aligned}(\mu_{A,B} \circ \gamma \circ \mathrm{Ind}_f(f))(a \otimes b) &= \textstyle\sum_{i=1}^n (\mu_{A,B} \circ (\mu_{A/B} \otimes \mathrm{id}_B))(a \otimes a_i \otimes b_i)\\ &= \textstyle\sum_{i=1}^n \mu_{A,B}(aa_i \otimes b_i) = \sum_{i=1}^n aa_i f(b_i)\\ &= af(b) = (\mu_{A,B} \circ \mathrm{id}_{A \otimes_B B})(a \otimes b).\end{aligned}$$

This establishes (3). Since $\mu_{A,B}$ is bijective, we get $\gamma \circ \mathrm{Ind}_f(f) = \mathrm{Ind}_f(\mathrm{id}_B)$. By (SF3), $\mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma) \circ f = \mathrm{id}_B$. Thus, f splits as an additive map. Since γ is left A-linear, $\mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma)$ is left B-linear. Now we show that $\mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma)$ is right B-linear. To this end, take $b \in B$ and let $m_b^B : B \to B$ and $m_b^A : A \to A$ denote the left B-linear maps given by right multiplication by b and $f(b)$, respectively. Since γ is right B-linear, we have $\gamma \circ \mathrm{Ind}_f(m_b^A) = \mathrm{Ind}_f(m_b^B) \circ \gamma$. By (SF3), $\mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma) \circ m_b^A = m_b^B \circ \mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma)$. Therefore, $\mathsf{R}_{A,B}^{\mathrm{Ind}_f}(\gamma)$ is right B-linear.

Now we show the "if" statement. Suppose that f is a split monomorphism in the category of B-bimodules. Then there is a B-bimodule map $s : A \to B$ such that $s \circ f = \mathrm{id}_B$. Suppose that M and N are firm left B-modules. Define $\mathsf{R}_{M,N}^{\mathrm{Ind}_f} : \mathrm{Hom}_A(\mathrm{Ind}_f(M), \mathrm{Ind}_f(N)) \to \mathrm{Hom}_B(M,N)$ in the following way. Take $g \in \mathrm{Hom}_A(\mathrm{Ind}_f(M), \mathrm{Ind}_f(N))$. Put

$$\mathsf{R}_{M,N}^{\mathrm{Ind}_f}(g) := \mu_{B,N} \circ (s \otimes \mathrm{id}_N) \circ g \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1}.$$

Then $\mathsf{R}_{M,N}^{\mathrm{Ind}_f}(g)$ is a morphism in ${}_B\mathrm{Mod}$. Now we show (SF1). Take $m \in M$, $k \in \mathbb{N}$, $b_i \in B$ and $m_i \in M$ with $\sum_{i=1}^k b_i m_i = m$. Let $g = \mathrm{Ind}_f(g')$ for some $g' \in \mathrm{Hom}_B(M,N)$. Then

$$\begin{aligned}
\mathsf{R}_{M,N}^{\mathrm{Ind}_f}(g)(m) &= \textstyle\sum_{i=1}^k (\mu_{B,N} \circ (s \otimes \mathrm{id}_N) \circ g)(f(b_i) \otimes m_i) \\
&= \textstyle\sum_{i=1}^k (\mu_{B,N} \circ (s \otimes \mathrm{id}_N) \circ (\mathrm{id}_A \otimes g'))(f(b_i) \otimes m_i) \\
&= \textstyle\sum_{i=1}^k (\mu_{B,N} \circ (s \otimes \mathrm{id}_N))(f(b_i) \otimes g'(m_i)) \\
&= \textstyle\sum_{i=1}^k \mu_{B,N}(b_i \otimes g'(m_i)) \\
&= \textstyle\sum_{i=1}^k b_i g'(m_i) = g'\left(\sum_{i=1}^k b_i m_i\right) = g'(m).
\end{aligned}$$

Now we show (SF3). Suppose that M, N, P are modules in ${}_B\mathrm{FMod}$. Take $g \in \mathrm{Hom}_A(A \otimes_B N, A \otimes_B P)$ and $h \in \mathrm{Hom}_A(A \otimes_B M, A \otimes_B N)$. We consider two cases.

Case 1: $g \in \mathrm{Ind}_f(\mathrm{Hom}_B(N,P))$ that is $g = \mathrm{id}_A \otimes g'$ for some $g' \in \mathrm{Hom}_B(N,P)$. Then

$$(s \otimes \mathrm{id}_P) \circ g \circ (f \otimes \mathrm{id}_N) \circ (s \otimes \mathrm{id}_N) = (\mathrm{id}_B \otimes g') \circ (s \otimes \mathrm{id}_N) = (s \otimes \mathrm{id}_P) \circ g.$$

Therefore,

$$\begin{aligned}
&\mathsf{R}_{N,P}^{\mathrm{Ind}_f}(g) \circ \mathsf{R}_{M,N}^{\mathrm{Ind}_f}(h) \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ (f \otimes \mathrm{id}_N) \circ \mu_{B,N}^{-1} \circ \mu_{B,N} \circ (s \otimes \mathrm{id}_N) \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ (f \otimes \mathrm{id}_N) \circ (s \otimes \mathrm{id}_N) \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} = \mathsf{R}_{M,P}^{\mathrm{Ind}_f}(g \circ h).
\end{aligned}$$

Case 2: $h \in \mathrm{Ind}_f(\mathrm{Hom}_B(M,N))$ that is $h = \mathrm{id}_A \otimes h'$ for some $h' \in \mathrm{Hom}_B(M,N)$. Then

$$(f \otimes \mathrm{id}_N) \circ (s \otimes \mathrm{id}_N) \circ h \circ (f \otimes \mathrm{id}_M) = (\mathrm{id}_A \otimes h') \circ (s \otimes \mathrm{id}_M) = h \circ (f \otimes \mathrm{id}_M).$$

Therefore,

$$\begin{aligned}
&\mathsf{R}_{N,P}^{\mathrm{Ind}_f}(g) \circ \mathsf{R}_{M,N}^{\mathrm{Ind}_f}(h) \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ (f \otimes \mathrm{id}_N) \circ \mu_{B,N}^{-1} \circ \mu_{B,N} \circ (s \otimes \mathrm{id}_N) \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ (f \otimes \mathrm{id}_N) \circ (s \otimes \mathrm{id}_N) \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} \\
= \ &\mu_{B,P} \circ (s \otimes \mathrm{id}_P) \circ g \circ h \circ (f \otimes \mathrm{id}_M) \circ \mu_{B,M}^{-1} = \mathsf{R}_{M,P}^{\mathrm{Ind}_f}(g \circ h).
\end{aligned}$$

This establishes (SF3). By Proposition 2.3, Ind_f is separable. □

4. Semisimple modules

From now on, A denotes a fixed left s-unital ring, and we consider the category ${}_A\mathrm{FMod}$ of firm left A-modules. Recall from Proposition 3.3 that

$$ {}_A\mathrm{FMod} = {}_A\mathrm{UMod} = {}_A\mathrm{SMod}. \tag{4} $$

Suppose that M and P are modules in ${}_A\mathrm{FMod}$ with $P \subseteq M$. Then P is called a submodule of M in ${}_A\mathrm{FMod}$. By (4), the quotient module M/P in ${}_A\mathrm{Mod}$ also belongs to ${}_A\mathrm{FMod}$. For a morphism $f : M \to N$ in ${}_A\mathrm{FMod}$, we define its kernel $\mathrm{Ker}(f) := \{m \in M \mid f(m) = 0\}$ and its image $\mathrm{Im}(f) := \{f(m) \mid m \in M\}$. By (4), $\mathrm{Ker}(f)$ is a submodule of M in ${}_A\mathrm{FMod}$ and $\mathrm{Im}(f)$ is a submodule of N in ${}_A\mathrm{FMod}$.

Proposition 4.1. *Suppose that $f : M \to N$ is a morphism in ${}_A\mathrm{FMod}$.*

(a) *f is a monomorphism $\Longleftrightarrow$ f is injective;*
(b) *f is an epimorphism $\Longleftrightarrow$ f is surjective;*
(c) *f is an isomorphism $\Longleftrightarrow$ f is bijective.*

Proof. (a) The implication ($\Leftarrow$) is trivial. Now we show ($\Rightarrow$). Suppose that f is a monomorphism. Define $g, h : \mathrm{Ker}(f) \to M$ by $g(m) = m$ and $h(m) = 0$ for $m \in \mathrm{Ker}(f)$. Then $fg = 0 = fh$. Since f is a monomorphism, $g = h$. Hence, $\mathrm{Ker}(f) = g(\mathrm{Ker}(f)) = h(\mathrm{Ker}(f)) = \{0\}$.

(b) The implication ($\Leftarrow$) is trivial. Now we show ($\Rightarrow$). Let f be an epimorphism. Define $g, h : N \to N/\mathrm{Im}(f)$ by letting g be the quotient map and h the zero map. Then $gf = 0 = hf$. Since f is an epimorphism, $g = h$, so that $\mathrm{Im}(f) = N$.

(c) This follows from (a) and (b). □

The next result shows that subobjects and quotient objects in ${}_A\mathrm{FMod}$ correspond to submodules and quotient modules in ${}_A\mathrm{FMod}$, respectively.

Proposition 4.2. *Let $M \in {}_A\mathrm{FMod}$. The following maps are bijections:*

$$ \gamma : \mathrm{Sub}_{{}_A\mathrm{FMod}}(M) \longrightarrow \{\textit{submodules of } M\}, \quad \gamma([f : P \to M]) = f(P), $$

$$ \delta : \mathrm{Quot}_{{}_A\mathrm{FMod}}(M) \longrightarrow \{\textit{quotient modules of } M\}, \quad \delta([q : M \to Q]) = Q. $$

Proof. The claim for γ follows from [28, Thm. 4.2]. Now we show the part for δ.

If $[q] = [q']$ in $\mathrm{Quot}(M)$, then $q' = \varphi q$ for some isomorphism $\varphi : Q \to Q'$, so $Q \cong Q'$, showing that δ is well defined. If $\delta([q]) = \delta([q'])$, then $Q \cong Q'$ and choosing an isomorphism $\varphi : Q \to Q'$ yields $q' = \varphi \circ q$, so $[q] = [q']$, proving injectivity. For any quotient module M/N, the canonical map $p : M \to M/N$ is an epimorphism and $\delta([p]) = M/N$, proving surjectivity. □

Let M be a module in ${}_A\mathrm{FMod}$. We say that M is simple in ${}_A\mathrm{FMod}$ if M is nonzero, and M has no submodules in ${}_A\mathrm{FMod}$ except the zero module and itself.

Proposition 4.3. *Let $M \in {}_A\mathrm{FMod}$. The following assertions are equivalent:*

(i) *M is simple in ${}_A\mathrm{FMod}$;*
(ii) *M is subobject simple in ${}_A\mathrm{FMod}$;*
(iii) *M is quotient simple in ${}_A\mathrm{FMod}$.*

Proof. By Proposition 4.2, categorical subobjects of M are in bijection with submodules of M. Likewise, categorical quotient objects of M are in bijection with quotient modules of M. Therefore $\mathrm{Sub}_{{}_A\mathrm{FMod}(M)}$ has exactly two elements if and only if M has exactly two submodules, and $\mathrm{Quot}_{{}_A\mathrm{FMod}(M)}$ has exactly two elements if and only if M has exactly two quotient modules, and these conditions are equivalent since the lattice of submodules and the lattice of quotient modules of M are anti-isomorphic via $N \mapsto M/N$. □

Lemma 4.4. *Suppose that M is a subobject semisimple module in ${}_A\mathrm{FMod}$ and N is a nonzero submodule of M. Then N contains a simple submodule.*

Proof. Take a nonzero $n \in N$. By Zorn's lemma, there is a maximal submodule P of N with $n \notin P$. Since $P \subseteq M$ and M is subobject semisimple in ${}_A\mathrm{FMod}$, the inclusion $P \hookrightarrow M$ splits. Hence $M = P \oplus P'$ for some submodule P' of M. Put $Q := N \cap P'$. Then $N = P \oplus Q$. Since $n \in N \setminus P$, we have $Q \neq 0$. We claim that Q is simple. Let $\{0\} \neq Q' \subseteq Q$ be a nonzero submodule. Since $Q' \subseteq M$ and M is subobject semisimple, the inclusion $Q' \hookrightarrow M$ splits. In particular, $Q = Q' \oplus Q''$ for some submodule Q'' of Q. Hence $N = P \oplus Q' \oplus Q''$. If $Q'' \neq 0$, then both $P \oplus Q'$ and $P \oplus Q''$ properly contain P, so by maximality of P both contain n. Thus $n \in (P \oplus Q') \cap (P \oplus Q'') = P$, a contradiction. Therefore $Q'' = 0$, so $Q' = Q$. Thus Q is simple. □

Let M be a module in ${}_A\mathrm{FMod}$. We say that M is semisimple in ${}_A\mathrm{FMod}$ if M is the direct sum of simple submodules in ${}_A\mathrm{FMod}$.

Proposition 4.5. *Let $M \in {}_A\mathrm{FMod}$. The following assertions are equivalent:*

(i) *M is semisimple in ${}_A\mathrm{FMod}$;*
(ii) *M is quotient semisimple in ${}_A\mathrm{FMod}$;*
(iii) *M is subobject semisimple in ${}_A\mathrm{FMod}$.*

Proof. (i)⇒(ii): Let $q : M \to N$ be an epimorphism. We consider two cases.

Case 1: $\mathrm{Ker}(q) = M$. Then $N = \{0\}$. Let $s : N \to M$ be the zero map. Then $q \circ s = \mathrm{id}_N$.

Case 2: $\mathrm{Ker}(q) \subsetneq M$. Since M is semisimple, $M = \oplus_{i\in I} M_i$ for some simple submodules M_i of M. Let S be the subset of nonzero submodules L of M such that $\mathrm{Ker}(q) \cap L = \{0\}$. Then S is nonempty. Indeed, by the assumptions, there is $i \in I$ with $M_i \not\subseteq \mathrm{Ker}(q)$, so that $M_i \cap \mathrm{Ker}(q) = \{0\}$, by simplicity of M_i. The set S, ordered by inclusion, is inductive, since the union of a chain in S again belongs to S. Thus, by Zorn's lemma, it has a maximal element P. Seeking a contradiction, suppose that $\mathrm{Ker}(q) + P \subsetneq M$. Then there is $j \in I$ with $M_j \not\subseteq \mathrm{Ker}(q) + P$ so that $M_j \cap (\mathrm{Ker}(q) + P) = \{0\}$ by simplicity of M_j. But then $P \subsetneq P + M_j$ and $\mathrm{Ker}(q) \cap (P + M_j) = \{0\}$ which violates the maximality of P, which is a contradiction. Therefore, $M = \mathrm{Ker}(q) \oplus P$. The restriction $q|_P : P \to N$ is injective since $P \cap \mathrm{Ker}(q) = \{0\}$, and surjective since $q(P) = q(\mathrm{Ker}(q)) + q(P) = q(\mathrm{Ker}(q) + P) = q(M) = N$, hence an isomorphism. Letting $s : N \to M$ be the composite of $(q|_P)^{-1}$ with the inclusion $P \hookrightarrow M$ yields $q \circ s = \mathrm{id}_N$. Hence, M is quotient semisimple.

(ii)⇒(iii): Suppose that M is quotient semisimple. Let $i : K \to M$ be a monomorphism. Consider the cokernel $p : M \to M/K$, which is an epimorphism. By hypothesis, there is $s : M/K \to M$ with $p \circ s = \mathrm{id}_{M/K}$. For any $m \in M$, we then have $m - s(p(m)) \in K$, so we may define a map $r : M \to K$ by $r(m) = m - s(p(m))$. Since $p(m - s(p(m))) = p(m) - p(s(p(m))) = p(m) - p(m) = 0$, we have $r(m) \in K$ for all $m \in M$. Thus $r : M \to K$ is well defined. It is clearly A-linear. Moreover, for $k \in K$, we have $p(k) = 0$, hence $r(k) = k$, so that $r \circ i = \mathrm{id}_K$. Hence, M is subobject semisimple.

(iii)⇒(i): Let M be subobject semisimple. If $M = \{0\}$, then M is the direct sum of an empty family of simple submodules. Assume now that $M \neq \{0\}$. By Lemma 4.4 and Zorn's lemma, there exists a nonempty family $\{N_i\}_{i\in I}$ of simple submodules such that $N = \oplus_{i\in I} N_i$ is maximal among direct sums of simple submodules of M. Since $N \subseteq M$ and M is subobject semisimple, we have $M = N \oplus P$ for some submodule P of M. Suppose that $P \neq \{0\}$. By Lemma 4.4, P contains a simple submodule P'. Since $P' \subseteq M$ and M is subobject semisimple, the inclusion $P' \hookrightarrow M$ splits. Hence $M = P' \oplus X$ for some submodule X of M. Thus, we get $P = P' \oplus (P \cap X)$. Hence $M = N \oplus P' \oplus (P \cap X)$, contradicting the maximality of the family $\{N_i\}_{i\in I}$. Therefore $P = \{0\}$, so $M = \bigoplus_{i\in I} N_i$. Hence M is semisimple. □

Proposition 4.6. *Let $M \in {}_A\mathrm{FMod}$. The following assertions hold:*

(a) *The module M is semisimple in ${}_A\mathrm{FMod}$ if and only if every submodule and every quotient module of M is semisimple in ${}_A\mathrm{FMod}$.*

(b) *Suppose* $M = \oplus_{i \in I} M_i$ *for some submodules* $(M_i)_{i \in I}$. *Then* M *is semisimple in* ${}_A\mathrm{FMod}$ *if and only if each* M_i *is semisimple in* ${}_A\mathrm{FMod}$.

Proof. (a) The "if" part is trivial, since M is both a submodule and a quotient module of itself. Now suppose that M is semisimple in ${}_A\mathrm{FMod}$. Let N be a submodule of M. We first show that N is semisimple. Let P be a submodule of N. By Proposition 4.5, M is subobject semisimple, so there exists a submodule Q of M such that $M = P \oplus Q$. Hence $N = N \cap M = N \cap (P \oplus Q) = P \oplus (N \cap Q)$. Thus every submodule of N is a direct summand of N, so N is subobject semisimple. By Proposition 4.5, N is semisimple. Now we show that M/N is semisimple. Again by Proposition 4.5, there exists a submodule R of M such that $M = N \oplus R$. Since R is a submodule of the semisimple module M, the first part shows that R is semisimple. Moreover, $M/N \cong R$. Hence M/N is semisimple.

(b) If M is semisimple, then each M_i is a submodule of M, so M_i is semisimple by (a). Conversely, suppose that each M_i is semisimple. For each $i \in I$, write $M_i = \oplus_{j \in J_i} S_{ij}$, where each S_{ij} is simple. Then $M = \oplus_{i \in I} M_i = \oplus_{i \in I} \oplus_{j \in J_i} S_{ij}$, so M is a direct sum of simple submodules. Hence M is semisimple in ${}_A\mathrm{FMod}$. □

We say that A is left semisimple if A is semisimple in ${}_A\mathrm{FMod}$.

Proposition 4.7. *The ring* A *is left semisimple if and only if every module in* ${}_A\mathrm{FMod}$ *is semisimple.*

Proof. The "if" direction is immediate. Now we show the "only if" direction. Suppose that A is left semisimple. Let M be a module in ${}_A\mathrm{FMod}$. Since M is unitary, we have $AM = M$ so that $M = \sum_{m \in M} Am$. Let $A^{(M)}$ denote the direct sum of $|M|$ copies of A. Define a surjective A-linear map $p : A^{(M)} \ni (a_m)_{m \in M} \mapsto \sum_{m \in M} a_m m \in M$. By Proposition 4.6(b), $A^{(M)}$ is semisimple. Therefore, by Proposition 4.6(a), $M \cong A^{(M)} / \mathrm{Ker}(p)$ is semisimple. □

Proof of Theorem 1.4. Let A and B be left s-unital rings with B left semisimple. Suppose that $f : B \to A$ is a left s-unital ring homomorphism such that A/B is separable. We wish to show that A is left semisimple. Let $M \in {}_A\mathrm{FMod}$. By Proposition 4.7, the module $\mathrm{Res}_f(M)$ is semisimple in ${}_B\mathrm{FMod}$. By Proposition 4.5, it is therefore subobject semisimple in ${}_B\mathrm{FMod}$. By Theorem 1.3, Res_f is separable. Clearly, Res_f preserves monomorphisms. By Proposition 2.11, M is subobject semisimple in ${}_A\mathrm{FMod}$. Hence, by Proposition 4.5, M is semisimple in ${}_A\mathrm{FMod}$. Therefore, by Proposition 4.7, A is left semisimple. □

Proof of Theorem 1.5. Let G be a finite group of order $n := |G|$. Let B be a ring with local units satisfying $nB = B$. Let A denote the group ring $B[G]$. We wish to show that A/B is separable. To this end, we will construct an A-bimodule splitting σ of the multiplication map $\mu_{A/B} : A \otimes_B A \to A$. Take $a \in A$ and an idempotent $e \in B$ such that $ae = a = ea$. Since $nB = B$, there exists $b \in B$ such that $nb = e$. Then $(ne)(ebe) = e(nb)e = e^2ee = e$ and, analogously, $(ebe)(ne) = e$. Hence ne is invertible in the unital ring eBe. Define $\sigma(a) := \sum_{g\in G} ag \otimes (ne)^{-1}g^{-1}$. Then

$$\begin{aligned}(\mu_{A/B} \circ \sigma)(a) &= \sum_{g\in G} ag(ne)^{-1}g^{-1} = \sum_{g\in G} a(ne)^{-1}gg^{-1}\\ &= \sum_{g\in G} a(ne)^{-1} = n\, a(ne)^{-1} = a(ne)(ne)^{-1} = ae = a.\end{aligned}$$

Therefore, σ is a splitting of $\mu_{A/B}$. Clearly, σ is left A-linear.

Next we show that σ is well defined. Suppose that $e' \in B$ is another idempotent such that $ae' = a = e'a$. Take an idempotent $f \in B$ such that $ef = fe = e$ and $e'f = fe' = e'$. Let $c = (nf)^{-1} \in fBf$, so that $(nf)c = f$. Then $(ne)c = e(nf)c = ef = e$, so $ec = (ne)^{-1}$. Since $c = (nf)^{-1}$, it follows that $e(nf)^{-1} = (ne)^{-1}$. Similarly, $e'(nf)^{-1} = (ne')^{-1}$. Therefore,

$$\begin{aligned}&\sum_{g\in G} ag \otimes (ne)^{-1}g^{-1} = \sum_{g\in G} ag \otimes e(nf)^{-1}g^{-1} = \sum_{g\in G} aeg \otimes (nf)^{-1}g^{-1}\\ &= \sum_{g\in G} ae'g \otimes (nf)^{-1}g^{-1} = \sum_{g\in G} ag \otimes e'(nf)^{-1}g^{-1} = \sum_{g\in G} ag \otimes (ne')^{-1}g^{-1}.\end{aligned}$$

Finally, we show that σ is right A-linear. Take $b \in B$ and $h \in G$. Choose an idempotent $e \in B$ such that $ae = a = ea$ and $be = b = eb$. Since $b(ne) = nb = (ne)b$, it follows that $(ne)^{-1}b = b(ne)^{-1}$. Hence

$$\begin{aligned}\sigma(a)bh &= \sum_{g\in G} ag \otimes (ne)^{-1}g^{-1}bh = \sum_{g\in G} ag \otimes b(ne)^{-1}g^{-1}h\\ &= \sum_{p\in G} ahp \otimes b(ne)^{-1}p^{-1} = \sum_{p\in G} abhp \otimes (ne)^{-1}p^{-1} = \sigma(abh),\end{aligned}$$

where $p := h^{-1}g$. Thus σ is right A-linear. Therefore, A/B is separable. □

Proof of Theorem 1.6. Let G be a finite group of order $|G|$. Let B be a semisimple ring with local units satisfying $|G|B = B$. Put $A := B[G]$. We wish to show that A is left semisimple. Since B is locally unital, the ring inclusion $B \to A$ is left s-unital. Therefore, by Theorem 1.4 and Theorem 1.5, A is left semisimple. □

Remark 4.8. It is easy to produce examples of nonunital rings B satisfying the hypotheses of Theorem 1.6 for any finite group G. Let K be a field of characteristic zero and let $B = K^{(\mathbb{N})}$ be the ring from Example 3.2(a). Then B is semisimple and nonunital, yet has local units, and satisfies $nB = B$ for every $n \in \mathbb{N}$. Hence Theorem 1.6 applies to the group ring $B[G]$ for any finite group G.

Moreover, one can even find simple nonunital rings B with local units such that $nB = B$ for all $n \in \mathbb{N}$. A standard example is the ring $\mathrm{FM}_{\mathbb{N}}(K)$ of finitary $\mathbb{N} \times \mathbb{N}$ matrices over K, that is, the ring consisting of all matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with $a_{ij} \in K$ such that $a_{ij} = 0$ for all but finitely many pairs $(i, j) \in \mathbb{N} \times \mathbb{N}$, over a field K of characteristic zero. This ring has local units, since any finite set of matrices is supported on finitely many rows and columns, and the diagonal idempotent corresponding to those rows and columns acts as a local unit for that set. We leave the simplicity verification to the reader.

Let A be a left s-unital ring. We say that A is left hereditary if for every projective left A-module M in ${}_A\mathrm{FMod}$, all A-submodules of M are again projective.

Theorem 4.9. *Let A and B be rings with A s-unital and B left hereditary. Let $f : B \to A$ be a left firm ring homomorphism with A/B separable. Suppose that* $\mathrm{Res}_f : {}_A\mathrm{FMod} \to {}_B\mathrm{FMod}$ *preserves projective modules. Then A is left hereditary.*

Proof. Let $M \in {}_A\mathrm{FMod}$ be projective and let N be a submodule of M. By the assumptions, $\mathrm{Res}_f(M)$ is projective in ${}_B\mathrm{FMod}$. Since B is left hereditary, also $\mathrm{Res}_f(N)$ is projective in ${}_B\mathrm{FMod}$. By Theorem 1.3, Res_f is separable. Clearly, Res_f preserves epimorphisms. Thus, by Proposition 2.22, N is projective in ${}_A\mathrm{FMod}$. Therefore, A is left hereditary. □

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References

[1] P. N. Ánh and L. Márki, *Morita equivalence for rings without identity*, Tsukuba J. Math., 11 (1987), 1-16.

[2] R. Baer, *Absolute retracts in group theory*, Bull. Amer. Math. Soc., 52 (1946), 501-506.

[3] G. Böhm and J. Gómez-Torrecillas, *Firm monads and firm Frobenius algebras*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 56 (2013), 281-298.

[4] T. Brzeziński, L. Kadison and R. Wisbauer, *On coseparable and biseparable corings*, in: *Hopf Algebras in Noncommutative Geometry and Physics*, S.

Caenepeel and F. Van Oystaeyen (eds.), Lecture Notes in Pure and Appl. Math., Dekker, New York, 239 (2005), 71-89.

[5] S. Caenepeel and F. Grandjean, *A note on Taylor's Brauer group*, Pacific J. Math., 186 (1998), 13-27.

[6] J. Cala, P. Lundström and H. Pinedo, *Object-unital groupoid graded rings, crossed products and separability*, Comm. Algebra, 49 (2021), 1676-1696.

[7] C. Christensen, *Groups with complemented normal subgroups*, J. London Math. Soc., 42 (1967), 208-216.

[8] F. DeMeyer and E. Ingraham, Separable Algebras over Commutative Rings, Lecture Notes in Math., 181, Springer, Berlin, 1971.

[9] D. S. Dummit and R. M. Foote, Abstract Algebra, 3rd ed., John Wiley & Sons, Inc., Hoboken, NJ, 2004.

[10] K. R. Fuller, *On rings whose left modules are direct sums of finitely generated modules*, Proc. Amer. Math. Soc., 54 (1976), 39-44.

[11] J. Haefner and G. Janusz, *Hereditary crossed products*, Trans. Amer. Math. Soc., 352 (2000), 3381-3410.

[12] F. C. Iglesias, J. G. Torrecillas and C. Năstăsescu, *Separable functors in graded rings*, J. Pure Appl. Algebra, 127 (1998), 219-230.

[13] L. Kadison, New Examples of Frobenius Extensions, University Lecture Series, 14, Amer. Math. Soc., Providence, RI, 1999.

[14] T. Y. Lam, A First Course in Noncommutative Rings, 2nd ed., Graduate Texts in Mathematics, 131, Springer-Verlag, New York, 2001.

[15] P. Lundström, *Crossed product algebras defined by separable extensions*, J. Algebra, 283 (2005), 723-737.

[16] P. Lundström, *Separable groupoid rings*, Comm. Algebra, 34 (2006), 3029-3041.

[17] S. Mac Lane, Categories for the Working Mathematician, 2nd ed., Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1998.

[18] L. Marín and V. Laan, *Firm homomorphisms of rings and semigroups*, Theory Appl. Categ., 39(22) (2023), 625-666.

[19] H. Maschke, *Über den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen*, Math. Ann., 50(4) (1898), 492-498.

[20] Y. Miyashita, *On Galois extensions and crossed products*, J. Fac. Sci. Hokkaido Univ. Ser. I, 21 (1970), 97-121.

[21] C. Năstăsescu, M. Van den Bergh and F. Van Oystaeyen, *Separable functors applied to graded rings*, J. Algebra, 123 (1989), 397-413.

[22] P. Nystedt, *A survey of s-unital and locally unital rings*, Rev. Integr. Temas Mat., 37 (2019), 251-260.

[23] P. Nystedt, J. Öinert and H. Pinedo, *Epsilon-strongly graded rings, separability and semisimplicity*, J. Algebra, 514 (2018), 1-24.

[24] D. Quillen, *Module theory over nonunital rings*, unpublished manuscript, 1997.

[25] M. D. Rafael, *Separable functors revisited*, Comm. Algebra, 18 (1990), 1445-1459.

[26] Th. Theohari-Apostolidi and H. Vavatsoulas, *On the separability of the restriction functor*, Algebra Discrete Math., 3 (2003), 95-101.

[27] H. Tominaga, *On s-unital rings*, Math. J. Okayama Univ., 18 (1976), 117-134.

[28] K. Väljako, *Monomorphisms in the category of firm modules*, Comm. Algebra, 48(4) (2020), 1528-1537.

[29] R. Wisbauer, *Separability in algebra and category theory*, Algebra and its applications, De Gruyter Proc. Math., De Gruyter, Berlin, (2018), 265-305.

Patrik Lundström

Department of Engineering Science

University West

SE-46186 Trollhättan, Sweden

e-mail: patrik.lundstrom@hv.se