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A SERRE-TYPE THEOREM FOR KRULL DOMAINS

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Abstract. In this paper we show that an integral domain D is a Krull domain if and only if D is a Mori domain, D satisfies Serre's condition (R_1) and every associated prime of a principal ideal has height one.

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1. Introduction

Applications and interests of Krull domains often have caused mathematicians to look for their characterizations (see for example [7], [8] and [9]). The aim of this short paper is to provide a new characterization of Krull domains. As an application, we deduce Krull's normality criterion [5, Theorem 4.1].

Let D be an integral domain with quotient field K and $F(D)$ be the set of nonzero fractional ideals of D. For $A \in F(D)$, let $A^{-1} = \{x \in K \mid xA \subseteq D\}$. Then $A^{-1} \in F(D)$, and so we can define $A_v = (A^{-1})^{-1}$. The t-operation is defined by $A_t = \bigcup\{I_v \mid I \subseteq A$ is a nonzero finitely generated ideal of $D\}$ and the w-operation is defined by $A_w = \{x \in K \mid xJ \subseteq A$ for some nonzero finitely generated ideal J of D with $J_v = D\}$. Clearly, $A \subseteq A_w \subseteq A_t \subseteq A_v$ for all $A \in F(D)$. An $A \in F(D)$ is called a *t-ideal* (resp., w-ideal) if $A_t = A$ (resp., $A_w = A$). A prime ideal of D is called a *t-prime ideal* (resp., *w-prime ideal*) if it is a t-ideal (resp., w-ideal). It is easy to see that a prime ideal $\mathfrak{p}$ of D where $\mathfrak{p}$ is minimal over an integral t-ideal (resp., w-ideal) is itself a t-ideal (resp., w-ideal). An $I \in F(D)$ is said to be *t-invertible* if $(II^{-1})_t = D$. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible.

An integral domain D is called to be a *Mori domain* (resp., *strong Mori domain (SM domain for short)*) if D satisfies the ascending chain condition on integral v-ideals (resp., w-ideals). Note that the class of Noetherian domains is properly contained in the class of SM domains, and the class of SM domains is properly contained in the class of Mori domains. It is well-known that D is a Mori domain if and only if, for every nonzero ideal I of D, there exists a finitely generated ideal $J \subseteq I$ such that $I^v = J^v$, and that every localization of a Mori domain is again Mori [7, Theorem 2.1].

Recall that D is a *Krull domain* if D has a nonempty collection of prime ideals $\mathfrak{p}_\alpha$ such that $D = \bigcap D_{\mathfrak{p}_\alpha}$, each $D_{\mathfrak{p}_\alpha}$ is a principal ideal domain, and every nonzero element of D is contained in only finitely many $\mathfrak{p}_\alpha$s. It is well-known that every localization of a Krull domain is again Krull by [5, Proposition 1.8]. Let D be an integral domain. Then D is a Krull domain if and only if D is a completely integrally closed Mori domain if and only if D is a PvMD and a Mori domain [7, Theorem 3.2], if and only if D is an integrally closed SM domain if and only if D is a PvMD and an SM domain [11, Theorem 2.8].

2. The results

For a Noetherian local commutative ring $(R, \mathfrak{m})$, the length of any maximal R-regular sequence in $\mathfrak{m}$ is independent of the sequence and is denoted by depth R.

Let R be a Noetherian ring. Recall Serre's conditions (R_i) and (S_i) for all non negative integers i:

(R_i) $R_{\mathfrak{p}}$ is regular for all prime ideals $\mathfrak{p}$ of R with $\operatorname{ht}(\mathfrak{p}) \leq i$, and

(S_i) $\operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{\operatorname{ht}(\mathfrak{p}), i\}$ for all prime ideals $\mathfrak{p}$ of R.

The Krull-Serre normality criterion states that a Noetherian domain D is integrally closed if and only if D satisfies (R_1) and (S_2) [5, Theorem 4.1].

The first attempt at a non-Noetherian analogue of Serre's condition (R_1) was given in [4]. A commutative ring R satisfies Serre's condition (R_1) if $R_{\mathfrak{p}}$ is a valuation domain whenever $\mathfrak{p}$ is a height-one prime ideal of R.

Next, these conditions were extended to SM domains in [3]. Let D be an SM domain and n a positive integer. Then D satisfies Serre's condition (R_n) if, for any w-prime ideal $\mathfrak{p}$ of height $m \leq n$, the localization $D_{\mathfrak{p}}$ is regular of dimension m; and D satisfies Serre's condition (S_n) if $\operatorname{depth}(D_{\mathfrak{p}}) \geq \min\{n, \operatorname{ht}(\mathfrak{p})\}$ for any w-prime ideals $\mathfrak{p}$ of D. Equivalently, these conditions can be stated for any w-prime ideal $\mathfrak{p}$. Clearly, if an SM domain D satisfies (R_n) (resp., (S_n)), then $D_{\mathfrak{p}}$ as a Noetherian domain satisfies (R_n) (resp., (S_n)) for any w-prime ideals $\mathfrak{p}$ of D.

For an ideal I of a commutative ring R, a prime ideal $\mathfrak{p}$ of R is said to be an *associated prime of* I if there exists $b \in R \setminus I$ such that $\mathfrak{p}$ is a minimal prime of $(I :_R b)$. In the sequel, $\operatorname{Ass}(I)$ denotes the set of all associated primes of I. We denote the set of associated primes of the zero ideal by $\operatorname{Ass}(R)$. It is well-known that for a Noetherian local commutative ring $(R, \mathfrak{m})$, one has $\mathfrak{m} \in \mathrm{Ass}(R)$ if and only if $\operatorname{depth} R = 0$.

We are now ready to state and prove our main result.

Theorem 2.1. *Let D be an integral domain. Then D is a Krull domain if and only if the following conditions hold*

(1) *D is a Mori domain,*
(2) *D satisfies Serre's condition (R_1), and*
(3) *Every associated prime of a principal ideal has height one.*

Proof. Assume first that D is a Krull domain. Then (1) and (2) hold by [7, Theorem 3.2] and [6, Proposition 43.7], respectively. It remains for us to prove (3). To do this, let $0 \neq a \in D$ and $\mathfrak{p} \in \mathrm{Ass}(aD)$. Then $\mathfrak{m} := \mathfrak{p}D_{\mathfrak{p}} \in \mathrm{Ass}(aD_{\mathfrak{p}})$. Thus there exists $x \in D_{\mathfrak{p}} \backslash aD_{\mathfrak{p}}$ such that $\mathfrak{m} \in \mathrm{Min}(aD_{\mathfrak{p}} : x)$, in particular, $\mathfrak{m} = \sqrt{(aD_{\mathfrak{p}} : x)}$. One notices that the ideal $(aD_{\mathfrak{p}} : x)$ is a t-ideal, hence so is $\mathfrak{m}$. Since D is a Mori domain, there exists a finite family $y_1, \ldots, y_n \in D_{\mathfrak{p}}$ such that $\mathfrak{m} = (y_1, \ldots, y_n)_t$. Whence there is a positive integer N such that $y_i^N \in (aD_{\mathfrak{p}} : x)$ for all i. It follows that there exists a positive integer l such that $(y_1, \ldots, y_n)^l \subseteq (aD_{\mathfrak{p}} : x)$. Therefore $\mathfrak{m}^l \subseteq (\mathfrak{m}^l)_t = (((y_1, \ldots, y_n)_t)^l)_t = ((y_1, \ldots, y_n)^l)_t \subseteq (aD_{\mathfrak{p}} : x)_t = (aD_{\mathfrak{p}} : x)$. Let ℓ be the least integer such that $\mathfrak{m}^\ell \subseteq (aD_{\mathfrak{p}} : x)$. This means that $\mathfrak{m}^\ell x a^{-1} \subseteq D_{\mathfrak{p}}$. Assume that $\mathfrak{m}^\ell x a^{-1} \subseteq \mathfrak{m}$. Then one has $\mathfrak{m}^{\ell-1} x a^{-1} \subseteq D_{\mathfrak{p}}$ since $D_{\mathfrak{p}}$ is completely integrally closed. This is a contradiction to the choice of ℓ. Therefore we have $\mathfrak{m}^\ell x a^{-1} = D_{\mathfrak{p}}$, which implies that $\mathfrak{m}$ is an invertible ideal of $D_{\mathfrak{p}}$. Hence $\mathfrak{m}$ is principal by [6, Proposition 7.4]. Since $D_{\mathfrak{p}}$ is a Krull domain, it follows from [7, Theorem 3.7] that $\operatorname{ht} \mathfrak{p} = \operatorname{ht} \mathfrak{m} = 1$.

Assume conversely that the conditions (1)-(3) hold. By [7, Theorem 3.2], to show that D is a Krull domain, it suffices to show that D is completely integrally closed. By (2) and (3), for every associated prime $\mathfrak{p}$, $D_{\mathfrak{p}}$ is a rank one valuation domain, in particular, $D_{\mathfrak{p}}$ is completely integrally closed by [6, Theorem 17.5]. One now notices that we have the representation $D = \bigcap D_{\mathfrak{p}}$ where $\mathfrak{p}$ ranges over associated primes of principal ideals from [1, Proposition 4]. Consequently, D is completely integrally closed. □

The following lemma shows that the third condition on the above theorem is exactly the same as the Serre's condition (S_2) in the case of Noetherian domains.

Lemma 2.2. *Let R be a Noetherian ring. If R satisfies Serre's condition (S_2), then every associated prime of a proper principal ideal has height at most one. The converse holds if R is an integral domain.*

Proof. Assume R satisfies Serre's condition (S_2), a is a non unit element of R and that $\mathfrak{p} \in \operatorname{Ass}(aR)$. Then $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(aR_\mathfrak{p})$; so that $\operatorname{depth}(R_\mathfrak{p}/aR_\mathfrak{p}) = 0$. Hence, by [2, Proposition 1.2.9], one has

$$0 = \operatorname{depth}(R_\mathfrak{p}/aR_\mathfrak{p}) \geq \min\{\operatorname{depth} R_\mathfrak{p} - 1, \operatorname{depth}(aR_\mathfrak{p})\}.$$

It follows that $\operatorname{depth}(aR_\mathfrak{p}) = 0$ if $\operatorname{depth} R_\mathfrak{p} > 1$. Hence $x\mathfrak{p}R_\mathfrak{p} = 0$ for some $0 \neq x \in aR_\mathfrak{p}$. This means that $\operatorname{depth} R_\mathfrak{p} = 0$, a contradiction. Thus $\operatorname{depth} R_\mathfrak{p} \leq 1$. It follows that $\operatorname{ht}\mathfrak{p} \leq 1$ since R satisfies Serre's condition (S_2) by assumption.

Assume now that R is an integral domain and for each nonzero element $a \in R$ and each $\mathfrak{p} \in \operatorname{Ass}(aR)$, $\operatorname{ht}\mathfrak{p} = 1$. Let $\mathfrak{p}$ be a prime ideal of R. There is nothing to prove if $\operatorname{ht}\mathfrak{p} \leq 1$. So assume that $\operatorname{ht}\mathfrak{p} \geq 2$. Choose a nonzero element a of $\mathfrak{p}$. Clearly, $\mathfrak{p} \notin \operatorname{Ass}(aR)$. Then $\mathfrak{p}R_\mathfrak{p} \notin \operatorname{Ass}(aR_\mathfrak{p})$, which in turn together with [2, Proposition 1.2.10(d)] yields that $\operatorname{depth} R_\mathfrak{p} - 1 = \operatorname{depth} R_\mathfrak{p}/aR_\mathfrak{p} \geq 1$. This completes the proof of the converse. □

The converse part of the above lemma is no longer true without the integral domain assumption. To end this, consider the ring $R = k[X, Y]/(X^2, XY)$ where k is a field.

We now combine Theorem 2.1 and Lemma 2.2 together with the fact that a Noetherian integrally closed domain is a Krull domain [6, Theorem 43.4] to yield Krull's normality criterion [5, Theorem 4.1] (see also [2, Theorem 2.2.22]).

Corollary 2.3. *(Krull's normality criterion) Let D be a Noetherian integral domain. Then D is integrally closed if and only if D satisfies Serre's conditions (R_1) and (S_2).*

One may observe that Lemma 2.2 is true for SM domains as follows.

Lemma 2.4. *Let D be an SM domain. Then D satisfies Serre's condition (S_2) if and only if every associated prime of a proper principal ideal has height one.*

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(Da)$ for some nonzero $a \in D$. Since D is an SM domain, $\mathfrak{p}$ is a w-prime ideal and so $D_\mathfrak{p}$ is a Noetherian domain [10, Proposition 4.6]. Arguing as in the proof of Lemma 2.2, one can completes the proof. □

For an SM domain D, it is shown that D is a Krull domain if and only if D satisfies Serre's conditions (R_1) and (S_2) in [3, Theorem 2.3]. Then we have the following improvement.

Corollary 2.5. *Let D be an integral domain. Then D is a Krull domain if and only if D is an SM domain, and satisfies Serre's conditions (R_1) and (S_2).*

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References

[1] J. W. Brewer and W. J. Heinzer, *Associated primes of principal ideals*, Duke Math. J., 41 (1974), 1-7.

[2] W. Bruns and J. Herzog, Cohen-Macaulay Rings, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[3] G. W. Chang and H. Kim, *Radical perfectness of prime ideals in certain integral domains*, J. Commut. Algebra, 9(1) (2017), 31-48.

[4] N. Epstein and J. Shapiro, *Perinormality—a generalization of Krull domains*, J. Algebra, 451 (2016), 65-84.

[5] R. M. Fossum, The Divisor Class Group of a Krull Domain, Results in Mathematics and Related Areas, 74, Springer-Verlag, New York-Heidelberg, 1973.

[6] R. Gilmer, Multiplicative Ideal Theory, Pure and Applied Mathematics, 12, Marcel Dekker, Inc., New York, 1972.

[7] B. G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra, 124 (1989), 284-299.

[8] S. Malik, J. L. Mott and M. Zafrullah, *On t-invertibility*, Comm. Algebra, 16(1) (1988), 149-170.

[9] J. L. Mott and M. Zafrullah, *On Krull domains*, Arch. Math. (Basel), 56(6) (1991), 559-568.

[10] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra, 25(4) (1997), 1285-1306.

[11] F. Wang and R. L. McCasland, *On strong Mori domains*, J. Pure Appl. Algebra, 135(2) (1999), 155-165.

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