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ON THE EQUATION $a^2x = a$ IN UNITAL RINGS

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ABSTRACT. In an arbitrary ring, the equation in the title defines the left strongly regular elements. Elements that are both left and right strongly regular are simply called strongly regular. If all elements of a ring are left strongly regular, then they are, in fact, strongly regular and this is the definition of strongly regular rings.

We provide a characterization of when a left strongly regular element is indeed strongly regular, based on an intrinsic condition. While we show that it is not possible to give 2×2 non-examples over $\mathbb{Z}$ or in certain matrix rings over $\mathbb{Z}_n$ for $n \in \{8, 9, 16\}$, we present two examples by George Bergman: a left strongly regular element that is not regular and a regular, left strongly regular element that is not strongly regular.

Further, we prove results for left strongly regular (square) matrices over various types of rings and propose a conjecture, strongly supported by computational evidence: over commutative rings, left strongly regular matrices are strongly regular.

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1. Introduction

An element a of a ring R is called *left* (respectively, *right*) *strongly regular* if $a \in a^2R$ (respectively, $a \in Ra^2$). It is called *strongly regular* if it is both left and right strongly regular. Notably, if a is strongly regular, meaning $a = a^2c = da^2$ for some $c, d \in R$, then one can choose $b = ac^2$ and by an argument of Azumaya [2, Lemma 1], it follows that $a = a^2b = ba^2$. Therefore, $a \in R$ is strongly regular if and only if there exists $b \in R$ such that $a = a^2b = ba^2$. The properties of strongly regular elements are summarized in [8].

The concept of strongly (von Neumann) regular rings dates back to Arens and Kaplansky in 1948 (see [1]), where such rings were defined as those in which, for every element $a \in R$, there exists $b \in R$ such that $a = a^2b$ (i.e., $a \in a^2R$). In the same paper, a *biregular* ring was defined as a ring in which every principal two-sided ideal is generated by a central idempotent. It was shown that this definition is actually left-right symmetric, and that strong regularity implies both regularity and biregularity.

In fact, in the seminal Arens-Kaplansky paper, using a global argument involving prime ideals and semiprime rings, the authors demonstrate that if all elements in a ring are left strongly regular, then the ring is reduced, Abelian, Dedekind finite, and all elements are strongly regular.

The original goal of this note was to provide examples of elements a in certain rings such that there exists $b \in R$ with $a^2 b = a$, but a is not regular, $ab \neq ba$, and neither of these elements are idempotents. Initially, the ring $\mathbb{M}_2(\mathbb{Z})$ (which is not regular) seemed like a promising candidate, but we discovered that such examples cannot be found there (see Theorem 3.4). Further attempts to find such examples in $\mathbb{M}_2(\mathbb{Z}_n)$ for $n \in \{8, 9, 16\}$ were also unsuccessful (see Proposition 3.9).

General results on left strongly regular elements are presented in Section 2. According to the definition, if $a^2 b = a$, to verify whether a is strongly regular, we must find an element $c \in R$ such that $ca^2 = a$. For a left strongly regular element a with $a = a^2 b$, we provide an intrinsic characterization for a to be strongly regular: $ab^2 a = ab$ (see Theorem 2.3).

The left strongly regular matrices are discussed in Section 3. We characterize the left strongly regular 2×2 matrices over commutative Bézout domains (see Theorem 3.2) and we show that the only (left) strongly regular 2×2 integral matrices are the units, the idempotents and the minus idempotents (see Theorem 3.4).

Among other results, we state a conjecture - strongly supported by computer verifications - that for a matrix A over a (say, commutative) ring R, if there exists a matrix B such that $A^2 B = A$, then there also exists a matrix C such that $A^2 C = A$ and $AC = CA$.

Finally, Section 4 provides several examples, including two constructed by George Bergman as valuable applications of his Diamond Lemma (see [3]), with his kind permission. One example (GB1) features a left strongly regular element that is not regular, while the other (GB2) showcases a regular, left strongly regular element that is not strongly regular.

For any positive integer n, E_{ij} denotes the $n \times n$ matrix with all zero entries except for the (i, j) entry, which equals 1. For a ring R, $U(R)$ denotes the set of units of R, and $N(R)$ denotes the set of nilpotents of R. $GL_n(R)$ denotes the general linear group of all invertible $n \times n$ matrices over R, and for a finite set

X, $|X|$ denotes the number of elements in X. An element a of a ring R is called *periodic* if $a^n = a$ for some positive integer $n \geq 2$. If $n = 3$ such an element is called a *tripotent*.

2. General results and characterizations

We start with some simple examples.

Lemma 2.1. (i) *If a is (left or right) strongly regular, so is $-a$.*

(ii) *Periodic elements and minus periodic elements are strongly regular. In particular, tripotents, idempotents and minus idempotents, all are strongly regular (see* [5]*).*

The next result (directly) shows how from the definition of strongly regular elements, follow the conclusions (for rings) from [1].

Lemma 2.2. (i) *If $a = a^2b = ba^2$, then $ab = ba$ (and $a = aba$). Conversely, $ab = ba$ implies $a^2b = ba^2$.*

(ii) *If $a = a^2b$ and $ab = ba$, then ab is an idempotent and a is (strongly) regular.*

(iii) *Conversely, if a is regular ($= aba$), then ab and ba are idempotents, which may not be equal. Moreover, a^2b and a may not be equal.*

Proof. For (i), multiplying $a = ba^2$ by b on the right gives $ab = b(a^2b) = ba$. The converse is obvious. For (ii), $(ab)^2 = a(ba)b = \underline{a(ab)}b = ab$. Finally, $a(ba) = a(ab) = a$. (iii) The positive statement is obvious. Nonexamples are given in the last section (see Subsection 4.2, example 1 and Subsection 4.4, example 1). □

As our first main result, we prove an intrinsic characterization of left strongly regular elements.

Theorem 2.3. *In an arbitrary ring R suppose $a^2b = a$ for some $a, b \in R$ (i.e., a is left strongly regular). Then a is strongly regular if and only if $ab^2a = ab$.*

Proof. Suppose $a^2b = a$ and consider $c = ab^2$. Then

(i) $a^2c = a^2(ab^2) = a(a^2b)b = a^2b = a$;

(ii) $ac = a^2b^2 = (a^2b)b = ab$.

Since $ca = ab^2a$, according to (ii), $ca = ac$ is equivalent to $ab = ab^2a$.

Moreover $ca^2 = a$ is equivalent to $ca = ac$. Indeed, one way ($\Rightarrow$) follows by right multiplication with c, the other way ($\Leftarrow$) follows by right multiplication with a. Both ways we use (i) (i.e., $a^2c = a$). Therefore, if $ab^2a = ab$, it follows $a = a^2c = ca^2$ and so a is strongly regular.

Conversely, suppose a is strongly regular and $a = a^2d = da^2$ for some $d \in R$. Moreover, suppose $a^2b = a$.

First, by Lemma 2.2, $ad = da$ and $a = ada$ (so a is regular and $ad = da$ is idempotent). Secondly, $ab = \underline{da^2}b = da = ad$ and so $aba = ada$. Finally $ab^2a = (ab)ba = \underline{ad}ba = d(aba) = d(ada) = da = ad = ab$. □

Remark 2.4. (1) If $a^2b = a$, then the condition $ba^2 = a$ is obviously sufficient for a to be strongly regular. Indeed, also directly, $ba^2 = a$ implies $ab = ab^2a$, as follows. According to the previous lemma, $a^2b = a = ba^2$ implies $ab = ba$ and $a = aba$. Hence $ab^2a = (ab)^2 = ab$. Examples show, this condition is not necessary (see Subsection 4.2, example 1). Moreover, also directly, if $a^2b = a$, then $ab^2a = ab$ implies by left multiplication with a that $aba = a$ (a is regular).

(2) The condition $ab^2a = ab$ for some a, b, does not imply $a^2b = a$. For example, take $A = \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 11 & 2 \\ 0 & 0 \end{bmatrix}$ over $\mathbb{Z}_{12}$. Then $AB^2A = \begin{bmatrix} 9 & 6 \\ 0 & 0 \end{bmatrix} = AB$ but $A^2B = \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix} \neq A$. Actually, since $9y+6w = 2$ has no solutions over $\mathbb{Z}_{12}$, A is not left strongly regular in $\mathbb{M}_2(\mathbb{Z}_{12})$. However, it is unit-regular: $\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 11 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 0 & 0 \end{bmatrix}$.

(3) A symmetric characterization holds for the right strongly regular elements which are strongly regular.

For the sake of completeness, we record some special (well-known) results on left strongly regular elements.

Proposition 2.5. *No nonzero nilpotent element is left strongly regular.*

Proof. From $a^2b = a$, we deduce $a^3b^2 = a^2b = a$ and also $a^nb^{n-1} = \cdots = a$. Hence if $a^n = 0$, then $a = 0$. □

Proposition 2.6. *In any reduced ring, left strongly regular elements are unit-regular.*

Proof. Suppose $a = a^2b$. Then $(a - aba)^2 = a^2 + aba^2ba - a^2ba - aba^2 = a^2 + aba^2 - a^2 - aba^2 = 0$, so $a = aba$, since R is reduced (i.e., has no nonzero nilpotents). Since reduced rings are Abelian (i.e., have only central idempotents), $e := ab$ is a central idempotent.

Consider $u = eb + \overline{e}$, $v = ea + \overline{e}$, where we denote by $\overline{e} = 1 - e$, the complementary idempotent. Then (since $ea = ae$) $vu = eaeb + \overline{e}^2 = e^2ab + \overline{e} = e + \overline{e} = 1$. Since reduced rings are also Dedekind finite (i.e., one-sided invertible elements are two-sided invertible), $uv = 1$ and so $u = v^{-1} \in U(R)$. Finally, from $a = a^2b = ae$, we get $a\bar{e} = 0$, so now $aua = a(eb + \bar{e})a = aba = a$, as desired. □

Proposition 2.7. *Let R be a Dedekind finite ring and $a \in R$ such that $a^2b = a = aba$ for some $b \in R$. Then a is unit-regular.*

Proof (See Ehrlich [7]). Denote by e the idempotent ab and consider its complementary idempotent $\bar{e}$. Then

(1) $eaeebe = ab\underline{aa}babbab = ab\underline{aab}bab = ababab = eee = e$;
(2) $(eae + \bar{e})(ebe + \bar{e}) = e + (\bar{e})^2 = 1$, that is, $u = eae + \bar{e}$ has a right inverse; as the ring is Dedekind finite, u is a unit;
(3) $a = aba = ea = ea^2b = eae$;
(4) $eu = e(eae + \bar{e}) = eae = a$, whence $e = au^{-1}$ and $a = ea = au^{-1}a$. □

Remark 2.8. (1) We can remove the Dedekind finite hypothesis if

$$ebeeae = abbabab\underline{aab} = (ab)(ba)^3 = e^4 = e,$$

for example if $e = ab = ba$, or else, if $ab^2a = ab$, the condition in the above characterization of left strongly regular elements that are strongly regular.

(2) We could wonder whether a regular, left strongly regular element actually is strongly regular, or even more restricted: if $a^2b = a = aba$, must a be strongly regular? George Bergman's example answers this question in the negative (see **GB2**, last section).

3. General results on left strongly regular matrices

In this section we consider *nonzero* matrices over commutative rings. We first gather some easy properties related to the equality $A^2B = A$, some of which hold for $n \times n$ matrices, for any positive integer n, others only for 2×2 matrices. To simplify the writing, the elements of a finite subset of a ring are called *coprime* if there is a linear combination of these which equals 1.

Lemma 3.1. (i) *If A is left strongly regular, so is $-A$.*
(ii) *No nonzero nilpotent matrix is left strongly regular.*
(iii) *For $n = 2$, if $\det(A) = 0$, $A^2B = A$ is equivalent to $Tr(A)AB = A$.*

Over commutative domains

(iv) *if A is left strongly regular, then A is a unit or $\det(A) = 0$.*
(v) *if A is left strongly regular and $n = 2$, then the entries of A are coprime, and $Tr(A)$ is a unit.*

(vi) *if A is left strongly regular and $n = 2$, then AB is idempotent.*

Proof. (i) See Lemma 2.1, in general.

(ii) See Proposition 2.5, in general.

(iii) We just use Cayley-Hamilton Theorem for $A^2 = Tr(A)A$.

(iv) Taking determinants, $A^2B = A$ implies $\det^2(A)$ divides $\det(A)$. As of course $\det(A) \mid \det^2(A)$, it follows that $\det(A)$ and $\det^2(A)$ are associated in divisibility (i.e., $\det^2(A) = \det(A)u$ for some unit u). By cancellation, if $\det(A) \neq 0$, then $\det(A)$ is a unit. Therefore the possible cases are $\det(A) = 0$ or else A is a unit. If A is a unit, then $B = A^{-1}$, obviously unique and so the pairs (A, B) are just all the pairs (A, A^{-1}).

(v) By (iii), start with $Tr(A)AB = A$ and take the traces on both sides. We obtain $Tr(A)(Tr(AB) - 1) = 0$. Restricting to 2×2 matrices, as already noticed, if $A \neq 0_2$, $Tr(A) \neq 0$ since otherwise (by Cayley-Hamilton Theorem, $A^2 = 0_2$) A is nilpotent and does not satisfy $A^2B = A$ (by (ii)). Therefore denoting $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, by cancellation, we get $Tr(AB) = ax + bz + cy + dw = 1$. Equivalently, A must have coprime entries.

Back to $Tr(A)AB = A$, it follows that $Tr(A)$ divides the entries of A, whence $Tr(A)$ is a unit.

(vi) As noticed in (v), $Tr(AB) = 1$. Since $\det(AB) = \det(A)\det(B) = 0$, it follows (again by Cayley-Hamilton Theorem) that AB is idempotent. □

Among the zero determinant matrices, the idempotent matrices A are obviously paired (for $A^2B = A$) with $B = I_n$ or $B = A$, but, as examples show (see Subsection 4.2, example 2), not only with these.

A commutative domain is called *Bézout* if the sum of two principal ideals is also a principal ideal. This means that Bézout's identity holds for every pair of elements, and that every finitely generated ideal is principal. Bézout domains are *GCD* (greatest common divisors exist).

As our second main result, over Bézout (commutative) domains we characterize the left strongly regular matrices 2×2 matrices.

Theorem 3.2. *Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{M}_2(D)$ for a Bézout domain D. Then A is left strongly regular if and only if A is a unit or else $\det(A) = 0$ and $\gcd(a^2 + bc, b(a+d))$ divides a and b.*

Proof. As already mentioned (see Lemma 3.1(iv)), the left strongly regular matrices are the invertible matrices A characterized by $\det(A) \in U(D)$ paired (only) with $B = A^{-1}$, and some of the singular matrices A (with $\det(A) = 0$). Searching for all matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc = 0$ such that $A^2B = A$ for some $B = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ reduces to the following two linear systems:

$$\begin{cases} (a^2+bc)x & + & b(a+d)z & = & a \\ c(a+d)x & + & (bc+d^2)z & = & c \end{cases}$$

and

$$\begin{cases} (a^2+bc)y & + & b(a+d)w & = & b \\ c(a+d)y & + & (bc+d^2)w & = & d. \end{cases}$$

Both systems have the same system matrix which is $M = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix}$. Clearly, $\det(M) = \det^2(A) = 0$ for the singular matrices we deal with, that is, the system matrices have rank 1 (we suppose $A \neq 0_2$).

It is well-known (see [4]) that a necessary condition for such linear systems to be solvable is that the (corresponding) augmented matrices also have rank 1. These are $\begin{bmatrix} a^2+bc & b(a+d) & a \\ c(a+d) & bc+d^2 & c \end{bmatrix}$ and $\begin{bmatrix} a^2+bc & b(a+d) & b \\ c(a+d) & bc+d^2 & d \end{bmatrix}$ and so for the solvability of the systems, the vanishing of 4 minors is necessary:
$\det \begin{bmatrix} a^2+bc & a \\ c(a+d) & c \end{bmatrix}$, $\det \begin{bmatrix} b(a+d) & a \\ bc+d^2 & c \end{bmatrix}$, $\det \begin{bmatrix} a^2+bc & b \\ c(a+d) & d \end{bmatrix}$, $\det \begin{bmatrix} b(a+d) & b \\ bc+d^2 & d \end{bmatrix}$.
Since, if $ad - bc = 0$, all these minors are also zero, the augmented matrices have also rank 1 and for solutions it suffices to choose the equations (say) $(a^2+bc)x + b(a+d)z = a$ for x and z, and $(a^2+bc)y + b(a+d)w = b$ for y and w. Over Bézout domains, these are linear Diophantine equations, which, as it is well-known, are solvable if and only if $\gcd(a^2+bc, b(a+d))$ divides a and b. □

A similar result ($\gcd(a^2+bc, c(a+d))$ divides a and c) holds for right strongly regular matrices.

Example 3.3. (over integers) (1) Let n be any nonzero integer. The matrix $A = \begin{bmatrix} 1 & n \\ n & n^2 \end{bmatrix}$ has $\det(A) = 0$ but $\gcd(a^2+bc, b(a+d)) = 1 + n^2$ does not divide 1 nor n. Hence the equation $A^2B = A$ has no integer solutions and so A is not left strongly regular. However, it is regular (take for example $B = E_{11}$) and even unit-regular (take for example $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$).

Actually, as $\det(A) = 0$, $ABA = Tr(AB)A$ (see [6]) and (with the notations of the previous proof) $Tr(AB) = x + nz + ny + n^2z$, whence $Tr(AB) = 1$ suffices for the (unit-)regularity of A ($x = 1, y = z = w = 0$ for the first B, $x = y = 1, z = -1, w = 0$ for the second B).

(2) As a positive example, take the idempotent $A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$. Here $\gcd(a^2+bc, b(a+d)) = 2$ divides both a and b. The corresponding Diophantine equations are now $2x + 2z = 2$ and $2y + 2w = 2$, with infinitely many solutions: $B = \begin{bmatrix} x & y \\ 1-x & 1-y \end{bmatrix}$ for any integers x, y.

As already mentioned in the Abstract, surprisingly, it turns out that, for 2×2 integral matrices, (only) the equality $A^2B = A$ suffices in order to determine the strongly regular matrices. More, these are precisely the units, the idempotents or else the minus idempotents and this is our third main result.

Theorem 3.4. *For any matrix $A \in \mathbb{M}_2(\mathbb{Z})$, the following conditions are equivalent.*

(i) *A is left strongly regular, i.e., there exists $B \in \mathbb{M}_2(\mathbb{Z})$ such that $A^2B = A$;*
(ii) *A is strongly regular, i.e., there exists $B \in \mathbb{M}_2(\mathbb{Z})$ such that $A^2B = A = BA^2$;*
(iii) *A is a unit or an idempotent or a minus idempotent.*

Proof. As (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious, it just remains to show that (i) $\Rightarrow$ (iii). According to Lemma 3.1(ii), it remains to show that if $\det(A) = 0$ and $A^2B = A$ for some B, then $A = \pm E$ for some nontrivial idempotent $E = E^2$ (i.e., not 0_2 nor I_2). Equivalently, it suffices to show that $Tr(A) \in \{\pm 1\}$. This follows from Lemma 3.1(v). □

We were not able to find a reference for the following two propositions.

Proposition 3.5. *Over any commutative ring R, let A be a left strongly regular 2×2 matrix. If $Tr(A)$ is a unit, then A is unit-regular.*

Proof. By Lemma 3.1(iii), $A^2B = A$ is equivalent to $Tr(A)AB = A$ and so $AB = Tr^{-1}(A)A$ (is independent of B). Hence $A = A^2B = Tr^{-1}(A)A^2 = ABA$ (i.e., A is regular). Now the conclusion follows from Proposition 2.7, since matrix rings over commutative rings are Dedekind finite. □

Recall the following well-known result: *all nonunits of a ring R are nilpotents (i.e., $R = U(R) \cup N(R)$) if and only if R is a local ring with nil Jacobson radical.* Then we can prove another result of the same sort.

Proposition 3.6. *Let R be a commutative local ring with nil Jacobson radical and $A \in \mathbb{M}_2(R)$. If A is left strongly regular, then A is unit-regular.*

Proof. Suppose $A^2B = A$ for some $A, B \in \mathbb{M}_2(R)$. The ring being local with nil radical, $\det(A)$ is a unit or is nilpotent. If $\det(A)$ is a unit, so is A. Hence we assume $\det(A) \in N(R)$ and let n be its degree of nilpotence. Taking determinants in $A^2B = A$, we get $\det^2(A)$ divides $\det(A)$. Hence $d := \det(A) = 0$ (indeed, $d = d^2r$ implies $d^{n-1} = d^n r$ for any positive integer n), so we have to show that A is unit-regular just in this case. We can suppose $A \neq 0_2$.

As $\det(A) = 0$, by Cayley-Hamilton Theorem, $A^2 = Tr(A)A$ and so $Tr(A)AB = A$. Taking traces we get $Tr(A)(Tr(AB) - 1) = 0$.

Case 1. $Tr(A)$ is a unit. Then by Proposition 3.5, A is unit-regular.

Case 2. $Tr(A)$ is nilpotent. As $A^2 = Tr(A)A$ implies $A^{n+1} = Tr^n(A)A$, it follows that A is nilpotent too, a contradiction (see Proposition 2.5). □

Strongly supported by computer, *at least* for matrices over $\mathbb{Z}_n$ with $n \in \{8, 9, 16\}$ and partly over $\mathbb{Z}$ (but completely verified by Theorem 3.4) we state the following

Conjecture 3.7. *Let A be a square matrix over a commutative ring R. If A is left strongly regular, then A is strongly regular.*

According to Lemma 2.2(ii), the following claim would suffice for a proof: if there exists $B \in \mathbb{M}_2(R)$ such that $A^2B = A$, then there exists $C \in \mathbb{M}_2(R)$ such that $A^2C = A$ and $AC = CA$.

Hint. As seen in the proof of Theorem 2.3, if $A^2B = A$, a possible candidate for C is AB^2 (which equals A for tripotents - including the $\pm$ idempotents).

Remark 3.8. As a final example in our paper shows (see **GB1**), the conjecture fails in general (a left strongly regular element may not be regular). However it holds (in general) for units u or tripotents t (including $\pm$ idempotents). Indeed,

$u^2b = u \Rightarrow b = u^{-1} \Rightarrow ub^2u = 1 = ub$;

$t^3 = t$ and $t^2b = t$ imply $tb = t^2 \Rightarrow tb^2t = t^2bt = t^2 = tb$.

Verifications by computer (see some details below) show that for the commutative local rings $R \in \{\mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_{16}\}$, any left strongly regular matrix of $\mathbb{M}_2(R)$, actually is strongly regular, and left strongly regular matrices turned out to be units or tripotents. Summarizing

Proposition 3.9. *Over $\mathbb{Z}_8$, the left strongly regular 2×2 matrices are all the invertible matrices (that is, 1536 including 176 invertible matrices of order two which are also tripotents) or else 384 tripotents with zero determinant (excluding the 176 invertible matrices of order two).*

Indeed, there are $|GL_2(\mathbb{Z}_8)| = (2^2-1)(2^2-2)4^4 = 1536$ invertible 2×2 matrices over $\mathbb{Z}_8$. The remaining 384 tripotents A have:
4096 B-s (all the matrices over $\mathbb{Z}_8$, that is 8^4) if $A = 0_2$, and 64 B-s each, if A is any zero determinant tripotent.

The results over $\mathbb{Z}_9$ or $\mathbb{Z}_{16}$ are similar.

We chose only local rings $\mathbb{Z}_n$, as every $\mathbb{Z}_n$ is a finite direct sum of local rings, matrix rings over finite direct sums of rings are isomorphic to finite direct sums of matrix rings and an element of a finite direct sum of rings is left strongly regular if and only if so are all its components.

It would be interesting to see to what extent this fact (the only left strongly regular matrices are units or tripotents) can be generalized, at least for 2×2 matrices over (more) general rings. A partial attempt over commutative local rings was made in the above Proposition 3.6.

4. Examples of left strongly regular matrices

4.1. Special integral matrices.

(i) *Diagonal* integral matrices $A = diag(a, b)$ paired with $B \in \mathbb{M}_2(\mathbb{Z})$. There are 4 units, namely $\pm I_2$ and $A = \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. As for zero determinant matrices, $a = 0$ or/and $b = 0$ so these are of form 0_2, aE_{11}, bE_{22}. Again $a^2 \mid a$ or $b^2 \mid b$, respectively, whence $a \in \{\pm 1\}$ or else $b \in \{\pm 1\}$ (that is $A = \pm E_{11}$ or $A = \pm E_{22}$). The corresponding matrices are paired with $B = \begin{bmatrix} \pm 1 & 0 \\ * & * \end{bmatrix}$ or with $B = \begin{bmatrix} * & * \\ 0 & \pm 1 \end{bmatrix}$ respectively, with any integers $*$.

(ii) *Upper triangular* matrices $A = diag(a, c) + bE_{12} = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ paired with $B \in \mathbb{M}_2(\mathbb{Z})$. There are infinitely many units with diagonal as in (i) and arbitrary $b \in \mathbb{Z}$, that is, $\pm I_2 + bE_{12}$ or else $\pm \begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$. As in case (i), units are only paired with their inverses.

As for zero determinant matrices, excepting the nilpotent matrices bE_{12} (which as already mentioned, if nonzero, cannot be left strongly regular), $a = 0$ or $c = 0$ so these are of form $aE_{11} + bE_{12}$ (with $a \neq 0$) or else $bE_{12} + cE_{22}$ (with $c \neq 0$).

4.2. Examples over $\mathbb{Z}_6$.

Since $\mathbb{Z}_6$ is unit-regular, $\mathbb{M}_n(\mathbb{Z}_6)$ is regular, so we don't expect to have not regular examples.

1) Consider $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ over $\mathbb{Z}_6$, which is a tripotent (so strongly regular) with coprime entries.

As $\det(A) = 0$, by Cayley-Hamilton Theorem, it follows that $A^2B = A$ is equivalent to $Tr(A)AB = A$ and $BA^2 = A$ is equivalent to $Tr(A)BA = A$. Since $Tr(A) = 5$ is an order 2 unit, i.e., $5^2 = 1$, $AB = Tr^{-1}(A)A = \begin{bmatrix} 5 & 4 \\ 4 & 2 \end{bmatrix} = 5A$ for all B. Actually, by computer, there exist 36 matrices B satisfying $A^2B = A$ and only 6 of these satisfy $BA = AB = Tr^{-1}(A)A$, or equivalently $BA^2 = A$. Therefore, for any of the other 30 matrices B, we have $BA \neq AB$ and $BA^2 \neq A$. As a sample, for $B = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$, $BA = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq AB$ and $BA^2 = \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix} \neq A$.

Remark 4.1. (i) This supports our Conjecture 3.7 for matrices: if there exists B such that $A^2B = A$, then there also exists C such $A^2C = A$ **and** $AC = CA$. As observed in the proof of Theorem 2.3, the possible candidate for C is AB^2. Indeed, for the previous example, $C = AB^2 = (AB)B = 5AB = 5^2A = A$.

(ii) It was already mentioned that over integral domains, $Tr(A)(Tr(AB) - 1) = 0$ for $A \neq 0_2$ implies $Tr(AB) = 1$ and so (as $\det(AB) = 0$), AB is idempotent. In order to have $Tr(AB) \neq 1$, zero divisors are needed. That's why we chose $\mathbb{Z}_6$. However, all AB and BA, in the previous example have trace $= 1$, so are (possibly different) idempotents. This also follows since for every regular element, $a = aba$, both ab and ba are idempotents.

(iii) If $A^2B = A$ and $AB = A^2$ (independent of B), then $A^3 = A$, that is, A is tripotent, so strongly regular.

2) Consider $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ where 2 is a tripotent in $\mathbb{Z}_6$ and $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ is a tripotent in $\mathbb{M}_2(\mathbb{Z}_6)$ (so A is also a tripotent in $\mathbb{M}_2(\mathbb{Z}_6)$).

Hence A is strongly regular, but with **not** coprime entries and $Tr(A) = 4$ is a nontrivial idempotent (so not a unit) of $\mathbb{Z}_6$. Surprisingly, A turns out to be an idempotent (by computation $A^2 = A$). Again, as $\det(A) = 0$, from Cayley-Hamilton's Theorem, $A^2 = Tr(A)A = 4A$, so $A^2B = A$ is equivalent to $4AB = A$. Now (again by computer) for all the 144 Bs, we have $AB = A$ (again independent of B) and only three possible BA (again $AB \neq BA$ may happen): $BA = A$ or $BA = \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix}$ or $BA = \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix}$ (each in $144 : 3 = 48$ cases). Actually, if $A^2B = A$ and $AB = A$, then $A^2 = A$, A is idempotent.

4.3. Upper triangular matrix rings.

It is well-known that the ring of upper triangular $n \times n$ matrices $\mathbb{T}_n(R)$, for any $n \geq 2$, is not regular (it contains a nilpotent ideal).

For $R \in \{\mathbb{Z}_8, \mathbb{Z}_9, \mathbb{Z}_{16}\}$ and $n = 2$, the computer has verified that if $A^2B = A$, then $AB^2A = AB$ for $A, B \in \mathbb{T}_2(R)$, that is, according to Theorem 2.3, left strongly regular matrices are strongly regular. This way, again the Conjecture 3.7 is supported by computer verifications.

The situation is similar to Proposition 3.9. Over $\mathbb{Z}_8$, there are 193 regular upper triangular matrices. These are precisely 129 tripotents (incl. the units of order two) and 64 units (not of order two).

4.4. Some special examples. In this subsection, we provide examples of some left strongly regular elements which satisfy only some or none of the properties of strongly regular elements.

Over $\mathbb{Z}_{12}$ take $A = diag(3, 4) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ and $B = diag(3, 1) + 3E_{21} = \begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$. Then $A^2B = diag(9, 4)B = diag(3, 4) = A$ and $AB = A^2 = diag(9, 4) = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \neq \begin{bmatrix} 9 & 0 \\ 9 & 4 \end{bmatrix} = diag(9, 4) + 9E_{21} = BA$.

Both AB, BA are different idempotents, and none is central (these are not scalar, i.e., of form nI_2). However, $A = AUA$ for unit $U = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$, so A is unit-regular. Note that A is a tripotent of $\mathbb{M}_2(\mathbb{Z}_{12})$, so it is strongly regular.

Supporting our Conjecture, there exists C such that $A^2C = CA^2 = A$ and obviously $C = A$.

Remark 4.2. Observe that in general (see Lemma 2.2), if $a^2b = a$ and $ab \neq ba$, then $ba^2 \neq a$. However this does not guarantee that a is not regular (see Subsection 4.2, example 1 or the previous example).

In closing, we present

GB1. George Bergman's example of *left strongly regular element that is not regular.*

Let R be the algebra over any commutative ring k (e.g., a field) presented by two generators a and b, and the relation

(1) $a^2b = a$.

By the Diamond Lemma (see [3]), applied to the single reduction

(2) $aab \longmapsto a$,

we get that

(3) a k-basis for R is given by the set of words in a and b which do not contain the substring aab.

Since the set of such words is closed under multiplication on the right by a, we see that

(4) Right multiplication by a gives a k-module isomorphism $R \cong Ra$.

Also, since the reduction (2) does not have on the right-hand side any occurrence of the empty monomial 1, we see that when we multiply a nonempty monomial by any monomial, we get a nonempty monomial, so

(5) The span of the set of all monomials other than 1 forms a 2-sided ideal I of R.

Since right multiplication by a increases the length of any monomial, we have

(6) $a \notin Ia$.

Now since $a \in I$, we have $aR \subseteq I$, so (6) gives

(7) $a \notin aRa$,

which, combined with (1), gives the example desired.

Summarizing, the Diamond Lemma applied to (2) shows (3), from which we get (7).

GB2. Somehow similar to the example above, here is an example of *regular, left strongly regular element which is not strongly regular* (actually more: elements a and b in a ring, with $a^2b = a$ and $aba = a$, such that a is not strongly regular), also constructed by George Bergman.

Consider the algebra S presented by generators a and b, and the two relations $a^2b = a$ and $aba = a$. It turns out that a is again non-strongly regular in S.

The argument is as follows.

Applying the Diamond Lemma to the pair of reductions

(8) $aab \longmapsto a$, $aba \longmapsto a$,

one finds that

(9) a k-basis for S is given by the set of words in a and b which do not admit application of either of the reductions of (8), i.e., which do not contain either of the substrings aab, aba.

It is easy to see that

(10) if a word in a and b ends in m (for a nonnegative integer m) a's, then the word one gets on reducing it using (8) ends in at least m a's.

Hence,

(11) There is no word u in a and b such that $ua^2 = a$.

Hence, since reducing a word always gives a word (not a linear combination of words),

(12) there is no element $c \in S$ such that $ca^2 = a$.

It is also easy to check that the algebra R of the previous example has no element c satisfying $aca = a$; so we could say that the elements a of R and S have differing degrees of "closeness to" strong regularity.

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