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ON RINGS WHOSE ATOMIC MODULES ARE INJECTIVE

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Abstract. We introduce a new class of rings close to von Neumann regular rings and V-rings. We call a ring R, a right atomic V-ring (or right AV-ring) if every atomic (in the sense of Zhou and Duns) right R-module is injective. We observe that a right semi-Artinian ring is a right AV-ring if and only if it is a right Σ-V-ring. Among other results, we show that every finite direct product of full matrix rings over Abelian regular rings is both a right and a left AV-ring. We also detect that there exist regular AV-rings (and hence Σ-V-rings) of infinite index (of nilpotency).

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1. Introduction

In this article, by ring, we always mean an associative ring with identity and by a right module we mean a unitary right R-module. Two right R-modules A and B are said to be *orthogonal* and denoted by $A \perp B$, if they do not have nonzero isomorphic submodules, that is, there are no nonzero submodules $A_1 \leq A$ and $B_1 \leq B$ such that $A_1 \cong B_1$. Let A and B be two R-modules, we say that A and B are parallel and denote by $A \parallel B$, if there does not exist a $0 \neq A_1 \leq A$ with $A_1 \perp B$, and also there does not exist a $0 \neq B_1 \leq B$ that $B_1 \perp A$. An R-module A is said to be *atomic* if $A \neq (0)$ and, for any $x, y \in A \setminus \{0\}$, xR and yR have isomorphic nonzero submodules. One can also say that an R-module A is atomic if and only if, for every $0 \neq B \leq A$, $B \parallel A$. Submodules and essential extensions of atomic modules are atomic. The reader is referred to [9], as the only available reference on the subject. It is easy to observe that if A is essential in B, then A and B are parallel. For example, $\mathbb{Z}$ and $\mathbb{Q}$ are parallel and both of them are atomic. A module A is called *iso-simple*[1] if every nonzero submodule of

[1]In the literature, iso-simple modules draw some attentions recently. Here and there, as the reader may very well expect, this concept, has been mentioned with different names. For example, Behboodi et al., called it "virtually-simple" (see [4] and [5]), Facchini and Nazemian called it "iso-simple" (see [11]) and even earlier it was called "self-similar" (We came across this name once

A is isomorphic to A, for example $\mathbb{Z}$ is an iso-simple $\mathbb{Z}$-module. It is clear that every iso-simple module is atomic. The converse, however is not true; $\mathbb{Q}$ is an atomic $\mathbb{Z}$-module, but it is clearly not iso-simple. Type dimension is the analogue of Goldie dimension (or uniform dimension). A module M is said to have finite type dimension n, denoted by $\text{t.dim}\, M = n$, if M contains an essential direct sum of n pairwise orthogonal atomic submodules of M. It is quite possible for a module to have infinite Goldie dimension while its type dimension is finite as we easily see when M is an infinite direct sum of a simple module. Just for this reason, type dimension is more applicable than Goldie dimension. Following [23], a module M is called p-Artinian, if for every ascending chain of submodules $M_1 \subseteq M_2 \subseteq \cdots$, there exists $n \in \mathbb{N}$ such that $M_n \parallel M_i$ for every $i \geq n$. Clearly, every atomic module is p-Artinian[2]. A wide range of concepts, such as simple modules, uniform modules, iso-simple modules, and compressible modules (they are embeddable in their nonzero submodules) are generalized in the framework of atomic modules. The main aim of this writing is to study rings over which every atomic right module is injective. We call these rings, right *AV-rings* (it is abbreviated the longer name "Atomic Villamayor rings").

In this paper, by a regular ring we always mean "regular" in the sense of von Neumann, i.e., a ring R is regular if and only if for every $a \in R$, there exists $b \in R$, such that $aba = a$. The book [14], is the standard text on regular rings and the reader is referred to this book which has a central role in the present study. Here by a right (left) duo ring, we mean a ring whose right (left) ideals are two sided. When a ring is right and left duo, we simply call it a duo ring. It is understood that all results stated for "the right case" can, with appropriate modifications, be formulated for "the left case" as well. A module M is Σ-injective, if every direct sum of copies of M is injective. In a ring R, an ideal K is called prime (resp., semi-prime) if $IJ \subseteq K$ (resp., $I^2 \subseteq K$) implies that $I \subseteq K$ or $J \subseteq K$ (resp., $I \subseteq K$), where I and J are ideals of R. A ring R is prime (resp., semi-prime) if the zero ideal, (0), is prime (resp., semi-prime). When K is a prime ideal of R, then R/K is a prime factor of R. An ideal P is said to be right primitive if there exists a simple right R-module S such that $P = \text{ann}_r(S)$. R is said to be right primitive if the ideal (0) is right primitive. When P is right primitive, then R/P is a right primitive ring. A right Kasch ring is a ring R over which every simple right

during our studies, but unfortunately we could not remember the name and the address of the reference later. We hope the author of the article will forgive us). In the present work, we adopt the name "iso-simple", which we believe is more expressive and descriptive.

[2]In [23, Example 5.5] the contrary is inadvertently stated-something that is clearly incorrect.

R-module is isomorphic with a right ideal of R. R is said to be right semi-Artinian if every nonzero right R-module contains a nonzero socle. In the literature, right semi-Artinian rings are also called right Loewy rings. In this article by $E(M)$ we always mean the injective hull of M.

2. Some basic facts on AV-rings

Villamayor rings play a central role in our discussion. We therefore begin with a brief review of V-rings, and subsequently examine the position of AV-rings among classes of rings that are closely related to this significant category.

It was I. Kaplansky, who proved that a commutative ring is regular if and only if its simple modules are injective. O. E. Villamayor studied rings over which every simple right module is injective. C. Faith named such rings V-rings after Villamayor. As far as we know, V-rings are divided into two classes. One class consists of V-domains. These are those amazing examples that were first constructed by J. Cozzens (with only one simple module up to isomorphism, see [7] and [8]) and those of Osofsky (with infinitely many non-isomorphic simple modules see [20]). The problem of finding hereditary Noetherian V-domain with only finitely many non-isomorphic simple modules has remained unsolved until not so long ago. Recently, in [2], a new class of hereditary Noetherian V-domains has been introduced whose non-isomorphic simple modules are (arbitrary) finite. These rings are constructed by using E.R. Kolchin's differential polynomials over a universal differential field (see [2], [7], [20] and [22]).

Another alternative source of Villamayor rings is to be found within the class of regular rings[3]. An early example illustrating this class is given below. Let V be an infinite dimensional right vector space over a field $\mathbb{F}$, and $Q = \mathrm{End}_{\mathbb{F}}(V)$. In general Q is not a right V-ring, though it is regular and right self-injective (see [1] for details). But Q has a subring that is a regular right V-ring. Let $J = \{x \in Q \mid \dim(xV) < \infty\}$ and $R = J + 1.\mathbb{F}$. Then R is the desired subring of Q (see [8] for detailed study of this ring). In this paper, this example is called Faith's example.

We conjecture that AV-rings are regular. Although, a partial answer to this conjecture has been given in this paper, we have not yet been able to prove it.

[3]Actually, there is another source of V-rings, using group rings. In [12, Theorem 3], the authors have shown that if $\mathbb{F}$ is either a field of char 0 with all roots of unity or a field of char p. Let G be a countable group. Then all simple $\mathbb{F}[G]$-modules are injective if and only if G is locally finite with no elements of order char $\mathbb{F}$ and G has an Abelian subgroup of finite index.

Since every simple module is atomic, obviously, every right AV-ring is a right V-ring. In fact, much more can be said. A ring R is said to be a right Σ-*V-ring* if every right homogeneous semisimple module is injective. The introduction and initial study of Σ-V-rings started in [16] and then it was followed in [3], [25] and [10]. We begin our paper with a useful result, i.e., Lemma 2.2, that will be invoked repeatedly throughout. Its proof requires a lemma known as Projection Argument.

Lemma 2.1. ([9, 2.3.3.], Projection Argument) *Let $\{M_\lambda\}_{\lambda\in\Lambda}$, be a family of right R-modules, and $0 \neq x \in E(\oplus M_\lambda)$. Then there exist $\lambda \in \Lambda$, a nonzero element $r \in R$ and $m_\lambda \in M_\lambda$ such that $0 \neq xrR \cong m_\lambda R$ with $\operatorname{ann}_r(xr) = \operatorname{ann}_r(m_\lambda)$. In other words, every nonzero submodule of $E(\oplus M_\lambda)$ contains a nonzero submodule isomorphic to a submodule of some M_λ.*

Lemma 2.2. *Let A be an atomic right R-module and Λ be a set. Then $A^{(\Lambda)}$ is atomic.*

Proof. Since submodules of atomic modules are atomic, it suffices to show that $E(A^{(\Lambda)})$ is atomic. Let $0 \neq x, y \in E(A^{(\Lambda)})$. Then by Lemma 2.1, there exist nonzero elements $r, s \in R$ and $a, b \in A$ such that $xrR \cong aR$ and $ysR \cong bR$. Since A is atomic, there exist nonzero submodules $A_1 \subseteq xrR$ and $B_1 \subseteq ysR$ such that $A_1 \cong B_1$. This implies that $E(A^{(\Lambda)})$ is atomic and hence so is $A^{(\Lambda)}$. □

Corollary 2.3. *Every homogeneous semisimple module is atomic.*

An immediate consequence of the above result is given as follows.

Proposition 2.4. *Every right AV-ring is a right Σ-V-ring.*

Proof. Note that every homogeneous semisimple module is atomic. So over a right AV-ring, every homogeneous semisimple module is injective. □

We may provisionally define *a right Σ-AV-ring* as a ring R for which every right atomic module is Σ-injective. The following proposition establishes that the class of right Σ-AV-rings coincides with that of right AV-rings, indicating that the process of "sigmatization" does not yield a genuinely new notion.

Proposition 2.5. *A ring R is a right AV-ring if and only if it is a right Σ-AV-ring.*

Proof. Let R be a right AV-ring and A be a right atomic module. It suffices to show that any direct sum of A is injective. According to Lemma 2.2, for any index set Λ, $A^{(\Lambda)}$ is atomic and hence injective because R is a right AV-ring and this completes the proof. □

A ring R is called right q.f.d relative to a module M if no cyclic right R-module contains an infinite direct sum of modules isomorphic to submodules of M. It has been observed (see [25] and its references) that if a right R-module M is Σ -injective, then R is right q.f.d relative to M.

Corollary 2.6. *If R is a right AV ring, then R is q.f.d. relative to every atomic right R-module.*

Proof. Every right atomic R-module A is Σ-injective by Proposition 2.5, therefore R is right q.f.d with respect to A. $\square$

The following result demonstrates that atomic modules over AV-rings are indeed homogeneous semisimple.

Proposition 2.7. *Let R be a right AV-ring and A be a right R-module. Then A is an atomic right R-module if and only if A is homogeneous semisimple.*

Proof. Let A be an atomic module. Then every submodule of A is a direct summand of A. This implies that A is semisimple and necessarily homogeneous due to A being atomic. The converse is evident. $\square$

Although AV-rings are Σ-V-rings, there exist Noetherian Σ-V-domains which are not AV-rings (see Example 2.11). We now aim to describe an important case in which Σ-V-rings become AV-rings. Although this condition is not necessary and sufficient, as we shall see, it accounts for many significant examples. Before we prove our next proposition, we need a useful lemma. In the sequel, by $\mathrm{soc}(M)$ we mean the socle of M, that is the direct sum of all simple submodules of M.

Lemma 2.8. *Let M be a right R-module. Then M is an atomic right R-module with nonzero socle if and only if $\mathrm{soc}(M)$ is (homogeneous) semisimple and essential.*

Proof. Suppose that M is atomic with $\mathrm{soc}(M) \neq 0$. Since $\mathrm{soc}(M)$ is also atomic, it cannot contain non-isomorphic simple modules, hence it is homogeneous. Now suppose that $\mathrm{soc}(M)$ does not intersect N, where N is a nonzero submodule of M. Since $N \parallel \mathrm{soc}(M)$, N must contain a simple submodule. Recalling that $\mathrm{soc}(M)$ contains all simple submodules of N, we get the desired contradiction. The verification of the converse is immediate. $\square$

Proposition 2.9. *Let R be a right semi-Artinian ring. Then the following are equivalent.*

(1) *R is a right AV-ring.*

(2) *R is a right Σ-V-ring.*

Proof. (1) $\Rightarrow$ (2) It is evident by Proposition 2.4.
(2) $\Rightarrow$ (1) Let R be a right Σ-V-ring and A be an atomic right R-module. Since R is right semi-Artinian, A has a nonzero socle. By Lemma 2.8, we conclude that $\mathrm{soc}(A)$ is homogeneous semisimple and essential in A. So it is injective by assumption. Hence the essentiality of socle implies that $A = \mathrm{soc}(A)$. □

It is now time to shed new light on AV-rings and their relation to Σ-V-rings through a series of illustrative examples. In particular, we will see that "the established assertion" that *right non-singular right Σ-V-rings have finite index of nilpotency* is not true.

Example 2.10. In [25, Corollary 6], it is asserted that

> **(P)** Every right nonsingular right Σ-V-ring has bounded index.

On the other hand, it is known that:

> **(Q)** A regular ring is a Σ-V-ring if and only if every prime factor ring is Artinian [16, Corollaire 2.7].

Furthermore, according to [14, Theorem 6.2], for a regular ring R, the condition that all primitive factor rings R/P are Artinian is equivalent to the condition that all prime factor rings R/P are Artinian.

Nevertheless, there exist regular rings for which every primitive factor ring is Artinian, yet the ring has infinite index. One such example is presented in the paragraph following Theorem 7.15 of [14]. Let $\mathbb{F}$ be a field, and for each $n \in \mathbb{N}$, let $R_n = M_n(\mathbb{F})$. Define R to be the $\mathbb{F}$-subalgebra of $\prod R_n$ generated by $\bigoplus R_n$ and the identity element. It is easy to verify that R has infinite index. Moreover, as shown in [14, Page 77], every primitive factor ring of R is Artinian. Hence, R is a regular (and thus nonsingular) Σ-V-ring of infinite index.

This provides a counterexample to the assertion (P) in [25]. In addition, according to Proposition 2.9, since R is semi-Artinian, it is an AV-ring. Therefore, there exist right non-singular (even regular) AV-rings of infinite index.

Example 2.11. The following examples illustrate several significant cases as well as certain borderline phenomena.

(1) There are semi-Artinian V-rings which are not Σ-V-ring, see Faith's example in the introduction of this paper.
(2) Let R be a domain. Then R is a division ring if and only if R is an AV-ring. To see this let R be an AV-ring. Since R is a domain, every right

ideal of R is atomic R-module. Hence every nonzero right ideal I is a direct summand, but R is a domain, i.e., $I = R$.

(3) We have already observed that an AV-domain is a division ring. This shows that Cozzens's examples ([6] and [8]), Osofsky's examples (see [20]) and those examples illustrated in [2] and [22], are not AV-rings. Whereas they are all Σ-V-rings. So these are in fact examples of right Noetherian Σ-V-rings that are not AV-rings.

To facilitate the classification and unified presentation of certain results established thus far, we now formulate the following proposition.

Proposition 2.12. *For a ring R, the following assertions are equivalent:*

(1) *R is a right AV-ring.*
(2) *Every atomic right R-module is injective.*
(3) *Every right R-module with finite type dimension is injective.*
(4) *Every atomic right R-module is (homogeneous) semisimple.*
(5) *Every R-module with finite type dimension is semisimple.*

In [23], the following lemma has been proved. The version given here is a little bit different from the main statement. If we replace the word "essential" with "parallel", the main statement is obtained. The authors of [23], actually proved the essentiality of the submodule. Since the article [23] is new and may not be accessible to everyone, a proof will be provided.

Lemma 2.13. [23, Proposition 5.4] *Let M be a p-Artinian module. Then M contains an essential submodule N, which is a direct sum of atomic modules.*

Proof. Let $\mathbb{A}$ be the set of all independent families of atomic submodules of M. It is clear that $\mathbb{A} \neq \emptyset$. According to Zorn's Lemma, $\mathbb{A}$ has a maximal element, say, $\mathcal{A}$. Since by our hypothesis, every submodule of M has an atomic submodule and since $\mathcal{A}$ is maximal, $N = \oplus_{A \in \mathcal{A}} A$ is essential in M. □

The following result is a counterpart of [10, Proposition 2.9].

Theorem 2.14. *Let R be a right Σ-V-ring. Then the following are equivalent.*

(1) *R is a right AV-ring.*
(2) *Every right R-module with finite type dimension is injective and (finitely generated) semisimple.*
(3) *Every p-Artinian right R-module is injective and semisimple.*

Proof. (1) $\Rightarrow$ (3) Let M be a p-Artinian module. By Lemma 2.13, M contains an essential submodule N that is a direct sum of atomic modules; that is $N = \oplus_{\lambda\in\Lambda} A_\lambda$, where each A_λ is atomic. Each A_λ is homogeneous semisimple and injective by Proposition 2.7 and our hypothesis. On the other hand, there exist only finitely many mutually orthogonal atomic modules. Consequently, the number of simple modules, up to isomorphism, is finite. This implies that $|\Lambda| < \infty$, due to M being finite type dimensional. Since each finite direct sum of injective modules is again injective, N is injective. Hence $M = N$, because of essentiality of N.
(3) $\Rightarrow$ (1) Every atomic module is, by definition, p-Artinian.
(1) $\Leftrightarrow$ (2) The verification is clear. □

Remark 2.15. We proved the above theorem in this way because we deliberately wanted to avoid this very new result that "every p-Artinian module has finite type dimension".

In Example 2.11, we have already seen that there exist Noetherian Σ-V-rings that are not AV-rings. We are now in a position to examine what happens when AV-rings become Noetherian or Kasch.

Proposition 2.16. *The following are equivalent for a ring R:*

(1) *R is semisimple Artinian.*
(2) *R is a right Noetherian right AV-ring.*
(3) *R is a right AV-ring and a right Kasch ring.*

Proof. (1) $\Rightarrow$ (3), (2) are clear.
(3) $\Rightarrow$ (1) Since every simple right R-module is projective, the proof is completed.
(2) $\Rightarrow$ (1) Let R be a right Noetherian right AV-ring. Let M be a finitely generated R-module. Then the type dimension of M is finite. This implies that M is injective due to R being a right AV-ring. Now by Osofsky's theorem, R is semisimple Artinian. □

As we have already pointed out elsewhere and corroborated by evidence and arguments such as Proposition 2.16, it seems that regularity is an intrinsic part of the nature of AV-rings. The next few results explore this connection in more detail. To prove the next result we need the following lemma.

Lemma 2.17. [13, Corollary 1.3.] *A ring R is regular if and only if R is semiprime and each prime factor ring of R is regular.*

Following Jacobson [17, Page 38], a right ideal is said to be *bounded* if it contains a nonzero two sided ideal. We say that a ring R is right *orthogonally bounded* if from each two orthogonal (nonzero) right ideals I and J, at least one of them is bounded.

Proposition 2.18. *Let R be a right orthogonally bounded right AV-ring. Then R is regular.*

Proof. Assume that R is a right AV-ring, hence it is right fully idempotent. So each factor ring of R is semiprime. Moreover, we observe that each prime factor ring of R is regular. Let P be a prime ideal of R, then $S = R/P$ is atomic. Otherwise, suppose that S has two nonzero right ideals I and J which are not parallel. This implies that there are nonzero right subideals $I_1 \subseteq I$ and $J_1 \subseteq J$ such that $I_1 \perp J_1$. Because R is right orthogonally bounded, we may suppose that either I_1 or J_1 is two sided. Hence $I_1 J_1 \subseteq I_1 \cap J_1 = 0$, due to I_1 and J_1 being orthogonal. Since S is a prime ring either $I_1 = 0$ or $J_1 = 0$, a contradiction. That is S is a right atomic right AV-ring. So S is injective. But $\mathrm{Jac}(S) = (0)$, because S is a right V-ring. Now by Utumi's theorem, $\frac{S}{Jac(S)} = S$ is regular. Finally, by Lemma 2.17, R is regular. $\square$

Corollary 2.19. *Let R be a right orthogonally bounded ring. If R is prime, then it is atomic.*

Proof. Suppose that R is not right atomic. Then there are two nonzero right ideals I and J, which are not parallel, i.e., they have two orthogonal subideals $I_1 \perp J_1$. By hypothesis, we may suppose that I_1 or J_1 are two sided. So without loss of generality suppose that I_1 is two sided, then $I_1 J_1 \subseteq I_1 \cap J_1 = 0$. Hence $I_1 = 0$ or $J_1 = 0$ due to R being prime. This provides a contradiction. So each two right ideals of R are parallel. $\square$

Although it is already known that right duo right V-rings are regular, it has not yet become clear to us whether right orthogonally bounded right V-rings (or even right Σ-V-rings) are regular. The following corollary is included simply to highlight a direct consequence of Proposition 2.18.

Corollary 2.20. *Let R be a right duo, right AV-ring. Then R is regular.*

The notion of iso-V-rings has been discussed in [10] and [21]. In the latter, however, iso-V-rings are referred to as strongly V-rings. In this paper, we adopt the terminology iso-V-rings, since right AV-rings are in fact "stronger" than what is termed strongly V-ring in [21]. The following result serves as a counterpart to [21, Proposition 3.5] whose proof has so far been implicitly indicated.

Proposition 2.21. *A ring R is a right AV-ring if and only if it is a right Σ-V-ring and all atomic right R-modules are (homogeneous) semisimple.*

3. Abelian regular rings, AV-rings and their relation

A regular ring R is said to be Abelian if idempotents of R are all central. Abelian regular rings are equivalently strongly regular. A ring R is strongly regular if for every $a \in R$, there exists $b \in R$ such that $a = a^2b$. For detailed study of Abelian regular rings, see [14, Chapter 5]. In this section, we first prove that every Abelian regular ring is a right and left AV-ring. With other results that we prove along the way, we are finally able to show that every finite direct product of full matrix rings over Abelian regular rings is a left and right AV-ring.

Generalization of localization to non-commutative algebra, has always attracted the attention of experts. The first candidate for such a generalization, was duo rings. The concept of "universal" localization has been examined and successfully generalized to duo rings in [24]. Though, universal localization of duo rings are not duo in general, in [18], it was shown that for a large class of duo rings, the universal localization is again a duo ring. Fortunately, the rings we are dealing with, that is Abelian regular rings, also belong to this class. Before we prove our next result we need the following interesting result. Following [18], a ring R is called *strongly duo* if it is a duo ring in which all ideals commute. We need a lemma from [18] and then prove our result.

Lemma 3.1. [18, Lemma 1] *A ring R is strongly duo if and only if for any two elements $x, y \in R$, there exist some elements $a, b \in R$, such that $xy = ayx = yxb$.*

A module M is said to be *t-indecomposable* if M is not a direct sum of two nonzero orthogonal submodules. Every atomic module is t-indecomposable; but the converse is not true. However, by [9, 4.3.16 Lemma] a module is atomic if and only if its injective hull is t-indecomposable. The idea of this proof was inspired by [6, Theorem 2.13].

Lemma 3.2. *Let R be an Abelian regular ring and M be a nonzero right R-module. Then the following are equivalent:*

(1) *M is homogeneous semisimple.*
(2) *M is t-indecomposable.*

Proof. (1) $\Rightarrow$ (2) is evident.
(2) $\Rightarrow$ (1) Since $M \neq (0)$, there exists a prime ideal $P \in \mathrm{Spec}(R) = \mathrm{Max}(R)$, such that the localization of M at P, i.e., $M_P \neq (0)$. Otherwise, for every prime ideal

P, $\mathrm{ann}(M) \cap R \setminus P \neq \emptyset$ and this is impossible because $\mathrm{ann}(M)$ is an intersection of a set of prime ideals due to R being a V-ring (or more generally, by Zorn's Lemma, $\mathrm{ann}(M)$ is contained in a maximal ideal P). Let $a \in P$. Then there exists an idempotent e such that $aR = eR$. Since $M_P \neq (0)$ and $(1-e) \notin P$, we have $M(1-e) \neq (0)$, therefore $M = Me \oplus M(1-e)$. We note that by Lemma 3.1, Me and $M(1-e)$ are (right) submodules of M. We also see that $Me \perp M(1-e)$. Suppose not, therefore there exist nonzero submodules $N \leq Me$ and $K \leq M(1-e)$ such that $N \cong K$. Therefore $\mathrm{ann}(N) = \mathrm{ann}(K)$, which is a contradiction because $K(1-e) = K$ and $N(1-e) = (0)$ because $N = Ne$. So $Me \perp M(1-e)$. Since M is t-indecomposable, $Me = (0)$, (i.e., $M(1-e) = M$). So M can be considered as an R/P-module. But R/P is a division ring by [14, Theorem 3.2 (b)], so M is a homogeneous semisimple R/P-module. Since R is regular, M is also homogeneous semisimple R-module. □

Corollary 3.3. *Let R be an Abelian regular ring and A be a nonzero R-module. Then the following assertions are equivalent.*

(1) *A is homogeneous semisimple.*
(2) *A is atomic.*
(3) *A is t-indecomposable.*

Recall that Abelian regular rings are Σ-V-rings because their primitive factor rings are division rings (and hence Artinian) (see [14, Chapter 5]).

Theorem 3.4. *Suppose that R is a duo ring. Then R is (Abelian) regular if and only if R is an AV-ring.*

Proof. ($\Leftarrow$) Every AV-ring is by definition a V-ring and duo V-rings are regular. ($\Rightarrow$) Since R is Abelian regular, every atomic R-module is homogeneous semisimple by Corollary 3.3. So R is injective as R is Σ-V-ring. Hence R is an AV-ring. □

It is worth mentioning that the assumption "duo ring" is not needed in the direction $\Rightarrow$ of the above proof.

Corollary 3.5. *Let R be a commutative ring. Then R is regular if and only if R is an AV-ring.*

We now prepare the groundwork for a generalization of Theorem 3.4. Having shown that every Abelian regular ring is an AV-ring, we proceed to demonstrate that finite direct products of matrices over Abelian regular rings are AV-rings as well. To do this, we need to show that being an AV-ring, is a Morita property.

Lemma 3.6. *Let R and S be equivalent rings via an equivalence $F\colon R$-$MOD \longrightarrow S$-MOD. The following statements hold.*

(1) *$A \perp B$ if and only if $F(A) \perp F(B)$.*
(2) *$A \parallel B$ if and only if $F(A) \parallel F(B)$.*

Proof. See [19, Proposition 1.2]. □

Proposition 3.7. *Let F be an equivalence between two categories R-MOD and S-MOD. If R is a right AV-ring, then S is a right AV-ring. In particular, $\mathrm{M}_n(R)$ is a right AV-ring, where R is a right AV-ring.*

Proof. Let A be an atomic module in R-MOD, then by Lemma 3.6, $F(A)$ is also atomic in S-MOD. It is well-known that injectivity is preserved by equivalences. □

Proposition 3.8. *Let R be a right AV-ring, and I be a two-sided ideal of R. Then R/I is a right AV-ring.*

Proof. Let A be a right R-module and $I \subseteq \mathrm{ann}(A)$. Then A is a right R/I-module. Suppose that A is atomic as a right module over R/I. We observe that A is atomic as an R-module too. To show this suppose that x, y are two nonzero elements of A_R. We know that xR and yR have isomorphic submodules, say, X and Y. Then X and Y are also isomorphic as R/I-modules. Since A is an injective R-module, it is injective as an R/I-module. □

Proposition 3.9. *Any finite direct product of right AV-rings is a right AV-ring.*

Proof. Let $R \cong R_1 \times \cdots \times R_n$, where R_i's are right AV-rings. Moreover, suppose that M is an atomic right R-module. For every $j \in \{1, \cdots, n\}$, define $T_j = \prod_{i=1}^n L_i$, where $L_j = R_j$ and $L_i = (0)$ for $i \neq j$. Then $M = M_1 \oplus \cdots \oplus M_n$, where $M_i = MT_i$, for all $i \in \{1, \cdots, n\}$. Every M_i is atomic as an R_i-module, according to Proposition 3.8. So each M_i is an injective R_i-module. This implies that each M_i is an injective R-module, because R is fully right idempotent (in fact R is a right V-ring). Finally M is an injective R-module. □

Remark 3.10. In Proposition 3.9, we cannot drop the "finite" condition. Goursaud and Jeremy ([15, Proposition 3.9]) have shown that $\prod_{n=1}^{\infty} \mathrm{M}_n(\mathbb{F}_n)$ is neither a right nor a left V-ring, where $\mathbb{F}_n$'s are fields. This implies that "infinite" direct products of matrix rings over Abelian regular rings are not even V-rings let alone AV-rings.

We are now ready to formulate and establish the central theorem of this section.

Theorem 3.11. *Let $R \cong \prod_{i=1}^{k} \mathrm{M}_{n_i}(R_i)$, where R_i's are all Abelian regular rings. Then R is a right (and left) AV-ring.*

Proof. According to Proposition 3.7, we have already observed that when R is a right AV-ring, $\mathrm{M}_n(R)$ is a right AV-ring. Also by Theorem 3.4, every Abelian regular ring is an AV-ring. So when R is Abelian regular, $\mathrm{M}_n(R)$ is a right and left AV-ring. Now by Proposition 3.9, R is a right (and left) AV-ring. $\square$

Corollary 3.12. *Let R be a regular and right self-injective ring, which has bounded index or (equivalently) primitive factor rings of R are Artinian. Then R is a right and left AV-ring.*

Proof. According to [14, Theorem 7.20], R is isomorphic to a finite direct product of full matrix rings over Abelian regular rings. Now by Theorem 3.11, R is a right (and left) AV-ring. $\square$

Corollary 3.13. *A ring R is a right self-injective, right AV-ring if and only if R is isomorphic to a finite direct product of matrix rings over right self-injective Abelian regular rings.*

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