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BICYCLIC UNITS IN FINITE LOOP ALGEBRA

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Abstract. Let $L = M(D_{2p}, 2)$ be the Moufang loop obtained from the dihedral group D_{2p} of order $2p$, where p is a prime number, and let F be a finite field of characteristic 2. In this article, we determine the bicyclic units in the loop algebra $F[L]$.

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1. Introduction

A bicyclic unit in an alternative loop ring $\mathbb{Z}L$ is an element of the form $u_{g,h} = 1 + (1-g)h\hat{g}$ where $g, h \in L,\ \hat{g} = 1 + g + g^2 + \ldots + g^{n-1},\ n$ is the order of g. Since $(1-g)\hat{g} = 0$, the element $u_{g,h}$ is of the type $1+\alpha$ with $\alpha^2 = 0, \alpha \in \mathbb{Z}L$. Hence, a bicyclic unit is indeed a unit with $(1+\alpha)^{-1} = 1 - \alpha$.

Bicyclic units were studied in the unit group of integral group rings of finite groups in several papers [3,4,5]. But for loop rings and loop algebras these are quite less unexplored. In the present article, we shall study the bicyclic units in the loop algebra of the Moufang loop obtained from the dihedral group over finite fields. Chein and Goodaire [1] proved that if $G = D_{2m}$, the dihedral group of order $2m$, then the loop $M(G, 2)$ is a RA2 loop. Vojtěchovský [6] gave the presentation of Moufang loops of the type $M(G, 2)$ with G a finite two generated group, as follows:

Theorem 1.1. [6, Theorem 3.1] *Let $G = \langle x, y \mid R\rangle$ be a presentation for a finite group G, where R is a set of relations in generators x, y. Then $M_{2n}(G, 2)$ is presented by*

$$\langle x, y, u \mid R, u^2 = (xu)^2 = (yu)^2 = (xy.u)^2 = e\rangle,$$

where e is the neutral element of G.

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Let $\mathcal{B}(F[M(D_{2p},2)])$ be the loop generated by the bicyclic units of $F[M(D_{2p},2)]$.

2. Determination of bicyclic units

To determine the bicyclic units, we first determine the trivial bicyclic units in $F[M(D_{2p},2)]$. An easy calculation implies that $u_{1,l} = u_{l,1} = u_{l,l} = 1 \;\forall\; l \in M(D_{2p},2)$.

Proposition 2.1. *$u_{a^i,l} = 1$ for all $0 \leq i \leq p-1$ and $\forall\; l \in M(D_{2p},2)$.*

Proof. From [2], it is known that $u_{g,h} = 1$ if and only if $h^{-1}gh = g^j$ for some j. Clearly, $h^{-1}gh = g^j \;\forall\; h \in D_{2p}$ and $g = a^i$ $(i = 0,1,2,\ldots,p-1)$. Furthermore, for $k \in D_{2p}$,

$$\begin{aligned}(ku)^{-1}a^i(ku) &= (ku)a^i(ku)\\ &= ((ka^{-i})u)(ku)\\ &= k^{-1}(ka^{-i})\\ &= a^{-i}.\end{aligned}$$

□

Proposition 2.2. *$u_{a^ib,hu} = 1$ for all $0 \leq i \leq p-1$ and $\forall\; h \in D_{2p}$.*

Proof. Here,

$$\begin{aligned}u_{a^ib,hu} &= 1 + (a^ib-1)hu(1+a^ib)\\ &= 1 + ((h(a^ib))u - hu)(1+a^ib)\\ &= 1 + (h(a^ib))u + ((h(a^ib))a^ib)u - hu - (h(a^ib))u\\ &= 1.\end{aligned}$$

□

Proposition 2.3. *$u_{gu,h} = 1$ for all $g \in D_{2p}$ and $h = a^ib$ $(0 \leq i \leq p-1)$.*

Proof. We have

$$\begin{aligned}u_{gu,h} &= 1 + (gu-1)h(1+gu)\\ &= 1 + ((gh^{-1})u - h)(1+gu)\\ &= 1 + (gh^{-1})u + h^{-1} - h - (gh)u.\end{aligned}$$

So, $u_{gu,h} = 1$ if $h^{-1} = h$, that is, $h = a^ib$ $(0 \leq i \leq p-1)$. □

Proposition 2.4. *$u_{gu,hu} = 1$ for all $g = a^ib$ $(0 \leq i \leq p-1)$ and $h \in \langle a \rangle$.*

Proof. Here,

$$\begin{aligned}u_{gu,hu} &= 1 + (gu-1)hu(1+gu)\\ &= 1 + (h^{-1}g - hu)(1+gu)\\ &= 1 + h^{-1}g - hu + (g(h^{-1}g))u - g^{-1}h.\end{aligned}$$

Therefore, $u_{gu,hu} = 1$ if $h^{-1}g = g^{-1}h$, that is, $g^{-1}hg^{-1} = h^{-1}$ which is true when $g = a^ib$ $(0 \leq i \leq p-1)$ and $h \in \langle a \rangle$. □

Proposition 2.5. $u_{a^iu,hu} = 1$ *for all* $1 \leq i \leq p-1$ *and* $h = a^jb$ $(0 \leq j \leq p-1)$.

Proof. Consider

$$\begin{aligned} u_{a^iu,hu} &= 1 + (a^iu - 1)hu(1 + a^iu) \\ &= 1 + (h^{-1}a^i - hu)(1 + a^iu) \\ &= 1 + h^{-1}a^i + (a^i(h^{-1}a^i))u - hu - a^{-i}h. \end{aligned}$$

Thus, $u_{a^iu,hu} = 1$ if $h^{-1}a^i = a^{-i}h$ which is true when $h = a^jb$ $(0 \leq j \leq p-1)$. □

We are left with the following non-trivial bicyclic units:

$\{u_{a^ib,a^j},\ u_{a^ib,a^kb},\ u_{a^ju,a^lu},\ u_{a^ju,a^l},\ u_{a^ibu,a^kbu},\ u_{a^ibu,a^l} | 0 \leq i,k \leq p-1, 1 \leq j,l \leq p-1\}$.

Next, we determine the distinct units among the above set.

Proposition 2.6. (i) $u_{a^ib,a^kb} = u_{a^ib,a^j}$ *if* $i \cong (k-j)(mod\ p)$.

(ii) $(u_{a^ib,a^j})^{-1} = u_{a^ib,a^l}$ *if* $l + j \cong 0(mod\ p)$.

Proof. (i) We have

$$\begin{aligned} u_{a^ib,a^kb} &= 1 + (a^ib - 1)a^kb(1 + a^ib) \\ &= 1 + (a^{i-k} - a^kb)(1 + a^ib) \\ &= 1 + a^{i-k} + a^{2i-k}b - a^kb - a^{-i+k} \end{aligned}$$

and

$$\begin{aligned} u_{a^ib,a^j} &= 1 + (a^ib - 1)a^j(1 + a^ib) \\ &= 1 + (a^{i-j}b - a^j)(1 + a^ib) \\ &= 1 + a^{i-j}b + a^{-j} - a^j - a^{i+j}b. \end{aligned}$$

Thus $u_{a^ib,a^kb} = u_{a^ib,a^j}$ when $i \cong (k-j)(\text{mod } p)$.

(ii) Also,

$$\begin{aligned} (u_{a^ib,a^j})^{-1} &= 1 - (a^ib - 1)a^j(1 + a^ib) \\ &= 1 - a^{i-j}b - a^{-j} + a^j + a^{i+j}b. \end{aligned}$$

Thus, $(u_{a^ib,a^j})^{-1} = u_{a^ib,a^l}$ if $l + j \cong 0(\text{mod } p)$. □

Proposition 2.7. (i) $u_{a^ju,a^lu} = u_{a^ju,a^m}$ *if* $j \cong (l-m)(mod\ p)$.

(ii) $(u_{a^ju,a^m})^{-1} = u_{a^ju,a^n}$ *if* $m + n \cong 0(mod\ p)$.

Proof. (i) We have

$$\begin{aligned} u_{a^ju,a^lu} &= 1 + (a^ju - 1)a^lu(1 + a^ju) \\ &= 1 + (a^{j-l} - a^lu)(1 + a^ju) \\ &= 1 + a^{j-l} + a^{2j-l}u - a^lu - a^{l-j} \end{aligned}$$

and

$$\begin{aligned} u_{a^ju,a^m} &= 1 + (a^ju - 1)a^m(1 + a^ju) \\ &= 1 + (a^{j-m}u - a^m)(1 + a^ju) \\ &= 1 + a^{j-m}u + a^{-m} - a^m - a^{j+m}u. \end{aligned}$$

Thus $u_{a^ju,a^lu} = u_{a^ju,a^m}$ when $j \cong (l-m)(\text{mod } p)$.

(ii) Also,

$$\begin{aligned}(u_{a^ju,a^m})^{-1} &= 1-(a^ju-1)a^m(1+a^ju)\\ &= 1-a^{j-m}u-a^{-m}+a^m+a^{j+m}u.\end{aligned}$$

Thus, $(u_{a^ju,a^m})^{-1}=u_{a^ju,a^n}$ if $m+n\cong 0(\text{mod } p)$. □

Proposition 2.8. (i) $u_{a^ibu,a^kbu}=u_{a^ibu,a^l}$ *if* $i\cong(k+l)(mod\ p)$.

(ii) $(u_{a^ibu,a^l})^{-1}=u_{a^ibu,a^r}$ *if* $l+r\cong 0(mod\ p)$.

Proof. (i) We have

$$\begin{aligned}u_{a^ibu,a^kbu} &= 1+(a^ibu-1)a^kbu(1+a^ibu)\\ &= 1+(a^{k-i}-a^kbu)(1+a^ibu)\\ &= 1+a^{k-i}+a^{2i-k}bu-a^kbu-a^{i-k}\end{aligned}$$

and

$$\begin{aligned}u_{a^ibu,a^l} &= 1+(a^ibu-1)a^l(1+a^ibu)\\ &= 1+(a^{i+l}bu-a^l)(1+a^ibu)\\ &= 1+a^{i+l}bu+a^{-l}-a^l-a^{i-l}bu.\end{aligned}$$

Thus $u_{a^ibu,a^kbu}=u_{a^ibu,a^l}$ when $i\cong(k+l)(\text{mod } p)$.

(ii) Also,

$$\begin{aligned}(u_{a^ibu,a^l})^{-1} &= 1-(a^ibu-1)a^l(1+a^ibu)\\ &= 1-a^{i+l}bu-a^{-l}+a^l+a^{i-l}bu.\end{aligned}$$

Thus, $(u_{a^ibu,a^l})^{-1}=u_{a^ib,a^r}$ if $l+r\cong 0(\text{mod } p)$. □

Furthermore, we have the following lemmas.

Lemma 2.9. *For* $l\in M(D_{2p},2)$, $l^{-1}u_{g,h}l\in\mathcal{B}(F[M(D_{2p},2)])$.

Proof. Consider

$$\begin{aligned}l^{-1}u_{g,h}l &= l^{-1}(1+(1-g)h(1+g+g^2+\ldots+g^{n-1}))l\\ &= 1+(l^{-1}-l^{-1}g)hll^{-1}(1+g+g^2+\ldots+g^{n-1})l\\ &= 1+(1-l^{-1}gl)l^{-1}hl(1+l^{-1}gl+l^{-1}g^2l+\ldots+l^{-1}g^{n-1}l)\\ &= u_{l^{-1}gl,l^{-1}hl}.\end{aligned}$$

□

Lemma 2.10. $M(D_{2p},2)\cap\mathcal{B}(F[M(D_{2p},2)])=\langle a\rangle$.

Proof. By the above lemma, we get that $\mathcal{B}(F[M(D_{2p},2)])\cap M(D_{2p},2)$ is a normal subloop of $M(D_{2p},2)$. Therefore, it is either trivial or $\langle a\rangle$.

Define a map

$$\phi: M(D_{2p},2)\to\langle h|h^2=1\rangle$$

by

$$\phi(l)=\begin{cases}1, & \text{if } l\in\langle a\rangle,\\ h, & \text{if } l\notin\langle a\rangle.\end{cases}$$

Note that it is a loop homomorphism and it can be extended to an algebra homomorphism ϕ' from $F[M(D_{2p},2)]$ to $F[\langle h\rangle]$. It is easy to see that the image of bicyclic units under ϕ' is 1, but the image of $b,\ a^ib,\ lu\ (0\leq i\leq p-1;\ l\in D_{2p})$ is $h(\neq 1)$. Hence, $M(D_{2p},2)\cap\mathcal{B}(F[M(D_{2p},2)])\neq M(D_{2p},2)$. Further, observe that

$$u_{a^jb,a^i}=1+(a^i+a^{-i})(1+a^jb)$$

which gives

$$\prod_{i=1}^{k}u_{ab,a^i}=ab(1+\widehat{D_{2p}})\text{ and }\prod_{i=1}^{k}u_{b,a^i}=b(1+\widehat{D_{2p}})\text{ where }k=\frac{p-1}{2}.$$

Thus $a=u_{ab,a}u_{ab,a^2}\dots u_{ab,a^k}u_{b,a^k}\dots u_{b,a^2}u_{b,a}$.
Hence,

$$M(D_{2p},2)\cap\mathcal{B}(F[M(D_{2p},2)])=\langle a\rangle.$$ □

Thus, using all above, we have proved the following:

Theorem 2.11. $\mathcal{B}(F[M(D_{2p},2)])=\{a^i,\ u_{a^ib,a^k},\ u_{a^iu,a^k},\ u_{a^ibu,a^k}\ :\ 0\leq i\leq p-1, 1\leq k\leq\frac{p-1}{2}\}$.

The cardinality of $\mathcal{B}(F[M(D_{2p},2)])$ is $\frac{p(3p-1)}{2}$.

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