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ON SEMIPERFECT F-INJECTIVE RINGS

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Abstract. A ring R is called right F-injective if every right R-homomorphism from a finitely generated right ideal of R to R extends to an endomorphism of R. R is called a right FSE-ring if R is a right F-injective semiperfect ring with essential right socle. The class of right FSE-rings is broader than that of right PF-rings. In this paper, we study and provide some characterizations of this class of rings. We prove that if R is left perfect, right F-injective, then R is QF if and only if R/S is left finitely cogenerated where $S = S_r = S_l$ if and only if R is left semiartinian, $Soc_2(R)$ is left finitely generated. It is also proved that R is QF if and only if R is left perfect, mininjective and $J^2 = r(I)$ for a finite subset I of R. Some known results are obtained as corollaries.

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1. Introduction

Throughout the paper, R represents an associative ring with identity $1 \neq 0$ and all modules are unitary R-modules. We write M_R (resp., ${}_RM$) to indicate that M is a right (resp., left) R-module. We also write J (resp., Z_r, S_r) for the Jacobson radical (resp., the right singular ideal, the right socle of R) and $E(M_R)$ for the injective hull of M_R. If X is a subset of R, the right (resp., left) annihilator of X in R is denoted by $r_R(X)$ (resp., $l_R(X)$) or simply $r(X)$ (resp., $l(X)$) if no confusion appears. If N is a submodule of M (resp., proper submodule) we denote by $N \leq M$ (resp., $N < M$). Moreover, we write $N \leq^e M$, $N \leq^{\oplus} M$ and $N \leq^{max} M$ to indicate that N is an essential submodule, a direct summand and a maximal submodule of M, respectively. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M.

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A ring R is called right F-injective (resp., P-injective) if every homomorphism from a finitely generated (resp., principal) right ideal to R is given by left multiplication by an element of R. R is called right mininjective if every homomorphism from a minimal right ideal to R is given by left multiplication by an element of R. R is called right Kasch if every simple right R-module embeds in R; or equivalently, $l(I) \neq 0$ for every maximal right ideal I of R. A ring R is called a QF-ring if it is right (or left) artinian and right (or left) self-injective. R is said to be a right PF-ring if the right R_R is an injective cogenerator in the category of right R-modules.

We refer to the following conditions on a module M_R:

C1: Every submodule of M is essential in a direct summand of M.

C2: Every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

C3: $M_1 \oplus M_2$ is a direct summand of M for any two direct summand M_1, M_2 of M with $M_1 \cap M_2 = 0$.

Module M_R is called extending (or CS) (resp., continuous) if it satisfies C1 (resp., both C1 and C2). R is called right CS (resp., continuous) if R_R is a CS-module (resp., continuous). A module M is called finitely continuous if it satisfies C2 and every finitely generated submodule is essential in a summand of M. R is called right finitely continuous if R_R is a finitely continuous module. A module M is called min-CS if every simple submodule is essential in a summand of M (see [24]). R is called right min-CS if R_R is a min-CS module. R is called a right IN-ring if $l(A \cap B) = l(A) + l(B)$ for all right ideals A and B of R.

General background material can be found in [1], [6], [7] and [24].

In this paper, we consider a generalization of right PF, namely the class of semiperfect right F-injective rings with essential right socle (called right FSE). We provide example of right FSE which are not right PF. Several characterizations of right FSE rings are provided. For instance, it is shown that R is right FSE if and only if R is left min-CS, right Kasch and right F-injective if and only if R is right finitely continuous, right finitely cogenerated, and right F-injective. In [5], Chen, Ding and Yousif proved that if R is left perfect, left and right F-injective then R is QF if and only if R/S is left finitely cogenerated where $S = S_r = S_l$ if and only if R is right perfect, $Soc_2(R)$ is left finitely generated. In this paper, we will prove that R is left perfect, right F-injective then R is QF if and only if R/S is left finitely cogenerated where $S = S_r = S_l$ if and only if R is left semiartinian and $Soc_2(R)$ is left finitely generated.

2. On semiperfect F-injective rings.

Lemma 2.1. *Let R be a right Kasch, right F-injective ring. Then*

(1) $rl(I) = I$ *for every right finitely generated ideal I of R. In particular, R is left P-injective.*
(2) $S_r = S_l$ *is essential in* ${}_RR$.
(3) $l(J)$ *is an essential left ideal.*
(4) $J = r(S) = rl(J)$, *where* $S = S_r = S_l$.
(5) $Z_r = Z_l = J$.
(6) xR *is minimal if and only if Rx is minimal, for every* $x \in R$.
(7) *Minimal left and right ideal are annihilators.*
(8) *The map* $K \longmapsto r(K)$ *gives a bijection from the set of all minimal left ideals of R onto the set of all maximal right ideals of R. The inverse map is defined by* $I \longmapsto l(I)$.
(9) *If* $l(T) = l(S)$, *where T and S are right ideals, with T is finitely generated, then* $T = S$.
(10) *If T_R is a finitely generated right ideal, and $l(T)$ is small in ${}_RR$, then T_R is essential in R_R.*
(11) $r(Rb \cap l(T)) = r(b) + T$ *for every finitely generated right ideal T of R and every* $b \in R$.

Proof. (2), (3), (4), (5), (6) by [4, Theorem 2.3].
(1). Let T be a right finitely generated ideal of R. Always $T \le rl(T)$. If $b \in rl(T)\backslash T$ let $T \le I \le^{max} (bR + T)$. Since R is right Kasch, we can find a monomorphism $\sigma : (bR + T)/I \to R$, and then define $\gamma : bR + T \to R$ via $\gamma(x) = \sigma(x + I)$. Since $bR + I$ is a right finitely generated ideal of R and R is right F-injective, it follows that $\gamma = c.$, where $c \in R$. Hence $cb = \sigma(b + I) \ne 0$ because $b \notin I$. But if $t \in T$ then $ct = \sigma(t + I) = 0$ because $T \le I$, so $c \in l(I)$. Since $b \in rl(T)$ this gives $cb = 0$, a contradiction. Thus $T = rl(T)$. It is clearly that R is left P-injective.
(7). Obvious because R is left and right mininjective.
(8). Let $K = Ra$ be a minimal left ideal. Then aR is a minimal right ideal, and so $r(K) = r(a)$ is a maximal right ideal. Clearly, $K = lr(K)$ since K is an annihilator. Note that R is right Kasch and right F-injective. Thus for all maximal right ideal T, $T = rl(T)$. So (8) follows.
(9). First $S \le rl(S) = rl(T) = T$ by (1). If $S < T$, by the same argument as in (1), we receive the contradiction. Thus $T = S$.
(10). Let $T \cap aR = 0$, where $a \in R$. Since R is right F-injective, $R = l(T \cap aR) =$

$l(T) + l(a)$. Thus $l(a) = R$ by hypothesis, so $a = 0$.
(11). Clear from (1). □

Recall that if M is a module, the submodules $Soc_1(M) \leq Soc_2(M) \leq ...$ are defined by setting $Soc_1(M) = Soc(M)$ and, if $Soc_n(M)$ has been specified, by $Soc_{n+1}(M)/Soc_n(M) = Soc(M/Soc_n(M))$.

Lemma 2.2. *Let R be a semilocal, right Kasch, right F-injective ring. Then*

(1) *R is left and right Kasch.*
(2) *R is left and right finitely cogenerated.*
(3) *$Soc_n(R_R) = Soc_n(_RR) = l(J^n) = r(J^n)$ for all $n \geq 1$.*

Proof. By Lemma 2.1, R is left and right P-injective. Then by [15, Lemma 5.49], R is right and left Kasch. Thus by [4, Theorem 2.8], R is left and right finitely cogenerated. Thus (3) follows from [15, Lemma 3.36]. □

Corollary 2.3. *Assume that R is a semiperfect, right F-injective ring in which $Soc(eR) \neq 0$ for every local idempotent e of R. Then*

(1) *$rl(I) = I$ for every right finitely generated ideal I of R.*
(2) *$S_r = S_l = S$ is essential in R_R and in $_RR$.*
(3) *$Soc(eR) = eS$ and $Soc(Re) = Se$ are simple for every local idempotent $e \in R$.*
(4) *If $e_1, ..., e_n$ are basic local idempotents, then $\{e_1S..., e_nS\}$ and $\{Se_1..., Se_n\}$ are systems of distinct representatives of the simple right and left R-modules, respectively.*
(5) *$Z_r = Z_l = J$.*
(6) *R is left and right finite dimensional.*

Proof. By the hypothesis R is right minfull (i.e., R is a semiperfect, right mininjective ring in which $Soc(eR) \neq 0$ for every local idempotent e of R), and then it is a right Kasch ring by [17, Theorem 3.7]. Hence by Lemma 2.1 and Lemma 2.2 we have (1), (2), (5) and (6). Thus R is also left minfull. It implies that $Soc(Re) = Se$ and $Soc(eR) = eS$ are simple for every local idempotent $e \in R$ by [17, Theorem 3.7], proving (3).
(4) follows from [17, Theorem 3.7 (7), (8)]. □

Corollary 2.4. *The following conditions are equivalent for a ring R.*

(1) *R is left finitely cogenerated, right Kasch and right F-injective.*
(2) *R is left finite dimensional, right Kasch and right F-injective.*

(3) *R is right Kasch, right F-injective and S_l is left finitely generated.*

(4) *R is semilocal, right Kasch and right F-injective.*

(5) *R is semilocal, right F-injective and $J = r(k_1, ..., k_n)$, where $k_i \in R$, $i = 1, ..., n$.*

(6) *R is right finitely cogenerated, right Kasch and right F-injective.*

(7) *R is right finite dimensional, right Kasch and right F-injective.*

(8) *R is right Kasch, right F-injective and S_r is a finitely generated left ideal.*

Proof. (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3). Obvious.
(3) $\Rightarrow$ (4). By the hypothesis and Lemma 2.1, $S_r = S_l$, say $S_r = Ra_1 \oplus ... \oplus Ra_n$ where Ra_i is a minimal left ideal of R, so a_iR is a minimal right ideal of R by Lemma 2.1(6) i.e., $r(a_i)$ is a maximal right ideal of R for all $i = 1, 2, ..., n$. Since R is right Kasch, $J = r(S_r) = \bigcap_{i=1}^{n} r(a_i)$. We construct a homomorphism
$\varphi : R/J = R/\bigcap_{i=1}^{n} r(a_i) \longrightarrow \bigoplus_{i=1}^{n} R/r(a_i)$ defined by $\varphi(r + \bigcap_{i=1}^{n} r(a_i)) = (r + r(a_i))_{i=1}^{n}$ for all $r \in R$. Then φ is a monomorphism. Hence R/J is semisimple or R is semilocal.
(4) $\Rightarrow$ (1), (6) and (8) by Lemma 2.2.
(4) $\Rightarrow$ (5). By Lemma 2.2 and R is semilocal.
(5) $\Rightarrow$ (4). By [5, Corollary 3.2].
(6) $\Rightarrow$ (7). Clearly.
(7) $\Rightarrow$ (4) follows because a right P-injective and right finite dimensional is semilocal by [16, Theorem 3.3].
(8) $\Rightarrow$ (4). By the same an argument as (3) $\Rightarrow$ (4). □

Next we consider ring which is semiperfect, right F-injective with essential right socle.

Theorem 2.5. *The following conditions are equivalent for a ring R.*

(1) *R is semiperfect, right Kasch and right F-injective.*

(2) *R is semiperfect, right F-injective and $S_r \leq^e R_R$.*

(3) *R is semiperfect, right F-injective and $S_r \leq^e {}_RR$.*

(4) *R is right finitely continuous, right finitely cogenerated and right F-injective.*

(5) *R is right min-CS, right finitely cogenerated and right F-injective.*

(6) *R is left min-CS, right Kasch and right F-injective.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). By the same argument as in [5, Theorem 3.3].
(3) $\Rightarrow$ (4). By [15, Theorem 1.48], R is right Kasch, and so by Lemma 2.1 and

Lemma 2.2, $S_l \leq^e R_R$ and $S_r \leq^e {}_RR$. Hence $rl(I)$ is essential in a summand of R_R for every right ideal I of R by [15, Lemma 4.2]. Furthermore, every right finitely generated ideal is a right annihilator by Lemma 2.1. Thus R is right finitely continuous.

(4) $\Rightarrow$ (5) is clearly. □

Definition 2.6. *A ring R is called a right FSE-ring if it satisfies the equivalent conditions in Theorem 2.5.*

Remark 2.7. *If R is right PF (i.e., R is semiperfect, right self-injective and $S_r \leq^e R_R$), then R is right FSE, however the converse is not true in general. There is a commutative FSE-ring which is not PF ([15, Example 5.45]): Let $R = F[x_1, x_2...]$, where F is a field and x_i are commuting indeterminants satisfying the relations: $x_i^3 = 0$ for all i, $x_ix_j = 0$ for all $i \neq j$, and $x_i^2 = x_j^2$ for all i and j. Then R is a commutative, semiprimary F-injective ring. But R is not a self-injective ring.*

A right R-module M is said to be F-injective if each R-momorphism $f : I \longrightarrow M$ from a right finitely generated ideal I into M extends to R; equivalently, $f = m.$, for some $m \in M$.

It is well-known that R is a right PF-ring iff $R = \bigoplus_{i=1}^{n} e_iR$ where $e_i^2 = e_i \in R$ and each e_iR is indecomposable injective with essential simple socle. We also have:

Theorem 2.8. *The following conditions are equivalent for ring R.*

(i) *R is right FSE.*

(ii) *R_R is finite direct sum $R = \bigoplus_{i=1}^{n} e_iR$ where $e_i^2 = e_i \in R$ and each e_iR is indecomposable F-injective with essential simple socle.*

Proof. (i) $\Rightarrow$ (ii). Since R is right FSE, $R = e_1R \oplus ... \oplus e_nR$ where $\{e_1, e_2, ..., e_n\}$ is a set of orthogonal local idempotents in R and each e_iR is indecomposable. But R is right finitely continuous, each e_iR is uniform. Hence $Soc(e_iR)$ is essential simple in e_iR for all $i = 1, 2, ..., n$ (because $Soc(e_iR) \neq 0$). Now we will prove that e_iR_R is F-injective. In fact, for every $i \in \{1, 2, ..., n\}$ we consider the following diagram:

$$\begin{array}{ccccc} I & \xrightarrow{f} & e_iR & \overset{\iota_i}{\underset{p_i}{\rightleftarrows}} & R \\ & \underset{\iota}{\searrow} & \bar{f}_i \uparrow & \underset{\bar{f}}{\nearrow} & \\ & & R & & \end{array}$$

with I_R is a finitely generated right ideal of R, ι, ι_i are the canonical inclusions and p_i is the canonical projection. Since R is right F-injective, there exists $\bar{f} : R \longrightarrow R$

such that $\bar{f} \circ \iota = \iota_i \circ f$. Let $\bar{f}_i = p_i \circ \bar{f}$. Therefore, $\bar{f}_i(x) = p_i \circ \bar{f}(x) = p_i \circ \iota_i \circ f(x) = f(x)$ for all $x \in I$. Thus e_iR_R is F-injective for each $i = 1, 2, ..., n$.

(ii) $\Rightarrow$ (i). Assume that $R = \bigoplus_{i=1}^{n} e_iR$ where $e_i^2 = e_i \in R$ and each e_iR is indecomposable F-injective with essential simple socle. We consider the following diagram:

$$\begin{array}{ccccc} I & \xrightarrow{f} & R & \xrightarrow{p_i} & e_iR \\ & \searrow_{\iota} & \bar{f}\uparrow & \nearrow^{\bar{f}_i} & \\ & & R & & \end{array}$$

with I_R is a finitely generated right ideal of R, ι, ι_i are the canonical inclusions and p_i is the canonical projection. Since each e_iR_R is F-injective, there is $\bar{f}_i : \ R \longrightarrow e_iR$ with $\bar{f}_i \circ \iota = p_i \circ f$ for each $i = 1, 2, ..., n$. Let $\bar{f} = \bigoplus_{i=1}^{n} \bar{f}_i : R \longrightarrow R$ via $\bar{f}(x) = \sum_{i=1}^{n} \bar{f}_i(x)$, for all $x \in R$. Then for all $x \in I$, $\bar{f}(x) = \sum_{i=1}^{n} \bar{f}_i(x) = \sum_{i=1}^{n} p_i(f(x)) = f(x)$. Thus R is right F-injective.

Now we will prove that R is semiperfect. In fact, let $0 \neq K \leq e_iR$. Since $Soc(e_iR) \leq^e e_iR$, $K \cap Soc(e_iR) \neq 0$. Furthermore, $Soc(e_iR)$ is simple. So $Soc(e_iR) = K \cap Soc(e_iR)$ i.e., $Soc(e_iR) \leq K$, from this it implies that $K \leq^e e_iR$. Hence e_iR is uniform for all $i = 1, 2, \ldots, n$.

Note that R is right C2-ring (since R is right F-injective) and R is finite direct sum of uniform right ideals. Thus, R is semiperfect by [15, Lemma 4.26]. On the other hand, $Soc(R_R) = \bigoplus_{i=1}^{n} Soc(e_iR) \leq^e \bigoplus_{i=1}^{n} e_iR = R$. Thus R is right FSE by Theorem 2.5. □

Proposition 2.9. *If R is a right FSE ring and R/S_r is right Goldie, then R is QF.*

Proof. Since R is right FSE, R is left and right mininjective. Thus by [17, Proposition 4.7] R is QF. □

Corollary 2.10. *If R is a right FSE-ring satisfying ACC on essential right ideals, then R is QF.*

Proof. Since R has ACC on essential right ideals, R/S_r is right noetherian by [6, 5.15]. Hence R/S_r is right Goldie. □

Corollary 2.11. *If R is a right F-injective, right CS ring and R/S_r is right Goldie, then R is QF.*

Proof. By the hypothesis, R is right continuous. So R is semiprimary by [22, Theorem]. Therefore, R is a right FSE ring. Thus R is QF.

A right FSE-ring is a left P-injective ring. Rutter ([20, Example 1]) has an example of a left P-injective ring satisfying ACC on left annihilators but not left F-injective, quasi-Frobenius. But the following proposition show that a right FSE-ring satisfying ACC on left annihilators is QF. □

Proposition 2.12. *Let R be a right FSE ring satisfying ACC on left annihilators. Then R is QF.*

Proof. By the hypothesis, R is right and left P-injective. So by [15, Proposition 5.15], R is left artinian. Hence R is QF by [21, Corollary 3] or [19, Theorem 2.7]. □

A ring R is called left semiartinian if every nonzero right R-module has an essential socle.

Now we consider a right F-injective, left perfect ring in which $Soc_2(R)$ is left finitely generated or R/S is left finitely cogenerated where $S = S_r = S_l$.

Lemma 2.13. (Osofsky's Lemma) *If R is a left perfect ring in which J/J^2 is right finitely generated, then R is right artinian.*

From the Osofsky's Lemma we have the following theorem extends the work in [5, Theorem 3.9 (2), (3)] and [15, Theorem 5.66 (2), (3)]

Theorem 2.14. *Let R be a left perfect, right F-injective ring. Then*

(1) *R is QF if and only if R/S is a finitely cogenerated left R-module where $S = S_r = S_l$.*

(2) *R is QF if and only if R is left semiartinian and $Soc_2(R)$ is a finitely generated left R-module.*

Proof. (1). Clearly R is right FSE, R is left and right Kasch and $S = S_r = S_l$. So $S = l(J)$ and $J = r(S_r) = r(S)$ (because R is semilocal and right Kasch). Hence $S = lr(S)$, then by [15, Lemma 1.40] R/S is torsionless as a left R-module. From this and the hypothesis, there exists a monomorphism $\phi : R/S \longrightarrow R^n$ for some positive integer n. Let $\phi(1 + S) = (a_1, ..., a_n)$, then $S = l(a_1, ..., a_n)$.

We have $J = r(S) = rl(a_1, ..., a_n) = a_1R + ... + a_nR$ by Lemma 2.1(1). Hence J_R is finitely generated. So is J/J^2. Therefore, R is right artinian by Lemma 2.13. Thus R is QF by Proposition 2.12.

(2). Since R is left semiartinian, R/S has an essential left socle. Note that $Soc(R/S) = Soc_2(R)/S$. If $Soc_2(R)$ is finitely generated left R-module, so is $Soc(R/S)$. Hence R/S is left finitely cogenerated. Thus by (1) R is QF. □

Corollary 2.15. [5, Corollary 3.10 (2), (3)] *Let R be a left perfect, right P-injective and right IN ring. Then*

(1) *R is QF if and only if R/S is a finitely cogenerated left R-module where $S = S_r = S_l$.*
(2) *R is QF if and only if R is left semiartinian and $Soc_2(R)$ is a finitely generated left R-module.*

The following theorem extends the work in [9, Theorem 2.7].

Theorem 2.16. *Let R be a left perfect, right F-injective ring. If $J^2 = r(A)$ for a finite subset A of R, then R is QF.*

Proof. Since R is semilocal, J/J^2 is a semisimple right R/J-module. Hence J/J^2 is a semisimple right R-module.

Let $J^2 = r(a_1, ..., a_n)$ where $a_i \in R,\ i = 1, 2, ..., n$. Define

$$\phi :\ R/J^2 \longrightarrow R^n,\ \text{via } \phi(a + J^2) = (a_1a, ..., a_na) \text{ for all } a \in R.$$

Then ϕ is a monomorphism. Hence way may regard J/J^2 as a submodule of R^n.

We have $J/J^2 = Soc(J/J^2) \leq Soc(R^n_R) = (Soc(R_R))^n = S^n_r$. On the other hand, R is right FSE, S_r is right finitely generated, so is $(S_r)^n$, as a direct summand of $(S_r)^n$, J/J^2 is right finitely generated. By Lemma 2.13, R is left artinian. Thus R is QF by Proposition 2.12. □

Corollary 2.17. [9, Theorem 2.7] *If R is left perfect, right self-injective, and if J^2 is the right annihilator of a finite subset of R, then R is QF.*

A ring R is called a right CPA-ring if every cyclic right R-module is a direct sum of a projective and an artinian module (see [13]).

Theorem 2.18. *If R is a right P-injective, right CPA-ring, then R is right artinian.*

Proof. By [13, Theorem 2.1], R has a direct decomposition

$$R_R = A \oplus U^{(1)} \oplus ... \oplus U^{(n)}$$

where A is an ideal of R such that A_R is artinian and each $U^{(i)}$ is a uniform right R-module with $Soc(U^{(i)}_R) = 0$. We will prove that $U^{(i)} = 0$ for every i. Assume on the contrary that $U^{(i)} \neq 0$ for some i. Take $0 \neq x \in U^{(i)}$, then $xR = P_R \oplus B_R$ where P_R is projective and B_R is artinian; however $Soc(xR) = 0$, it follows that $B = 0$. i.e., xR is projective. Then by [16, Corollary 1.2], xR is a direct summand of R. So $R = xR \oplus I$ where $I \leq R_R$. Therefore

$$U^{(i)} = (xR \oplus I) \cap U^{(i)} = xR \oplus (I \cap U^{(i)}).$$

On the other hand, since $xR \neq 0$ and $U^{(i)}$ is uniform, $I \cap U^{(i)} = 0$. So $U^{(i)} = xR$ for each $0 \neq x \in U^{(i)}$, showing $U^{(i)}$ is simple, a contradiction to $Soc(U^{(i)}) = 0$. Hence $U^{(i)} = 0,\ i = 1, 2, ..., n$. It implies that $R = A$ i.e., R is right artinian. □

Corollary 2.19. *If R is a right F-injective, right CPA-ring, then R is QF.*

Now we provide a generalization of Theorem 2.16 and [5, Corollary 2.10(2)].

Theorem 2.20. *Let R be a left perfect, left and right mininjective ring in which $J^2 = r(A)$ for a finite subset A of R, then R is QF.*

Proof. By using technique of proving Theorem 2.16, we have $J/J^2 = Soc(J/J^2) \leq Soc(R_R^n) = (Soc(R_R))^n = S_r^n$. On the other hand, R is right minfull by hypothesis. So $S_r = S_l \leq^e R_R$. For each local idempotent $e \in R$, $Soc(Re) = S_l \cap Re = S_r \cap Re = S_r e \neq 0$ by [15, Theorem 3.12]. Hence R is left minfull. It follows that S_r is right finitely generated by [15, Proposition 3.17], so is $(S_r)^n$, as a direct summand of $(S_r)^n$, J/J^2 is right finitely generated. By Lemma 2.13, R is left artinian. Thus R is QF. □

Recall that a ring R is called right pseudo-coherent if $r(S)$ is finitely generated for every finite subset S of R.

Theorem 2.21. *Assume that R is a left perfect, left and right mininjective ring. If R is right (or left) pseudo-coherent, then R is QF.*

Proof. If R is left perfect, left and right mininjective ring, then R is right minfull and $S_r = S_l$. On the other hand, for each local idempotent $e \in R$ then $Soc(eR) = S_r \cap eR = S_l \cap eR = eS_l$. From this and [15, Theorem 3.12], it implies that R is left and right minfull. Thus $S = S_r = S_l$ is a finitely generated left and right ideal and R is left and right Kasch.

Clearly, $J \leq l(S)$. Let M be any maximal left ideal. Then $R/M \overset{\phi}{\cong} I$ where I is a minimal left ideal since R is left Kasch. If $x \in l(S)$, then $xI \leq xS = 0$. Thus $0 = xI = x\phi(R/M) = \phi(x(R/M))$, and so $x(R/M) = 0$, whence $x \in M$. Therefore it follows that $l(S) \leq J$, and hence $J = l(S)$. Similarly, we have $J = r(S)$ because R is right Kasch. By hypothesis, R is left (or right) pseudo-coherent, and so J is a finitely generated left (or right) ideal. If J is a finitely generated right R-module, then J/J^2 is too. Consequently, R is right artinian by Lemma 2.13. If J is a finitely generated left R-module, then J is nilpotent by [15, Lemma 5.64] or [14, Ex. 9, P.305], and hence R is semiprimary. So R is left artinian by Lemma 2.13. Thus R is QF. □

Corollary 2.22. *If R is a left perfect, right F-injective and right (or left) pseudo-coherent ring, then R is QF.*

Proof. By hypothesis, R is right FSE. Therefore R is left and right mininjective. Thus R is QF. □

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