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ON DIVISOR GRAPHS OF VALUATION DOMAINS AND LOCAL ARTINIAN PRINCIPAL IDEAL RINGS

John D. LaGrange

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Abstract. For an element x of a commutative ring R, let $\Gamma_x(R)$ be the graph whose vertices are the elements of R that divide x such that distinct vertices r and s are adjacent if and only if $rs = x$. If x is a nonzero nonirreducible nonunit of an integral domain R, then $\Gamma_x^{\mathcal{C}}(R)$ is the graph whose vertices are the associate classes of divisors of x that are neither units nor associates of x such that distinct vertices A and B are adjacent if and only if rs divides x for some (and hence every) $r \in A$ and $s \in B$. The graphs $\Gamma_x(R)$ are considered when R is a local Artinian principal ideal ring, and $\Gamma_x^{\mathcal{C}}(R)$ is examined when R is a valuation domain. For example, it is shown that a finite local ring R is a principal ideal ring if and only if there exists $x \in R$ such that every connected component of $\Gamma_x(R)$ is a star graph of a prescribed cardinality. Moreover, it is proved that an integral domain R is a discrete valuation ring if and only if its collection of graphs $\Gamma_x^{\mathcal{C}}(R)$ consists precisely of a single-vertex graph, along with every (up to isomorphism) graph that is realizable as the compressed 0-divisor graph of a local Artinian principal ideal ring. Certain graphs associated with partially ordered abelian groups have an essential role in the work.

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1. Introduction

Since the seminal works [5,9] on zero-divisor graphs appeared in the late twentieth century, a number of graph-theoretic models of divisibility in rings and other algebraic structures have been explored. To list a few, zero-divisor graphs of semigroups, partially ordered sets, and groupoids were introduced in [12,13,18,23], and annihilating ideal graphs and dot product graphs of commutative rings were studied in [1,10]. While these constructions were based on "products" that equal zero, graphs involving more general divisibility relations have been considered in [8,11,16,18,19,20,25,28].

Among commutative rings with identity, it is well-known that every local principal ideal ring is either Artinian or a discrete valuation ring. In fact, R is a local principal ideal ring if and only if it is the homomorphic image of a discrete valuation ring [14, Theorem 1]. The purpose of this paper is to analyze graph-theoretic models of divisibility in these rings, as well as more general valuation domains. Furthermore, certain graphs associated with partially ordered abelian groups are considered so that integral domains can be studied via groups of divisibility.

1.1. The divisor graphs. To describe the graphs that are relevant in this paper, let R be a commutative ring with $1 \neq 0$. The *x-divisor graph* of R is the graph $\Gamma_x(R)$ (or Γ_x when there is no risk of confusion) whose vertices are the elements of $V(\Gamma_x) = d(x) = \{r \in R \mid rs = x \text{ for some } s \in R\}$ such that two distinct vertices $r, s \in R$ are adjacent if and only if $rs = x$. These graphs were considered for commutative von Neumann regular rings in [19, Section 2], and were extended to commutative groupoids and quasigroups in [18,20]. Note that Γ_0 agrees with the graph that was defined in [9], and more generally, if x is idempotent, then Γ_x agrees with the *idempotent-divisor graph* that was defined in [16].

When R is an integral domain, the graphs $\Gamma_x(R)$ are well understood. In fact, by [19, Proposition 1], for any commutative ring R, if $x \in R$ is not a zero-divisor, then every connected component of Γ_x is a complete graph on either one or two vertices (see Lemma 2.1(3)). This suggests that x-divisor graphs are limited in terms of distinguishing algebraic structure between integral domains. Instead, more information can be obtained by using graphs defined by the rule that vertices r and s are adjacent whenever rs divides x. This idea was first used in [11], where the vertices represented associate classes of irreducible elements that divide x, assuming that x can be factored into irreducibles. In [19], the definition was extended to include associate classes of nonirreducible divisors of x, which broadens the scope of integral domains to which these graphs can be applied (e.g., using the extended definition, x need not have any irreducible divisors).

To give a precise definition, let R be an integral domain, let $U(R)$ be the group of units of R, and let $\mathrm{irr}(R)$ be the set of irreducible elements of R. Denote the (multiplicative) semigroup of reducible elements of R by $R^\circ = R \setminus (\mathrm{irr}(R) \cup U(R) \cup \{0\})$. If $x \in R^\circ$, then define the *(compressed) divisor graph* $\Gamma_x^{\mathcal{C}}(R)$ (or, $\Gamma_x^{\mathcal{C}}$ if there is no risk of confusion) associated with x to be the graph whose vertex-set $V(\Gamma_x^{\mathcal{C}})$ is comprised of the associate-equivalence classes $rU(R)$ of elements $r \in d(x) \setminus (xU(R) \cup U(R))$ such that two distinct vertices $rU(R)$ and $sU(R)$ are adjacent if and only if $rs \in d(x)$ (the nomenclature is clarified in the discussion that follows Proposition 2.2).

To illustrate, one can check that if $\mathbb{Z}$ is the usual group of integers, then $\Gamma_{12}^{\mathcal{C}}(\mathbb{Z})$ is a path of length three on $V(\Gamma_{12}^{\mathcal{C}}(\mathbb{Z})) = \{2U(\mathbb{Z}),\ 3U(\mathbb{Z}),\ 4U(\mathbb{Z}),\ 6U(\mathbb{Z})\}$. More generally, it is shown in [19, Theorem 7(3)] that R is a UFD if and only if $\Gamma_x^{\mathcal{C}}(R)$ is a finite graph with a dominant clique for every $x \in R^\circ$. In this case, if x is square-free, then $\Gamma_x^{\mathcal{C}}(R)$ is isomorphic to the zero-divisor graph (as defined in [5]) of a finite Boolean ring by [19, Proposition 4].

Finally, "groups of divisibility" (defined in Section 1.3) of integral domains are applied in Sections 4 and 5, making it useful to associate graphs to *partially ordered commutative monoids*; that is, commutative monoids M endowed with a partial order $\leq$ such that if $a, b \in M$ with $a \leq b$, then $a+c \leq b+c$ for every $c \in M$. In this paper, the partially ordered submonoids given by the positive cones $G^+ = \{g \in G \mid g \geq 0\}$ of partially ordered abelian groups G are of primary interest.

Let P be a partially ordered set that contains a least element 0. An element $x \neq 0$ is an *atom* if $\{y \in P \mid y \leq x\} = \{0, x\}$, and otherwise x will be called *nonminimal*. If M is a partially ordered commutative monoid such that $x \geq 0$ for every $x \in M$, and if $0 \neq x \in M$ is nonminimal, then define $\Gamma_{\leq x}(M)$ (or, $\Gamma_{\leq x}$ if there is no risk of confusion) to be the graph whose vertices are the elements of the interval $(0, x) = \{y \in M \mid 0 < y < x\}$ such that distinct vertices a and b are adjacent if and only if $a + b \leq x$. Similarly, $\Gamma_{\geq x}$ is the graph on $(0, x)$ such that distinct vertices a and b are adjacent if and only if $a + b \geq x$.

For free abelian groups $G \cong \bigoplus_I \mathbb{Z}$ with the usual product order, the graphs $\Gamma_{\leq x}(G^+)$ were used in [19, Theorem 7(2)] to characterize UFDs. The "dual" graphs $\Gamma_{\geq n}(\mathbb{Z}^+)$ for the group of integers were used in [4, Example 4.8] to represent "compressed zero-divisor graphs" of the rings $\mathbb{Z}_q$ for every prime-power q. This is generalized in Section 3 of the current study, where $\Gamma_{\geq n}(\mathbb{Z}^+)$ is shown to model compressed 0-divisor graphs of local Artinian principal ideal rings (Theorem 3.1). Moreover, the divisor graphs $\Gamma_x^{\mathcal{C}}(R)$ of discrete valuation rings R are proved to be isomorphic to $\Gamma_{\leq n}(\mathbb{Z}^+)$ in Section 4 (Theorem 4.2). Proposition 2.2 unifies these results by observing that $\Gamma_{\leq x}(G^+)$ and $\Gamma_{\geq x}(G^+)$ are isomorphic, further linking the structures of local Artinian principal ideal rings and discrete valuation rings (see Corollary 4.3).

1.2. The rings. In this paper, rings are always commutative with $1 \neq 0$. A local Artinian principal ideal ring will be referred to as a *special principal ideal ring*, or an *SPIR* for brevity. Several well-known properties of SPIRs and valuation domains will be freely assumed in the forthcoming arguments. These properties are outlined in the following discussion.

Let R be a commutative ring with $1 \neq 0$. If $\emptyset \neq A \subseteq R$, then let $\operatorname{ann}(A) = \{r \in R \mid ra = 0$ for every $a \in A\}$, and set $\operatorname{ann}(\{a\}) = \operatorname{ann}(a)$. If R is Noetherian and local with principal maximal ideal $\mathfrak{m} = (a)$, then every $0 \neq r \in R$ can be written as $r = ua^k$ for some $u \in U(R)$ and $0 \leq k \in \mathbb{Z}$ (we take $0^0 := 1$ in case R is a field), and hence R is a principal ideal ring [14, Proposition 4]. In this case, if R is Artinian, then there exists $0 \leq t \in \mathbb{Z}$ such that $\mathfrak{m}^t \neq (0) = \mathfrak{m}^{t+1}$ (actually, such t exists in any local Artinian ring, e.g., [7, Proposition 8.4]). Thus, if R is an SPIR, then $R = \{ua^k \mid u \in U(R),\ k \in \{0, 1, \ldots, t+1\}\}$ for some $0 \leq t \in \mathbb{Z}$. Hence, if $k \in \{0, 1, \ldots, t\}$, then $a^k \notin \mathfrak{m}^{k+1}$, and $\operatorname{ann}(a^k) = \operatorname{ann}(a^k U(R)) = \mathfrak{m}^{t+1-k}$. Moreover, every ideal of R is of the form $\mathfrak{m}^k$ for some $k \in \{0, 1, \ldots, t+1\}$.

Recall that an integral domain R is a *valuation domain* if either $x \in R$ or $x^{-1} \in R$ for every nonzero x in the field of fractions of R. Any two ideals of a valuation domain are comparable under inclusion. In particular, every valuation domain R is quasilocal; that is, R has a unique maximal ideal (in this paper, the term "local" carries the additional assumption that R is Noetherian). A Noetherian valuation domain that is not a field is called a *discrete valuation ring* (more briefly, a *DVR*). Equivalently, a DVR is a local PID that is not a field. It follows that if R is not a field, then R is a DVR if and only if there exists $a \in R$ such that $R \setminus \{0\} = \{ua^k \mid u \in U(R)$ and $0 \leq k \in \mathbb{Z}\}$.

1.3. Summary of results and notation. Throughout, the positive integers, integers, integers modulo n, rational numbers, and real numbers will be denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, and $\mathbb{R}$, respectively. If R is an integral domain with field of fractions K, then the *group of divisibility* $G(R) = (K \setminus \{0\})/U(R)$ is a directed partially ordered (multiplicative) abelian group under the partial order given by $rU(R) \leq sU(R)$ if and only if $s/r \in R$ (it is directed since if $a/b, c/d \in K \setminus \{0\}$, then $a/bU(R), c/dU(R) \leq acU(R)$). Given any $x \in K \setminus \{0\}$, note that $x \in R$ if and only if $U(R) \leq xU(R)$. Hence, from the discussion at the end of Section 1.2, it is easy to check that an integral domain R is a DVR if and only if $G(R)^+ \cong \mathbb{Z}^+$.

Let Γ be a simple graph. The vertex-set of Γ will be denoted by $V(\Gamma)$, and the *neighborhood* of a vertex $v \in V(\Gamma)$ is the set $N_\Gamma(v) = \{u \in V(\Gamma) \mid u$ and v are adjacent in $\Gamma\}$ (which may be denoted by $N(v)$ when there is no risk). The cardinality $\deg_\Gamma(v) = |N_\Gamma(v)|$ of the neighborhood of v is called the *degree* of v in Γ.

The complete graph of order n, and the complete bipartite graph with partite sets of cardinalities m and n will be denoted by K_n and $K_{m,n}$, respectively. Similarly, as $\Gamma_{\geq n}(\mathbb{Z}^+) \cong \Gamma_{\leq n}(\mathbb{Z}^+)$ (Proposition 2.2), it will be convenient to have a purely graph-theoretic model (on an abstract set of vertices) that depicts these constructions.

Thus, given any $2 \leq n \in \mathbb{Z}$, let $\Gamma_{[n]}$ denote the graph to which $\Gamma_{\leq n}(\mathbb{Z}^+)$ (and $\Gamma_{\geq n}(\mathbb{Z}^+)$) is isomorphic.

Section 2 records some preliminary results and definitions that are needed in the sequel. Section 3 provides characterizations of $\Gamma_x(R)$ for elements x of an SPIR $(R, \mathfrak{m})$. Theorem 3.1 gives the general structure of Γ_0 (e.g., it is observed that the "compression" of Γ_0 is isomorphic to $\Gamma_{[n]}$ for some $3 \leq n \in \mathbb{Z}$), and then Theorem 3.6 completely characterizes the generators of $\mathfrak{m}$ by showing that if R is not a field, then $\mathfrak{m} = (a)$ if and only if Γ_a is a disjoint union of $|\mathfrak{m} \setminus \mathfrak{m}^2|$ copies of $K_{1,|R/\mathfrak{m}|}$ (where cardinalities are not necessarily finite). Theorem 3.9 proves that a finite commutative local ring R with $1 \neq 0$ is a principal ideal ring if and only if either R is a field, or there exists $a \in R$ such that every connected component of Γ_a is isomorphic to $K_{1,|R/\mathfrak{m}|}$.

In Sections 4 and 5, the focus is shifted to integral domains and partially ordered abelian groups. It is proved in Theorem 4.2 that R is a DVR if and only if $\{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ consists precisely (up to isomorphism) of the graphs $\Gamma_{[n]}$ (i.e., there exists a bijection $\psi : \{\Gamma_{[n]} \mid 2 \leq n \in \mathbb{Z}\} \to \{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ such that $\Gamma_{[n]} \cong \psi(\Gamma_{[n]})$ for every $2 \leq n \in \mathbb{Z}$). In Theorem 5.8, it is shown that a directed partially ordered abelian group G is totally ordered if and only if $\{\Gamma_{\leq x}(G^+) \mid 0 \neq x \in G^+$ is nonminimal$\}$ is totally ordered under the usual "subgraph relation". As a corollary, an integral domain R is a valuation domain if and only if its compressed divisor graphs $\{\Gamma_x^{\mathcal{C}} \mid x \in R^\circ\}$ are totally ordered under the subgraph relation (Corollary 5.10).

2. Preliminary results

This section extends the Introduction in order to disclose some technical aspects of the definitions, and to record ideas that will be used to support the main results. The reader may wish to proceed to Section 3, and refer to the current section as needed.

In Section 1.1, the graphs $\Gamma_x^{\mathcal{C}}$ were motivated by the claim that x-divisor graphs Γ_x are limited in terms of distinguishing algebraic structure between integral domains. The following lemma clarifies this assertion.

Lemma 2.1. *The following statements are valid for every commutative ring R with $1 \neq 0$.*

(1) *The inclusion $U(R) \subseteq V(\Gamma_x)$ holds for every $x \in R$.*
(2) *If $x \in R \setminus U(R)$, then every element of $U(R)$ is adjacent to exactly one vertex in Γ_x.*

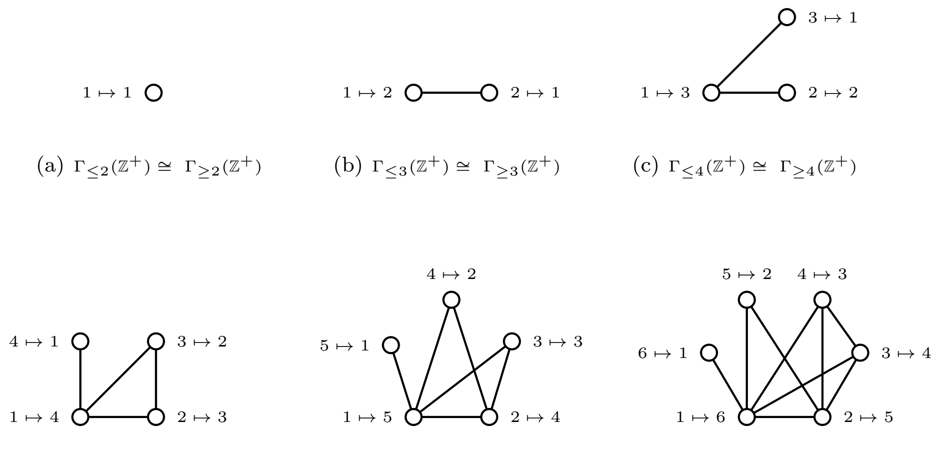
(3) [19, Proposition 1] *If $x \in R$ is not a zero-divisor, then every connected component of Γ_x is a complete graph on either one or two vertices.*

Proof. The first statement follows since if $u \in U(R)$, then $u(u^{-1}x) = x$. The assertion in (2) holds since the equality $ur = x$ holds for $u \in U(R)$ if and only if $r = u^{-1}x$, and if $u^{-1}x = u$, then $x = u^2 \in U(R)$. The statement in (3) is easily verified, as in [19, Proposition 1]. □

The remainder of this section elaborates on some of the graph-theoretic definitions presented in Section 1. The notation $\Gamma_{[n]}$ was introduced in Section 1.3 to represent the purely graph-theoretic model to which the graphs $\Gamma_{\leq n}(\mathbb{Z}^+)$ and $\Gamma_{\geq n}(\mathbb{Z}^+)$ are isomorphic. The following proposition confirms, more generally, that these constructions do render isomorphic graphs (see Figure 1).

Proposition 2.2. *Let G be a partially ordered abelian group. If $0 \neq x \in G^+$ is nonminimal, then $\Gamma_{\leq x}(G^+) \cong \Gamma_{\geq x}(G^+)$.*

Proof. Clearly $(0, x) \to (0, x)$ by $a \mapsto x - a$ is bijective. If $a, b \in (0, x)$, then $a + b \leq x$ if and only if $(x - a) + (x - b) = 2x - (a + b) \geq 2x - x = x$. Hence, the mapping $\Gamma_{\leq x}(G^+) \to \Gamma_{\geq x}(G^+)$ by $a \mapsto x - a$ is a graph-isomorphism. □



(a) $\Gamma_{\leq 2}(\mathbb{Z}^+) \cong \Gamma_{\geq 2}(\mathbb{Z}^+)$ (b) $\Gamma_{\leq 3}(\mathbb{Z}^+) \cong \Gamma_{\geq 3}(\mathbb{Z}^+)$ (c) $\Gamma_{\leq 4}(\mathbb{Z}^+) \cong \Gamma_{\geq 4}(\mathbb{Z}^+)$

(d) $\Gamma_{\leq 5}(\mathbb{Z}^+) \cong \Gamma_{\geq 5}(\mathbb{Z}^+)$ (e) $\Gamma_{\leq 6}(\mathbb{Z}^+) \cong \Gamma_{\geq 6}(\mathbb{Z}^+)$ (f) $\Gamma_{\leq 7}(\mathbb{Z}^+) \cong \Gamma_{\geq 7}(\mathbb{Z}^+)$

FIGURE 1. The graphs $\Gamma_{\leq n}(\mathbb{Z}^+)$ and $\Gamma_{\geq n}(\mathbb{Z}^+)$ for $n \in \{2, 3, 4, 5, 6, 7\}$, where $a \mapsto b$ indicates that the vertex is given by a in $\Gamma_{\leq n}(\mathbb{Z}^+)$, and it is given by b in $\Gamma_{\geq n}(\mathbb{Z}^+)$.

Let Γ be a simple graph. It was proved in [4, Theorem 2.1] that the relation $\equiv$ on $V(\Gamma)$ defined by $v \equiv w$ if and only if $N_\Gamma(v)\setminus\{w\} = N_\Gamma(w)\setminus\{v\}$ is an equivalence relation on $V(\Gamma)$. Define the *compression* $\mathcal{C}(\Gamma)$ of Γ to be the graph whose vertices are the equivalence classes $[v] = \{w \in V(\Gamma) \mid v \equiv w\}$ of vertices $v \in V(\Gamma)$ such that distinct vertices $[v_1]$ and $[v_2]$ are adjacent in $\mathcal{C}(\Gamma)$ if and only if some (and hence every) element of $[v_1]$ is adjacent to some (and hence every) element of $[v_2]$ in Γ. In [19, Theorem 3], it is shown that if R is an integral domain such that $U(R) \neq \{1\}$, then for every $x \in R^\circ$, the graph $\Gamma_x^{\mathcal{C}}(R)$ is the compression of the graph whose vertices are the elements of $d(x) \setminus (xU(R) \cup U(R))$ such that distinct r and s are adjacent if and only if $rs \in d(x)$.

Let R be a commutative ring with $1 \neq 0$. Define the *compressed* 0*-divisor graph* of R to be the graph $(\Gamma_0)_E(R)$ whose vertices are the equivalence classes $\widetilde{r} = \{x \in R \mid \operatorname{ann}(x) = \operatorname{ann}(r)\}$ $(r \in R)$ such that two distinct vertices $\widetilde{r}$ and $\widetilde{s}$ are adjacent if and only if $rs = 0$. This extends the construction that was first given in [27, (3.5)], where the vertices included only the equivalence classes $\widetilde{r}$ such that r is a nonzero zero-divisor.

By Proposition 2.4(1) given below, if $|R| > 4$, then $(\Gamma_0)_E(R)$ is indeed a "compression". More generally, the next result exploits the relationship between the equivalence classes in this paper.

Proposition 2.3. *Let R be a commutative ring with $1 \neq 0$. If $r, s \in R$, then the following statements hold.*

(1) *If R is an SPIR, then $rU(R) = sU(R)$ if and only if* $\operatorname{ann}(r) = \operatorname{ann}(s)$.
(2) [4, Theorem 2.4] *If $|R| > 4$, then $r \equiv s$ in $\Gamma_0(R)$ if and only if* $\operatorname{ann}(r) = \operatorname{ann}(s)$.

Proof. Suppose that R is an SPIR with maximal ideal $\mathfrak{m} = (a)$, and let $0 \leq t \in \mathbb{Z}$ such that $\mathfrak{m}^t \neq (0) = \mathfrak{m}^{t+1}$. Let $r, s \in R$, say $r = ua^j$ and $s = va^k$ $(u, v \in U(R)$, $j, k \in \{0, 1, \ldots, t+1\})$. Note that $t+1-j, t+1-k \in \{0, 1, \ldots, t+1\}$, so $\mathfrak{m}^{t+1-j} = \mathfrak{m}^{t+1-k}$ if and only if $t+1-j = t+1-k$. That is, $\operatorname{ann}(r) = \operatorname{ann}(s)$ if and only if $j = k$. Hence, $\operatorname{ann}(r) = \operatorname{ann}(s)$ if and only if $rU(R) = sU(R)$. This proves (1). The statement in (2) holds by [4, Theorem 2.4]. □

The following proposition further unifies the constructions presented in this paper.

Proposition 2.4. *Let R be a commutative ring with $1 \neq 0$, and let $\Gamma_{[n]}$ be the graph defined in Section 1.3. The following statements hold.*

(1) *If $|R| > 4$, then $\mathcal{C}(\Gamma_0(R)) = (\Gamma_0)_E(R)$.*

(2) [19, Proposition 3] *If R is an integral domain and $x \in R^\circ$, then* $\Gamma^{\mathcal{C}}_x(R) = \Gamma_{\leq xU(R)}(G(R)^+)$.

(3) [4, Example 4.8] *If* $2 \leq n \in \mathbb{N}$, *then* $\mathcal{C}(\Gamma_{[n+1]}) \cong \Gamma_{[n]}$.

Proof. The result in (1) follows immediately by Proposition 2.3(2). The claims in (2) and (3) are confirmed in [19, Proposition 3] and [4, Example 4.8], respectively. □

3. Graphs of special principal ideal rings

In this section, the structures of x-divisor graphs of SPIRs are investigated. The generators a of the maximal ideal $\mathfrak{m}$ are completely determined by Γ_a. Furthermore, among finite local rings, SPIRs are completely characterized by their x-divisor graphs.

The main results of this section involve Γ_x for $x \neq 0$. Nevertheless, the next theorem, which characterizes the structure of $\Gamma_0(R)$ for every SPIR R, is presented for the sake of completeness. Note that it expands on [6, Theorem 3.2], where it is observed that the zero-divisor graph of R (in the sense of [5]) is a "threshold graph" when R is a finite SPIR.

Recall that $\widetilde{r} = \{x \in R \mid \mathrm{ann}(x) = \mathrm{ann}(r)\}$. Also, Proposition 2.3(1) will be applied in the following proof without reference.

Theorem 3.1. *Suppose that $(R, (a))$ is an SPIR. Let $0 \leq t \in \mathbb{Z}$ such that $(a)^t \neq (0) = (a)^{t+1}$ (where $0^0 := 1$ in case R is a field), and let $\Gamma_{[n]}$ be the graph defined in Section 1.3. The following statements hold (see Figure 2).*

(1) $(\Gamma_0)_E(R) \cong \Gamma_{[t+3]}(\mathbb{Z}^+)$.

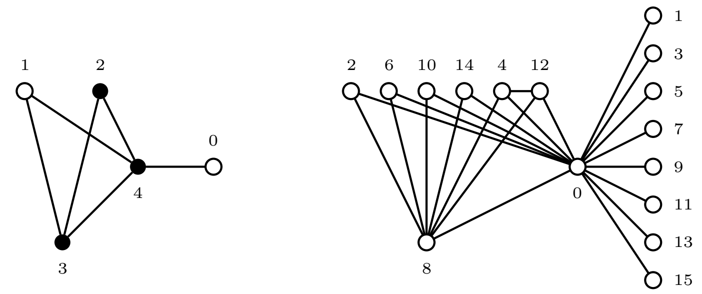
(2) *The structure of $\Gamma_0(R)$ is obtained from $(\Gamma_0)_E(R)$ as follows.*

  (i) *If $r, s \in R$ such that $\widetilde{r}$ and $\widetilde{s}$ are adjacent in $(\Gamma_0)_E(R)$, then r and s are adjacent in $\Gamma_0(R)$.*

  (ii) *If $(t+1)/2 \leq k \leq t+1$, then $\widetilde{a^k}$ induces a complete subgraph of $\Gamma_0(R)$.*

  (iii) *If $0 \leq k < (t+1)/2$, then $\widetilde{a^k}$ induces a totally disconnected subgraph of $\Gamma_0(R)$.*

Proof. Recall that $\Gamma_{\geq t+3} \cong \Gamma_{[t+3]}$ (Proposition 2.2). It is straightforward to check that $\varphi : \Gamma_0(R) \to \Gamma_{\geq t+3}(\mathbb{Z}^+)$ by $\varphi(ua^k) = k+1$ ($u \in U(R)$ and $k \in \{0, \dots, t+1\}$) is a surjection that reflects adjacency, and it preserves adjacency between $r, s \in V(\Gamma_0)$ such that $\varphi(r) \neq \varphi(s)$. But for any $u, v \in U(R)$ and $k, l \in \{0, \dots, t+1\}$, the equality $\varphi(ua^k) = \varphi(va^l)$ holds if and only if $k = l$, if and only if $\mathrm{ann}(ua^k) = \mathrm{ann}(va^l)$. Therefore, the mapping $(\Gamma_0)_E \to \Gamma_{\geq t+3}$ by $\widetilde{a^k} \mapsto k+1$ ($k \in \{0, \dots, t+1\}$) is an isomorphism of graphs. Also, (2)(ii) and (2)(iii) follow since ua^k and va^k are adjacent in $\Gamma_0(R)$ if and only if $ua^k \neq va^k$ and $2k \geq t+1$, i.e., the distinct elements of $\widetilde{a^k}$ induce a complete subgraph of $\Gamma_0(R)$ if $k \geq (t+1)/2$, and they induce a totally disconnected subgraph if $k < (t+1)/2$. Note that (2)(i) is clear by the definition of $(\Gamma_0)_E$. $\square$



(a) $\Gamma_{\geq 6}(\mathbb{Z}^+)$ $\big(\cong (\Gamma_0)_E(\mathbb{Z}_{16})\big)$ (b) $\Gamma_0(\mathbb{Z}_{16})$

FIGURE 2. Compressed and noncompressed 0-divisor graphs of $\mathbb{Z}_{16}$. Solid vertices of $\Gamma_{\geq 6}(\mathbb{Z}^+)$ correspond to complete subgraphs of $\Gamma_0(\mathbb{Z}_{16})$, while hollow vertices correspond to totally disconnected subgraphs.

Of course, if R is a field, then $\Gamma_0(R) \cong K_{1,|R\setminus\{0\}|}$. In general, however, the cardinalities of the complete and totally disconnected subgraphs from Theorem 3.1(2) are unknown, and this is the only remaining obstacle to a full characterization of zero-divisor graphs of SPIRs.

Question 3.2. *Let R be an SPIR. What can be said about the cardinalities of the complete and totally disconnected subgraphs of $\Gamma_0(R)$ from Theorem 3.1(2)(ii) and (iii)?*

Let R be an SPIR. In Theorem 3.4, it will be shown that if $0 \neq b \in R \setminus U(R)$ and $u \in U(R)$, then the connected component of Γ_b containing ub is isomorphic to $K_{1,|R/(b)|}$. Upon recalling some arithmetic of infinite cardinal numbers, there will be little inconvenience in relaxing any assumptions of "finiteness" on $R/(b)$.

For the sake of precision, recall that the *cardinality* of a set A can be defined as the least ordinal number that is in bijective correspondence with A. If $k \in \mathbb{N}$ and $A_1, \ldots, A_k$ are sets, then the *product* of their cardinalities is defined by $|A_1| \cdots |A_k| = |\prod_{i=1}^k A_i|$, where $\prod_{i=1}^k A_i$ is the usual Cartesian product of $A_1, \ldots, A_k$. If at least one of these sets A_i is infinite, then $|A_1| \cdots |A_k| = \max\{|A_i| \mid i \in \{1, \ldots, k\}\}$. For any set A, let $|A|^0 = |\{\emptyset\}|$, and if $k \in \mathbb{N}$, then denote the k-fold product $|A|\cdots|A|$ by $|A|^k$. It is clear that $|A|^k \cdot |A| = |A|^{k+1}$ for every $0 \leq k \in \mathbb{Z}$. The reader is referred to [26] for a more extensive treatment of set theory.

The next lemma is well known (its first assertion is Lagrange's Theorem for groups that need not be finite), and handles the cardinal arithmetic that is needed in the forthcoming results. We write (AC) in the statement of the lemma to remind the reader that the result relies on the axiom of choice.

Lemma 3.3. (AC) *Suppose that G is a group, and R is a commutative ring with $1 \neq 0$.*

(1) *If H is a subgroup of G, then $|G| = |G/H| \cdot |H|$.*
(2) *If $\mathfrak{m}$ is a principal maximal ideal of R, then $|R/\mathfrak{m}|^k = |R/\mathfrak{m}^k|$ for every $0 \leq k \in \mathbb{Z}$.*

Proof. For every coset $C \in G/H$, designate a unique element $g_C \in C$. Then one easily verifies that the mapping $G/H \times H \to G$ by $(C, h) \mapsto g_C + h$ is bijective. The statement in (2) is trivial if $k = 0$. Suppose that $|R/\mathfrak{m}|^k = |R/\mathfrak{m}^k|$ for some $0 \leq k \in \mathbb{Z}$. The equality $|R/\mathfrak{m}| = |\mathfrak{m}^k/\mathfrak{m}^{k+1}|$ holds since $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is a one-dimensional $R/\mathfrak{m}$ vector space, and therefore $|R/\mathfrak{m}|^{k+1} = |R/\mathfrak{m}|^k \cdot |R/\mathfrak{m}| = |R/\mathfrak{m}^k| \cdot |\mathfrak{m}^k/\mathfrak{m}^{k+1}| = |(R/\mathfrak{m}^{k+1})/(\mathfrak{m}^k/\mathfrak{m}^{k+1})| \cdot |\mathfrak{m}^k/\mathfrak{m}^{k+1}| = |R/\mathfrak{m}^{k+1}|$, where the last equality follows by (1). □

Let $(R, \mathfrak{m})$ be an SPIR. Lemma 2.1(3) reveals the possible connected components of Γ_u for every $u \in U(R)$, and Theorem 3.1 characterizes the structure of Γ_0 (although, see Question 3.2). Now, the graphs Γ_b are investigated for $0 \neq b \in R \setminus U(R)$.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be an SPIR. If $0 \neq b \in \mathfrak{m}$ and $u \in U(R)$, then the connected component of Γ_b containing ub is isomorphic to $K_{1,|R/(b)|}$. Moreover, $\deg_{\Gamma_b}(ub) = |R/(b)|$, and ub is the only vertex of this component that is not a unit of R.*

Proof. Throughout, let $t \in \mathbb{N}$ such that $\mathfrak{m}^t \neq (0) = \mathfrak{m}^{t+1}$. Since $0 \neq b \in R \setminus U(R)$, there exists $k \in \{1, \ldots, t\}$ such that $\mathfrak{m}^k = (b)$.

The inclusion $u^{-1} + \mathfrak{m}^{t+1-k} \subseteq N_{\Gamma_b}(ub)$ is easily verified. Moreover, if $r \in N_{\Gamma_b}(ub)$, then $ru - 1 \in \operatorname{ann}(b) = \mathfrak{m}^{t+1-k}$, and thus $r \in u^{-1} + \mathfrak{m}^{t+1-k}$. This shows that $N_{\Gamma_b}(ub) = u^{-1} + \mathfrak{m}^{t+1-k}$.

By Lemma 2.1(2), every element of $u^{-1} + \mathfrak{m}^{t+1-k} \subseteq U(R)$ has degree equal to 1 in Γ_b. It follows that the connected component of Γ_b containing ub is isomorphic to $K_{1,|u^{-1}+\mathfrak{m}^{t+1-k}|}$, and ub is the only vertex of this component that is not a unit of R. But $|u^{-1} + \mathfrak{m}^{t+1-k}| = |\mathfrak{m}^{t+1-k}| = |\mathfrak{m}^{t+1-k}/\mathfrak{m}^{t+2-k}| \cdot |\mathfrak{m}^{t+2-k}/\mathfrak{m}^{t+3-k}| \cdots |\mathfrak{m}^t| =$

$|R/\mathfrak{m}|^k = |R/\mathfrak{m}^k|$, where the second and fourth equalities hold by Lemma 3.3, and the third equality is verified by noting that $\mathfrak{m}^{t+j-k}/\mathfrak{m}^{t+(j+1)-k}$ is a one-dimensional $R/\mathfrak{m}$-vector space for every $j \in \{1, \ldots, k\}$. □

Remark 3.5.

(1) The necessity of the condition "$b \in R \setminus U(R)$" in Theorem 3.4 is seen by letting $b = u = 1$, and Theorem 3.1 shows that "$b \neq 0$" cannot be omitted.
(2) The connected component of $\Gamma_8(\mathbb{Z}_{16})$ that contains 2 is isomorphic to $K_{2,4}$, which shows that Theorem 3.4 fails if ub is replaced by an arbitrary vertex of Γ_b.
(3) More generally, let $(R,(a))$ be an SPIR with $(a)^t \neq (0) = (a)^{t+1}$ where $2 \leq t \in \mathbb{Z}$, and suppose that $b \in (a)^2$. The statement in (2) can be extended to show that if the connected component of Γ_b containing a has at least three vertices, then this component contains a cycle. Indeed, if this component has at least three vertices, then it contains a vertex $r \in R \setminus \{a, a+a^t\}$ such that either $ra = b$ or $r(a+a^t) = b$. Clearly $r \notin U(R)$ (e.g., since $a \notin (a)^2$), and hence $ra^t = 0$. Thus, $r(a+a^t) = ra = b$, and it follows that a cycle in Γ_b can be found among the distinct elements of $\{a, r, a+a^t, r+a^t\}$.

The next result characterizes the generators of the maximal ideal of an SPIR in terms of x-divisor graphs.

Theorem 3.6. *Let $(R,\mathfrak{m})$ be an SPIR that is not a field. If $a \in R$, then $\mathfrak{m} = (a)$ if and only if every connected component of Γ_a is isomorphic to $K_{1,|R/\mathfrak{m}|}$. In this case, Γ_a is a disjoint union of $|\mathfrak{m} \setminus \mathfrak{m}^2|$ copies of $K_{1,|R/\mathfrak{m}|}$.*

Proof. Throughout, let $t \in \mathbb{N}$ such that $\mathfrak{m}^t \neq (0) = \mathfrak{m}^{t+1}$, and $a \in R$ with $\mathfrak{m} = (a)$. Note that $V(\Gamma_a) = R \setminus \mathfrak{m}^2$, where the inclusion "$\subseteq$" is clear since $a \notin \mathfrak{m}^2$, and "$\supseteq$" follows since every element of $R \setminus \mathfrak{m}^2$ is of the form u or ua for some $u \in U(R)$. Thus, to determine Γ_a, only the elements of $R\setminus\mathfrak{m}$ and $\mathfrak{m}\setminus\mathfrak{m}^2$ need to be considered.

Observe that the subgraph of Γ_a induced by $R \setminus \mathfrak{m} = U(R)$ has no edges since $a \notin U(R)$. But every element of $R\setminus\mathfrak{m}$ has degree equal to 1 in Γ_a by Lemma 2.1(2), so every element of $R\setminus\mathfrak{m}$ belongs to a connected component that contains an element of $\mathfrak{m}\setminus\mathfrak{m}^2$. Since every element of $\mathfrak{m}\setminus\mathfrak{m}^2$ has the form ua for some $u \in U(R)$, Theorem 3.4 implies that every connected component of Γ_a is isomorphic to $K_{1,|R/\mathfrak{m}|}$, and that each of these components contains exactly one member of $\mathfrak{m} \setminus \mathfrak{m}^2$. That is, Γ_a is a disjoint union of $|\mathfrak{m} \setminus \mathfrak{m}^2|$ copies of $K_{1,|R/\mathfrak{m}|}$.

It only remains to verify the "if" part of the first statement. Suppose that $b \in R$ such that $\mathfrak{m} \neq (b)$, i.e., $b \in \mathfrak{m}^2 \cup U(R)$. By Lemma 2.1(3), if $b \in U(R)$, then every connected component of Γ_b is isomorphic to either K_1 or $K_{1,1} \ncong K_{1,|R/\mathfrak{m}|}$. Thus, assume $b \in \mathfrak{m}^2$.

To the contrary, suppose that every connected component of Γ_b is isomorphic to $K_{1,|R/\mathfrak{m}|}$. Since $K_{1,|R/\mathfrak{m}|}$ has at least three vertices, Remark 3.5(3) implies that $t = 1$, and hence $b \in \mathfrak{m}^2 = (0)$. This implies that $\mathfrak{m}$ induces a clique in $\Gamma_0 = \Gamma_b$, and since Γ_b is assumed to have no cycles, it must be the case that $\mathfrak{m} = \{0, a\}$. Therefore, R is isomorphic to either $\mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$, which leads to the contradiction $\Gamma_0 \cong K_{1,3} \ncong K_{1,2} \cong K_{1,|R/\mathfrak{m}|}$. □

Remark 3.7. In Theorem 3.6, it is not enough to stipulate that every connected component of Γ_a is a star graph. For example, if $R = \mathbb{Z}_4$, then (0) and (3) = R are not maximal, but $\Gamma_0 \cong K_{1,3}$ and $\Gamma_3 \cong K_{1,1}$. Also, if $R = \mathbb{Z}_8$, then (4) is not maximal, but $\Gamma_4 \cong K_{1,1} \sqcup K_{1,4}$ (this is an example where the connected component of Γ_b containing a does not have at least three vertices; cf. Remark 3.5(3)).

This section is closed by providing conditions that guarantee a finite local ring is a principal ideal ring. The following lemma is used to accomplish this goal.

Lemma 3.8. *Let R be a commutative ring with $1 \neq 0$. If $a \in R$ and $t \in \mathbb{N}$ such that* $\operatorname{ann}(a) = (a^t)$*, then* $\operatorname{ann}(\operatorname{ann}(a)) = (a)$*. In particular, if $b \in R$ such that $a^t b = 0$, then $b \in (a)$.*

Proof. The containment $(a) \subseteq \operatorname{ann}(\operatorname{ann}(a))$ always holds, so the hypotheses imply that it only remains to verify the "in particular" statement. Note that if $ab = 0$, then $b \in \operatorname{ann}(a) = (a^t) \subseteq (a)$. Hence, the result holds if $t = 1$.

Assume $t > 1$. By way of induction, suppose that $k \in \{1, \ldots, t-1\}$ such that if $b \in R$ with $a^k b = 0$, then $b \in (a)$. Let $b \in R$ such that $a^{k+1} b = 0$. Then $a^k b \in \operatorname{ann}(a) = (a^t)$, so pick $r \in R$ such that $a^k b = a^t r$. Since $a^k(b - a^{t-k} r) = 0$, the induction hypothesis implies that $b - a^{t-k} r \in (a)$. Therefore, $b \in (a)$, and the result follows by induction. □

Theorem 3.9. *A finite commutative local ring $(R, \mathfrak{m})$ with $1 \neq 0$ is a principal ideal ring if and only if either R is a field, or there exists $a \in R$ such that every connected component of Γ_a is isomorphic to $K_{1,|R/\mathfrak{m}|}$.*

Proof. The "only if" statement holds by Theorem 3.6. Conversely, assume that R is not a field, and let $a \in R$ such that every connected component of Γ_a is isomorphic to $K_{1,|R/\mathfrak{m}|}$. Since R is finite, it is enough to show that $\mathfrak{m} = (a)$ (see Section 1.2).

As noted in the proof of Theorem 3.6, $a \notin U(R)$ by Lemma 2.1(3). Also, $a \neq 0$ since Γ_0 is connected on $|R| \neq |R/\mathfrak{m}| + 1$ vertices. Therefore, $a \in \mathfrak{m} \setminus \{0\}$, so there exists $t \in \mathbb{N}$ such that $a^t \neq 0 = a^{t+1}$.

It is clear that $1 + (a^t) \subseteq 1 + \mathrm{ann}(a) \subseteq N_{\Gamma_a}(a)$, and thus $|(a^t)| \leq |\mathrm{ann}(a)| \leq \deg_{\Gamma_a}(a)$. If K is the kernel of the R-module homomorphism $R \to (a^t)$ defined by $r \mapsto a^t r$, then

$$|R/\mathfrak{m}| \leq |R/K| = |(a^t)| \leq |\mathrm{ann}(a)| \leq \deg_{\Gamma_a}(a) \leq |R/\mathfrak{m}|,$$

where the last inequality holds since a belongs to a copy of $K_{1,|R/\mathfrak{m}|}$. Therefore, $(a^t) \subseteq \mathrm{ann}(a)$ have equal finite cardinalities, and hence $\mathrm{ann}(a) = (a^t)$. Similarly, these inequalities imply $\mathfrak{m} = K$. Thus, $a^t\mathfrak{m} = a^t K = (0)$, and the inclusion $\mathfrak{m} \subseteq (a)$ follows by Lemma 3.8. By maximality, $\mathfrak{m} = (a)$. □

Example 3.10. The assumption of finiteness cannot be omitted in Theorem 3.9. For example, $R = \mathbb{Q}[X,Y]/(X,Y)^2$ is a local Artinian ring whose maximal ideal $\mathfrak{m} = (\overline{X}, \overline{Y})$ is not principal. However, if Γ is any connected component of $\Gamma_{\overline{X}}$, then $V(\Gamma) = \{\overline{uX}\} \cup (\overline{u^{-1}} + \mathfrak{m})$ for some $u \in \mathbb{Q}$. In particular, every connected component of $\Gamma_{\overline{X}}$ is isomorphic to $K_{1,|\mathfrak{m}|} = K_{1,|\mathbb{Q}\times\mathbb{Q}|} = K_{1,|\mathbb{Q}|} = K_{1,|R/\mathfrak{m}|}$.

4. Graphs of discrete valuation rings

In the remainder of this paper, the graphs $\Gamma_{\leq x}(G^+)$ are examined for partially ordered abelian groups G, and then results concerning integral domains are given in terms of $\Gamma_x^{\mathcal{C}}(R)$ by using Proposition 2.4(2) with $G = G(R)$ being the group of divisibility of R. In the current section, discrete valuation rings R are characterized in terms of the graphs $\Gamma_x^{\mathcal{C}}(R)$. The following lemma will have an important role.

Lemma 4.1. *If G is a partially ordered abelian group, then $G^+ \cong \mathbb{Z}^+$ (as partially ordered monoids) if and only if there exists a bijection $\psi : \{\Gamma_{[n]} \mid 2 \leq n \in \mathbb{Z}\} \to \{\Gamma_{\leq x}(G^+) \mid 0 \neq x \in G^+$ is nonminimal$\}$ such that $\Gamma_{[n]} \cong \psi(\Gamma_{[n]})$ for every $2 \leq n \in \mathbb{Z}$.*

Proof. Since $\Gamma_{[n]} \cong \Gamma_{\leq n}(\mathbb{Z}^+)$ (Section 1.3), the "only if" statement is trivial. Conversely, suppose that the stated bijection ψ exists. Since $\psi(\Gamma_{[2]}) \cong \Gamma_{[2]} \cong K_1$, let $g \in G^+$ such that $V(\psi(\Gamma_{[2]})) = \{g\}$.

It is enough to prove that $G^+ = \{ng \mid 0 \leq n \in \mathbb{Z}\}$. The inclusion "$\supseteq$" is trivial. For the reverse inclusion, let $x \in G^+$. If $x = 0$, then the result follows by letting $n = 0$. Suppose that $0 \neq x \in G^+$. Since ψ is bijective, $x \in V(\Gamma_{\leq 2x}(G^+)) = V(\psi(\Gamma_{[n]}))$ for some $2 \leq n \in \mathbb{Z}$. Thus, it is sufficient to prove $V(\psi(\Gamma_{[n]})) = \{g, \ldots, (n-1)g\}$ for every $2 \leq n \in \mathbb{Z}$.

It is first shown that $\psi(\Gamma_{[2]}) = \Gamma_{\leq 2g}(G^+)$. To justify this claim, observe that if $y \in G^+$ such that $\psi(\Gamma_{[2]}) = \Gamma_{\leq y}(G^+)$, then the inequalities $0 < g < y$ imply $0 < y - g < y$. Hence, $y - g \in V(\Gamma_{\leq y}(G^+)) = \{g\}$. That is, $y = 2g$.

Proceeding by induction, let $2 \leq n \in \mathbb{Z}$ such that $\psi(\Gamma_{[n]}) = \Gamma_{\leq ng}(G^+)$ and $V(\Gamma_{\leq ng}(G^+)) = \{g, \ldots, (n-1)g\}$. Since $\psi(\Gamma_{[n+1]})$ is the only member of $\{\Gamma_{\leq x}(G^+) \mid 0 \neq x \in G^+$ is nonminimal$\}$ that has order n, it is enough to confirm the equality $V(\Gamma_{\leq (n+1)g}(G^+)) = \{g, \ldots, ng\}$.

The inclusion "$\supseteq$" is clear, so let $h \in V(\Gamma_{\leq (n+1)g}(G^+))$. That is, $0 < h < (n+1)g$. If h and g are adjacent in $\Gamma_{\leq (n+1)g}(G^+)$, then $h + g \leq (n+1)g$, which implies that either $h = ng \in \{g, \ldots, ng\}$, or $0 < h < ng$. In the latter case, $h \in V(\Gamma_{\leq ng}(G^+)) = \{g, \ldots, (n-1)g\} \subseteq \{g, \ldots, ng\}$.

Suppose that h and g are not adjacent in $\Gamma_{\leq (n+1)g}(G^+)$. The desired containment holds if $h = g$, so assume that $h \neq g$. Also, since $h < (n+1)g$, the inequality $h - g < ng$ holds. Thus, to complete the proof, it will be shown that $0 < h - g$. Indeed, this implies that $h - g \in V(\Gamma_{\leq ng}(G^+)) = \{g, \ldots, (n-1)g\}$, and hence $h \in \{2g, \ldots, ng\} \subseteq \{g, \ldots, ng\}$.

It is clear that, for any $2 \leq m \in \mathbb{Z}$, the neighborhoods of any two nonadjacent vertices in $\Gamma_{\leq m}(\mathbb{Z}^+) \cong \Gamma_{[m]}$ are comparable under inclusion. Thus, in $\Gamma_{\leq (n+1)g}(G^+)$ (being in the image of ψ), either $N(h) \subseteq N(g)$, or $N(g) \subseteq N(h)$. In the former case, since $(n+1)g - h$ is adjacent to h, the inequality $((n+1)g - h) + g \leq (n+1)g$ holds. That is, $0 \leq h - g$, and the result follows since it is assumed that $h \neq g$.

Suppose that $N(g) \subseteq N(h)$. Then ng is adjacent to h, and hence $ng + h \leq (n+1)g$, i.e., $h \leq g$. But then $h \in V(\Gamma_{\leq 2g}(G^+)) = \{g\}$ (as shown above), contradicting the assumption that $h \neq g$. □

The following theorem characterizes DVRs in terms of their compressed divisor graphs. It provides a link with compressed 0-divisor graphs of local Artinian principal ideal rings, which is observed in the subsequent corollary.

Theorem 4.2. *An integral domain R is a discrete valuation ring if and only if $\{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ consists precisely (up to isomorphism) of the graphs $\Gamma_{[n]}$ $(2 \leq n \in \mathbb{Z})$. Specifically, R is a DVR if and only if there exists a bijection $\psi : \{\Gamma_{[n]} \mid 2 \leq n \in \mathbb{Z}\} \to \{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ such that $\Gamma_{[n]} \cong \psi(\Gamma_{[n]})$ for every $2 \leq n \in \mathbb{Z}$. In this case, if $\mathfrak{m} = (a)$ is the maximal ideal of R and $x = ua^n$ $(u \in U(R)$, $2 \leq n \in \mathbb{Z})$, then $\Gamma_x^{\mathcal{C}}(R) \cong \Gamma_{[n]}$.*

Proof. Recall that R is a DVR if and only if $G(R)^+ \cong \mathbb{Z}^+$ (see the discussion at the beginning of Section 1.3). Hence, the first claim follows from Proposition 2.4(2) and Lemma 4.1. To verify the last statement of the theorem, let $(R, (a))$ be a DVR.

The mapping $G(R) \to \mathbb{Z}$ by $a^nU(R) \mapsto n$ is an isomorphism of totally ordered abelian groups. Hence, if $2 \leq n \in \mathbb{Z}$, then $\Gamma_{\leq a^nU(R)}(G(R)^+) \cong \Gamma_{\leq n}(\mathbb{Z}^+) \cong \Gamma_{[n]}$. Therefore, if $x = ua^n$ ($u \in U(R)$, $2 \leq n \in \mathbb{Z}$), then Proposition 2.4(2) implies that $\Gamma_x^{\mathcal{C}}(R) = \Gamma_{ua^n}^{\mathcal{C}}(R) \cong \Gamma_{\leq ua^nU(R)}(G(R)^+) = \Gamma_{\leq a^nU(R)}(G(R)^+) \cong \Gamma_{[n]}$. □

Note that for every $0 \leq t \in \mathbb{Z}$, there exists an SPIR with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}^t \neq (0) = \mathfrak{m}^{t+1}$ (e.g., consider the ring $\mathbb{Z}_{2^{t+1}}$). Hence, the next corollary is an immediate consequence of Theorems 3.1 and 4.2.

Corollary 4.3. *Let $\mathscr{C}$ be the set of all (up to graph-isomorphism) compressed* 0-*divisor graphs (as defined prior to Proposition 2.3) of local Artinian principal ideal rings. An integral domain R is a discrete valuation ring if and only if there exists a bijection $\psi : \mathscr{C} \cup \{K_1\} \to \{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ such that $G \cong \psi(G)$ for every $G \in \mathscr{C} \cup \{K_1\}$.*

Remark 4.4.

1. The "$G^+ \cong \mathbb{Z}^+$" part of Lemma 4.1 cannot be extended to "$G \cong \mathbb{Z}$" (although, cf. Theorem 5.8 and Remark 5.9). For example, the relation $(a, b) \leq (x, y)$ if and only if $a \leq x$ and $b = y$ makes $G = \mathbb{Z} \oplus \mathbb{Z}_2$ into a partially ordered abelian group such that $G^+ \cong \mathbb{Z}^+$.
2. Lemma 4.1 does not generalize to arbitrary partially ordered abelian groups. That is, two partially ordered abelian groups G_1 and G_2 can have the "same" graphs $\Gamma_{\leq x}$ even if $G_1^+ \ncong G_2^+$. This claim is illustrated in the next example.

Example 4.5. Let $\mathbb{Q} \subseteq G_1, G_2 \subseteq \mathbb{R}$ be any two nonisomorphic countable subgroups of the partially ordered abelian group $\mathbb{R}$ (e.g., we could have $G_1 = \mathbb{Q}$ and $G_2 = \mathbb{Q} + \mathbb{Q}\sqrt{2}$). Then $G_1^+ \ncong G_2^+$ have the "same" graphs $\Gamma_{\leq x}$; that is, there exists a bijection $\psi : \{\Gamma_{\leq x}(G_1^+) \mid 0 < x \in G_1^+\} \to \{\Gamma_{\leq x}(G_2^+) \mid 0 < x \in G_2^+\}$ such that $\Gamma_{\leq x}(G_1^+) \cong \psi(\Gamma_{\leq x}(G_1^+))$ for every $0 < x \in G_1^+$.

To prove this claim, it is enough verify the stronger assertion that $\Gamma_{\leq x}(G_1^+) \cong \Gamma_{\leq y}(G_2^+)$ for every $0 < x \in G_1^+$ and $0 < y \in G_2^+$. This is easily accomplished by extending any isomorphism $(0, x/2) \to (0, y/2)$ of totally ordered sets (intervals are taken in G_i, as defined in Section 1.1, but such an isomorphism exists, for instance, by Cantor's Isomorphism Theorem) to a function $\varphi : (0, x) \to (0, y)$ by letting $\varphi(x/2) = y/2$, and $\varphi(g) = y - \varphi(x - g)$ for every $g \in (x/2, x)$. One readily checks that φ is an isomorphism of totally ordered sets such that $\varphi(g) = y - \varphi(x - g)$ for every $g \in (0, x)$. Thus, if $a, b \in (0, x)$, then $a + b \leq x$ if and only if $a \leq x - b$, if and only if $\varphi(a) \leq \varphi(x-b) = y - \varphi(b)$, if and only if $\varphi(a)+\varphi(b) \leq y$. Therefore, $\varphi : \Gamma_{\leq x}(G_1^+) \to \Gamma_{\leq y}(G_2^+)$ is an isomorphism of graphs.

Incidentally, the above results address the often-studied problem of providing conditions under which an algebraic object is completely determined (up to isomorphism) by the graph to which it is associated. This problem has been considered extensively (especially for zero-divisor graphs) in the literature (e.g., [2,3,15,17,21,22,24,29]). For example, [3, Theorem 4.1] and [2, Theorem 5] reveal finite commutative rings that are isomorphic whenever their zero-divisor graphs are isomorphic, and a complete characterization is given in [21, Theorem 1.3] for partially ordered sets P such that $P \cong Q$ whenever their zero-divisor graphs (as defined in [23]) are isomorphic.

In the present situation, Lemma 4.1 shows that the partially ordered monoid $\mathbb{Z}^+$ is completely determined (among positive cones of partially ordered abelian groups) by its graphs $\Gamma_{\leq n}(\mathbb{Z}^+)$, while Example 4.5 shows that this result can fail in general. Hence, the following question arises naturally.

Question 4.6. *Let M be a partially ordered commutative monoid such that $x \geq 0$ for every $x \in M$. Under what conditions is M completely determined (as in Lemma 4.1) by the graphs $\Gamma_{\leq x}(M)$?*

5. Graphs of valuation domains

The graphs $\Gamma_x^{\mathcal{C}}(R)$ are now considered for valuation domains R. Throughout, if Γ_1 and Γ_2 are graphs, then we will write $\Gamma_1 \subseteq \Gamma_2$ if Γ_1 is a subgraph of Γ_2.

Let a and b be vertices of a simple graph Γ. Recall from Section 2 that $a \equiv b$ if and only if $N_\Gamma(a)\setminus\{b\} = N_\Gamma(b)\setminus\{a\}$, and the resulting equivalence class containing a is denoted by $[a]$. Define a relation $\leq_{\mathcal{C}}$ on $V(\mathcal{C}(\Gamma))$ by $[a] \leq_{\mathcal{C}} [b]$ if and only if $N_\Gamma(a)\setminus\{b\} \subseteq N_\Gamma(b)\setminus\{a\}$ (equivalently, $N_\Gamma(a)\setminus\{b\} \subseteq N_\Gamma(b)$).

It is easy to check that properties (i) and (ii) below hold for the graphs $\Gamma_{\leq n}(\mathbb{Z}^+)$.

(i) $\Gamma_{\leq 2} \subseteq \Gamma_{\leq 3} \subseteq \cdots$.

(ii) If $n \geq 2$, then $[n-1] \leq_{\mathcal{C}} \cdots \leq_{\mathcal{C}} [1]$ in $V(\mathcal{C}(\Gamma_{\leq n}))$.

In particular, by Theorem 4.2, if R is a DVR, then (i) and (ii) are satisfied by $\Gamma_x^{\mathcal{C}}(R)$ for every $x \in R^\circ$.

In this section, observations (i) and (ii) are generalized to provide a characterization of valuation domains. First, it is shown that the relation $\leq_{\mathcal{C}}$ is a partial order on $V(\mathcal{C}(\Gamma))$, after which some observations are recorded in cases where $\Gamma_x^{\mathcal{C}} \cong \Gamma_y^{\mathcal{C}}$.

Lemma 5.1. *If Γ is a simple graph, then $\leq_{\mathcal{C}}$ is a partial order on $V(\mathcal{C}(\Gamma))$.*

Proof. It is straightforward to check that $\leq_{\mathcal{C}}$ is reflexive and antisymmetric. The following argument that $\leq_{\mathcal{C}}$ is transitive is a close mimicry of the proof given in [4, Theorem 2.1] that shows $\equiv$ is transitive.

Let $a, b, c \in V(\Gamma)$ such that $[a] \leq_{\mathcal{C}} [b]$ and $[b] \leq_{\mathcal{C}} [c]$. The result is easily checked if either $a = b$, $a = c$, or $b = c$. Thus, suppose that a, b, and c are mutually distinct. The assumed relations imply $N_\Gamma(a) \setminus \{b, c\} \subseteq N_\Gamma(b) \setminus \{a, c\} \subseteq N_\Gamma(c) \setminus \{a, b\}$. Hence, the result will follow if the containment $b \in N_\Gamma(a)$ implies $b \in N_\Gamma(c)$.

Let $b \in N_\Gamma(a)$. Then $a \in N_\Gamma(b) \setminus \{c\} \subseteq N_\Gamma(c) \setminus \{b\}$, so $c \in N_\Gamma(a) \setminus \{b\} \subseteq N_\Gamma(b) \setminus \{a\}$. Therefore, $b \in N_\Gamma(c)$. □

The next example observes that an integral domain R may contain nonassociate $x, y \in R^\circ$ that have precisely the same proper divisors (i.e., such that $\Gamma_x^{\mathcal{C}}$ and $\Gamma_y^{\mathcal{C}}$ have precisely the same vertices), and also $\Gamma_x^{\mathcal{C}} \cong \Gamma_y^{\mathcal{C}}$.

Example 5.2. Consider the integral domain $R = \mathbb{Q} + X\mathbb{R}[X]$, and let $x = r_1 X^2$ and $y = r_2 X^2$ for any $r_1, r_2 \in \mathbb{R} \setminus \{0\}$ such that $r_1/r_2 \notin \mathbb{Q}$. Then x and y are nonassociate, but $V(\Gamma_x^{\mathcal{C}}) = V(\Gamma_y^{\mathcal{C}}) = \{aXU(R) \mid a \in \mathbb{R}\setminus\{0\}\}$. Moreover, if $i \in \{1, 2\}$ and $a, b \in \mathbb{R} \setminus \{0\}$ such that $(aX)(bX) \in d(r_i X^2)$, then $r_i \in ab\mathbb{Q}$, so $bXU(R) = a^{-1} r_i XU(R)$. This shows that no vertex of $\Gamma_{r_i X^2}^{\mathcal{C}}$ can have degree greater than one. Since $aXU(R) = a^{-1} r_i XU(R)$ if and only if $a^2 \in r_i U(R) = r_i(\mathbb{Q} \setminus \{0\})$, it follows that $\Gamma_{r_i X^2}^{\mathcal{C}}$ is the disjoint union of a countably infinite set of isolated vertices (of the form $aXU(R)$ where $a^2 \in r_i(\mathbb{Q} \setminus \{0\})$) and $|\mathbb{R}|$ disjoint copies of the complete graph K_2 (each consisting of two vertices $aXU(R)$ and $a^{-1} r_i XU(R)$ where $a^2 \notin r_i \mathbb{Q}$). Hence, $\Gamma_x^{\mathcal{C}} \cong \Gamma_y^{\mathcal{C}}$.

On the other hand, the next two observations show that if $\Gamma_x^{\mathcal{C}} = \Gamma_y^{\mathcal{C}}$, then x and y are necessarily associates.

Theorem 5.3. *Let G be a partially ordered abelian group. If $0 \neq x, y \in G^+$ are nonminimal, then $\Gamma_{\leq x} = \Gamma_{\leq y}$ if and only if $x = y$.*

Proof. The "if" portion is trivial. Conversely, suppose that $\Gamma := \Gamma_{\leq x} = \Gamma_{\leq y}$. If a is an isolated vertex of Γ, then $x = 2a = y$ (otherwise, for example, a is adjacent to $x - a$), and the result follows. Thus, assume that Γ has no isolated vertices.

Let a and b be adjacent vertices of Γ. In particular, $a + b \leq x$ holds by the definition of $\Gamma_{\leq x}$. Notice that $2b > 0$ since $b > 0$, so it cannot happen that both $x = 2(a+b)$ and $x = 2a$. If $x \neq 2(a+b)$, then either $x = a+b$, or $a+b$ and $x-a-b$ are adjacent in $\Gamma = \Gamma_{\leq x}$. Similarly, if $x \neq 2a$, then a and $x - a$ are adjacent in $\Gamma = \Gamma_{\leq x}$. It will be observed that each of these three cases imply $x \leq y$.

Note that $a+b \leq y$ holds by the definition of $\Gamma_{\leq y}$, so the inequality $x \leq y$ is immediate if $x = a+b$. On the other hand, if either $a+b$ and $x-a-b$ are adjacent in $\Gamma = \Gamma_{\leq y}$, or a and $x-a$ are adjacent in $\Gamma = \Gamma_{\leq y}$, then the inequality $x \leq y$ follows again by the definition of $\Gamma_{\leq y}$. By interchanging x and y, a symmetric argument shows that $y \leq x$, and the proof is complete. □

By Proposition 2.4(2) and Theorem 5.3, the next corollary follows immediately.

Corollary 5.4. *Let R be an integral domain. If $x, y \in R^\circ$, then $\Gamma_x^{\mathcal{C}} = \Gamma_y^{\mathcal{C}}$ if and only if x and y are associates.*

The following results generalize the observations given in (i) and (ii) prior to Lemma 5.1, providing a characterization of valuation domains in terms of their compressed divisor graphs. We proceed in the context of partially ordered abelian groups.

Lemma 5.5. *Let G be a partially ordered abelian group. If $0 \neq x, y \in G^+$ are nonminimal such that $x \leq y$, then $\Gamma_{\leq x} \subseteq \Gamma_{\leq y}$.*

Proof. If a and b are adjacent vertices of $\Gamma_{\leq x}$, then $a+b \leq x \leq y$, and it follows that a and b are adjacent vertices of $\Gamma_{\leq y}$. □

Remark 5.6. The converse of Lemma 5.5 can fail. For example, $\Gamma_4^{\mathcal{C}}(\mathbb{Z}) \subseteq \Gamma_6^{\mathcal{C}}(\mathbb{Z})$ but 4 does not divide 6. Equivalently, if G is the group of divisibility of the integral domain $\mathbb{Z}$, then $\Gamma_{\leq 4U(\mathbb{Z})}(G^+) \subseteq \Gamma_{\leq 6U(\mathbb{Z})}(G^+)$ but $4U(\mathbb{Z}) \not\leq 6U(\mathbb{Z})$. On the other hand, we have the following result.

Lemma 5.7. *Let G be a partially ordered abelian group. If the set $\{\Gamma_{\leq x}(G^+) \mid 0 \neq x \in G^+$ is nonminimal$\}$ is totally ordered under $\subseteq$, then the following statements hold.*

1. *G^+ is totally ordered.*
2. *If $0 \neq x, y \in G^+$ are nonminimal, then $x \leq y$ if and only if $\Gamma_{\leq x} \subseteq \Gamma_{\leq y}$.*

Proof. It is first observed that (2) follows from (1). If $x \leq y$, then $\Gamma_{\leq x} \subseteq \Gamma_{\leq y}$ by Lemma 5.5. Conversely, suppose that $\Gamma_{\leq x} \subseteq \Gamma_{\leq y}$. If $\Gamma_{\leq x} = \Gamma_{\leq y}$, then $x = y$ by Theorem 5.3. If $\Gamma_{\leq x} \subsetneq \Gamma_{\leq y}$, then $x \lneq y$ since, otherwise, $y \leq x$ by (1), and thus $\Gamma_{\leq y} \subseteq \Gamma_{\leq x}$ by Lemma 5.5. Hence, it remains to show that (1) holds.

Let a and b be distinct elements of G^+. It is clear that a and b are comparable if $0 \in \{a, b\}$, so assume that $0 \notin \{a, b\}$. The result will be established by considering the case where $2a \neq 2b$, followed by the case where $2a = 2b$.

Suppose that $2a \neq 2b$. By hypothesis, it can be assumed that $\Gamma_{\leq 2a} \subseteq \Gamma_{\leq 2b}$, and either $\Gamma_{\leq a+b} \subseteq \Gamma_{\leq 2b}$ or $\Gamma_{\leq 2b} \subseteq \Gamma_{\leq a+b}$. If $\Gamma_{\leq a+b} \subseteq \Gamma_{\leq 2b}$, then $a+b \leq 2b$ (because

a and b are adjacent vertices of $\Gamma_{\leq a+b}$, and are therefore adjacent vertices of $\Gamma_{\leq 2b}$). Hence, $a \leq b$.

Suppose that $\Gamma_{\leq 2b} \subseteq \Gamma_{\leq a+b}$. The inclusion $\Gamma_{\leq 2a} \subseteq \Gamma_{\leq 2b}$ implies that the vertex a of $\Gamma_{\leq 2a}$ is also a vertex of $\Gamma_{\leq 2b}$. In particular, $a < 2b$, and hence $2b - a > 0$. Since $a \neq 2b - a$ (because $2a \neq 2b$) and $a + (2b - a) = 2b$, it follows that a and $2b - a$ are adjacent vertices of $\Gamma_{\leq 2b}$, and are therefore adjacent vertices of $\Gamma_{\leq a+b}$. Thus, $a + (2b - a) \leq a + b$, i.e., $b \leq a$.

To complete the proof, assume that $2a = 2b$. Then $3a \neq 3b$ since $a \neq b$. By hypothesis, it can be assumed that $\Gamma_{\leq 3a} \subseteq \Gamma_{\leq 3b}$. Hence, the equality $a + 2b = a + 2a = 3a$ implies a and $2b$ are adjacent in $\Gamma_{\leq 3a} \subseteq \Gamma_{\leq 3b}$, so $a + 2b \leq 3b$, i.e., $a \leq b$. But $a \neq b$, so $a \lneq b$, which implies $2a \lneq 2b$. This contradicts that $2a = 2b$, so the case where $2a = 2b$ is void. □

The next theorem is the main result on partially ordered abelian groups.

Theorem 5.8. *The following statements are equivalent for a directed partially ordered abelian group G.*

(1) *G is totally ordered.*
(2) *The set $\{\Gamma_{\leq x}(G^+) \mid 0 \neq x \in G^+$ is nonminimal$\}$ is totally ordered under $\subseteq$.*
(3) *$V(\mathcal{C}(\Gamma_{\leq x}(G^+)))$ is totally ordered under $\leq_{\mathcal{C}}$ for every nonminimal $0 \neq x \in G^+$.*

Proof. Note that (1) implies (2) by Lemma 5.5. To show that (2) implies (3), let $0 \neq x \in G^+$ be nonminimal, and let a and b be vertices of $\Gamma_{\leq x}$. Lemma 5.7 shows that the inequality $b \leq a$ can be assumed. Therefore, if $q \in N_{\Gamma_{\leq x}}(a) \setminus \{b\}$, then $q+b \leq q+a \leq x$, and hence $q \in N_{\Gamma_{\leq x}}(b) \setminus \{a\}$. Thus, $N_{\Gamma_{\leq x}}(a) \setminus \{b\} \subseteq N_{\Gamma_{\leq x}}(b) \setminus \{a\}$, i.e., $[a] \leq_{\mathcal{C}} [b]$.

It remains to verify that (3) implies (1). Since G is directed, it is sufficient to prove that G^+ is totally ordered (because if $a, b \in G$ and $c \in G$ with $c - a, c - b \geq 0$, then the total order on G^+ implies the inequality $c - b \leq c - a$ can be assumed, so that $a \leq b$). Hence, let $a, b \in G^+$. If $0 \in \{a, b\}$, then a and b are obviously comparable, so assume that $0 \notin \{a, b\}$. Also, the result is trivial if $b = 2a$, so assume that $b \neq 2a$.

Set $x = b + 2a$. If $3a \leq x$, then $a \leq b$, so assume that $3a \not\leq x$. Then $2a \in N_{\Gamma_{\leq x}}(b) \setminus \{a\}$ while $2a \notin N_{\Gamma_{\leq x}}(a)$. Hence, the hypothesis implies that $N_{\Gamma_{\leq x}}(a) \setminus \{b\} \subseteq N_{\Gamma_{\leq x}}(b) \setminus \{a\}$. Then $a + b \in N_{\Gamma_{\leq x}}(a) \setminus \{b\} \subseteq N_{\Gamma_{\leq x}}(b) \setminus \{a\}$, which implies that $a + 2b \leq x$. That is, $a + 2b \leq b + 2a$, i.e., $b \leq a$. □

Remark 5.9. The proof of Theorem 5.8 shows that statements (2) and (3) are both equivalent to the condition "G^+ is totally ordered" even if G is not directed. The "directed" hypothesis of Theorem 5.8 is necessary only to extend the condition in (1) from G^+ to all of G. Indeed, it is necessary; e.g., as noted in Remark 4.4(1), if $G = \mathbb{Z} \times \mathbb{Z}_2$ with $(a, b) \leq (c, d)$ if and only if $a \leq c$ and $b = d$, then $G^+ \cong \mathbb{Z}^+$ even though G is not totally ordered.

Recall that an integral domain R is a valuation domain if and only if $G(R)$ is totally ordered. Hence, the following characterization of valuation domains is an immediate corollary of Theorem 5.8 and Proposition 2.4(2).

Corollary 5.10. *The following statements are equivalent for an integral domain R.*

(1) *R is a valuation domain.*
(2) *$\{\Gamma_x^{\mathcal{C}}(R) \mid x \in R^\circ\}$ is totally ordered under $\subseteq$.*
(3) *$V(\mathcal{C}(\Gamma_x^{\mathcal{C}}))$ is totally ordered under $\leq_{\mathcal{C}}$ for every $x \in R^\circ$.*

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References

[1] M. Abdulla and A. Badawi, *On the dot product graph of a commutative ring II*, Int. Electron. J. Algebra, 28 (2020), 61-74.
[2] S. Akbari and A. Mohammadian, *On the zero-divisor graph of a commutative ring*, J. Algebra, 274 (2004), 847-855.
[3] D. F. Anderson, A. Frazier, A. Lauve and P. S. Livingston, *The zero-divisor graph of a commutative ring, II*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, New York, Basel, 220 (2001), 61-72.
[4] D. F. Anderson and J. D. LaGrange, *Some remarks on the compressed zero-divisor graph*, J. Algebra, 447 (2016), 297-321.
[5] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra, 217(2) (1999), 434-447.
[6] G. Arunkumar, P. J. Cameron, T. Kavaskar and T. Tamizh Chelvam, *Induced subgraphs of zero-divisor graphs*, Discrete Math., 346(10) (2023), 113580 (9 pp).

[7] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[8] M. Axtell and J. Stickles, *Irreducible divisor graphs in commutative rings with zero divisors*, Comm. Algebra, 36(5) (2008), 1883-1893.

[9] I. Beck, *Coloring of commutative rings*, J. Algebra, 116(1) (1988), 208-226.

[10] M. Behboodi and Z. Rakeei, *The annihilating-ideal graph of commutative rings II*, J. Algebra Appl., 10(4) (2011), 741-753.

[11] J. Coykendall and J. Maney, *Irreducible divisor graphs*, Comm. Algebra, 35(3) (2007), 885-895.

[12] F. R. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup*, Semigroup Forum, 65(2) (2002), 206-214.

[13] R. Halaš and M. Jukl, *On Beck's coloring of posets*, Discrete Math., 309(13) (2009), 4584-4589.

[14] T. W. Hungerford, *On the structure of principal ideal rings*, Pacific J. Math., 25 (1968), 543-547.

[15] V. Joshi and A. Khiste, *The zero divisor graphs of Boolean posets*, Math. Slovaca, 64(2) (2014), 511-519.

[16] C. F. Kimball and J. D. LaGrange, *The idempotent-divisor graphs of a commutative ring*, Comm. Algebra, 46(9) (2018), 3899-3912.

[17] J. D. LaGrange, *Complemented zero-divisor graphs and Boolean rings*, J. Algebra, 315(2) (2007), 600-611.

[18] J. D. LaGrange, *The x-divisor pseudographs of a commutative groupoid*, Int. Electron. J. Algebra, 22 (2017), 62-77.

[19] J. D. LaGrange, *Divisor graphs of a commutative ring*, in Advances in Commutative Algebra, Trends in Mathematics, Birkhäuser/Springer, Singapore, (2019), 217-244.

[20] J. D. LaGrange, *Divisor graphs and isotopy invariants of commutative quasigroups*, J. Combin. Theory Ser. A, 187 (2022), 105577 (16 pp).

[21] J. D. LaGrange, *Intervals of posets of a zero-divisor graph*, Math. Slovaca, 74(4) (2024), 803-818.

[22] D. Lu and T. Wu, *The zero-divisor graphs which are uniquely determined by neighborhoods*, Comm. Algebra, 35(12) (2007), 3855-3864.

[23] D. Lu and T. Wu, *The zero-divisor graphs of posets and an application to semigroups*, Graphs Combin., 26(6) (2010), 793-804.

[24] A. Mohammadian, *On zero-divisor graphs of Boolean rings*, Pacific J. Math., 251(2) (2011), 375-383.

[25] C. P. Mooney, *τ-Irreducible divisor graphs in commutative rings with zero-divisors*, Int. Electron. J. Algebra, 20 (2016), 45-69.

[26] J. D. Monk, Introduction to Set Theory, McGraw-Hill Book Co., New York-London-Sydney, 1969.

[27] S. B. Mulay, *Cycles and symmetries of zero-divisors*, Comm. Algebra, 30(7) (2002), 3533-3558.

[28] A. Sharma and D. K. Basnet, *Nil clean divisor graph*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 41(1) (2021), 146-153.

[29] M. Tavakkoli, A. Borumand Saeid and N. Shajareh Poursalavati, *Classification of posets using zero-divisor graphs*, Math. Slovaca, 68(1) (2018), 21-32.

John D. LaGrange
School of Mathematics and Sciences
Lindsey Wilson College
Columbia, KY 42728-1223
e-mail: lagrangej@lindsey.edu