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# SEMISTAR OPERATIONS ON ALMOST PSEUDO-VALUATION DOMAINS

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Abstract. We characterize when an almost pseudo-valuation domain has a finite number of semistar operations.

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## 1. Introduction

The notion of a star operation is classical, and that of a Kronecker function ring which arises by a star operation is also classical. The notions of star operations, semistar operations, and their Kronecker function rings of integral domains have been well-known. Let $D$ be an integral domain, $K$ be its quotient field, and $\mathrm{F}(D)$ be the set of non-zero fractional ideals of $D$. A mapping $I \longmapsto I^\star$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ is called a star operation on $D$ if, for every $x \in K \setminus \{0\}$ and $I, J \in \mathrm{F}(D)$, the following conditions are satisfied: (1) $(x)^\star = (x)$; (2) $(xI)^\star = xI^\star$; (3) $I \subseteq I^\star$; (4) $(I^\star)^\star = I^\star$; (5) $I \subseteq J$ implies $I^\star \subseteq J^\star$. The Kronecker function ring of $D$ with respect to a star operation $\star$ on $D$ was first defined by L.Kronecker [7] and further investigated by W.Krull [8]. Let $\mathrm{F}'(D)$ be the set of non-zero $D$-submodules of $K$. A mapping $I \longmapsto I^\star$ from $\mathrm{F}'(D)$ to $\mathrm{F}'(D)$ is called a semistar operation on $D$ if, for every $x \in K \setminus \{0\}$ and $I, J \in \mathrm{F}'(D)$, the following conditions are satisfied: (1) $(xI)^\star = xI^\star$; (2) $I \subseteq I^\star$; (3) $(I^\star)^\star = I^\star$; (4) $I \subseteq J$ implies $I^\star \subseteq J^\star$. We refer to M.Fontana and K.Loper [2] and [3] and F.Halter-Koch [5] for notions of star operations, semistar operations, and their Kronecker function rings.

Let $\Sigma(D)$ (resp., $\Sigma'(D)$) be the set of star operations (resp., semistar operations) on $D$. In this paper, we are interested in the cardinalities $|\Sigma(D)|$ and $|\Sigma'(D)|$, especially, when $|\Sigma'(D)| < \infty$.

Let $D$ be an integrally closed domain. Then $D$ has only a finite number of semistar operations if and only if $D$ is a finite dimensional Prüfer domain with only a finite number of maximal ideals [11, (5.2)].

Let $V$ be a valuation domain with dimension $n$, $v$ be a valuation belonging to $V$, and $\Gamma$ be its value group. Let $M = P_n \supsetneqq P_{n-1} \supsetneqq \cdots\cdots P_1 \supsetneqq (0)$ be the prime ideals of $V$, let $\{0\} \subsetneqq H_{n-1} \subsetneqq \cdots \subsetneqq H_1 \subsetneqq \Gamma$ be the convex subgroups of $\Gamma$, and let $m$ be an integer with $n+1 \leq m \leq 2n+1$. Then the following conditions are equivalent: (1) $|\Sigma'(V)| = m$; (2) The maximal ideal of $V_{P_i}$ is principal for exactly $2n+1-m$ of $i$; (3) $\dfrac{\Gamma}{H_i}$ has a least positive element for exactly $2n+1-m$ of $i$ [9].

In [12], we studied star operations and semistar operations on a pseudo-valuation domain $D$. We gave conditions for $D$ to have only a finite number of semistar operations, and showed that conditions for $|\Sigma'(D)| < \infty$ reduce to conditions for related fields. In this paper, we will study star operations and semistar operations on almost pseudo-valuation domains, and will prove the following,

**Main Theorem** Let $D$ be an almost pseudo-valuation domain which is not a pseudo-valuation domain, $P$ its maximal ideal, $V = (P : P)$, $M$ be the maximal ideal of $V$ and set $K = \dfrac{V}{M}$ and $k = \dfrac{D}{P}$. Then $|\Sigma'(D)| < \infty$ if and only if one of the following conditions holds:

(1) $K$ is an infinite field, $K = k$, $\dim(D) < \infty$, and either $P = M^2$ or $P = M^3$.

(2) $K$ is a finite field, $\dim(D) < \infty$, and $P = M^n$ for some integer $n \geq 2$.

The paper consists of six sections. Section 2 contains preliminary results, Section 3 is the case where $K = k$ and $\min v(M)$ exists, Section 4 is the case where $K = k$ and $P = M^2$ or $P = M^3$, Section 5 is the case where $K = k$ and $P = M^n$ with $n \geq 4$, and Section 6 is the case where $K \supsetneqq k$.

## 2. Preliminary results

For the general ideal theory, especially for star operations on integral domains, we refer to R.Gilmer [4]. Thus, for every $I, J \in \mathrm{F}(D)$, we set $(I : J) = \{x \in \mathrm{q}(D) \mid xJ \subseteq I\}$, where $\mathrm{q}(D)$ denotes the quotient field of $D$, set $I^{-1} = (D : I)$, and set $I^{\mathrm{v}} = (I^{-1})^{-1}$. If $I = I^{\mathrm{v}}$, then $I$ is called divisorial. By [4, Theorem (34.1)], $I^{\mathrm{v}}$ is the intersection of principal fractional ideals of $D$ containing $I$, the mapping $I \longmapsto I^{\mathrm{v}}$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ is a star operation on $D$, and is called the v-operation, and for every star operation $\star$ on $D$ and for every $I \in \mathrm{F}(D)$, we have $I^{\star} \subseteq I^{\mathrm{v}}$. The identity mapping $I \longmapsto I^{\mathrm{d}} = I$ on $\mathrm{F}(D)$ is a star operation on $D$, and is called the d-operation.

Let $I$ be an ideal of a domain $D$. If, for elements $a, b \in \mathrm{q}(D)$, $ab \in I$ and $b \notin I$ imply $a \in I$, then $I$ is called strongly prime. If every prime ideal of $D$ is strongly prime, then $D$ is called a pseudo-valuation domain (or, a PVD). We refer to J.Hedstrom and E.Houston [6] for a PVD. Thus, every PVD is a local domain, that is, $D$ has only one maximal ideal. If $D$ is a local domain with maximal ideal strongly prime, then $D$ is a PVD.

For elements $a, b \in \mathrm{q}(D)$, if $ab \in I$ and $b \notin I$ imply $a^n \in I$ for some positive integer $n$, then $I$ is called strongly primary. If every prime ideal of $D$ is strongly primary, then $D$ is called an almost pseudo-valuation domain (or, an APVD). We refer to A.Badawi and E.Houston [1] for the notion of an APVD. Thus, every APVD is a local domain. Let $P$ be the maximal ideal of $D$, then $V = (P : P)$ is a valuation domain, $P$ is a primary ideal of $V$, and $P$ is primary to the maximal ideal of $V$. If $D$ is a local domain with maximal ideal strongly primary, then $D$ is an APVD.

In this section, let $D$ be an APVD which is not a PVD, $P$ be the maximal ideal of $D$, $V = (P : P)$, $M$ be the maximal ideal of $V$, $v$ be a valuation belonging to the valuation domain $V$, $\Gamma$ be the value group of $v$, $K = \dfrac{V}{M}$, and $k = \dfrac{D}{P}$.

We note that $P$ is not strongly prime and hence $P \subsetneqq M$. For, if $P$ is strongly prime, then $D$ is a PVD by [6, Theorem 1.4]; a contradiction to our assumption that $D$ is not a PVD.

The following Lemmas 2.1, 2.2 and 2.3 appear in [10, Lemmas 15 and 16 and Theorem 17].

**Lemma 2.1.** (1) $V = P^{-1}$.

(2) $P = P^{\mathrm{v}}$.

(3) *The set of non-maximal prime ideals of $D$ coincides with the set of non-maximal prime ideals of $V$, and* $\dim(V) = \dim(D)$.

Since $((I^{-1})^{-1})^{-1} = I^{-1}$ for every $I \in \mathrm{F}(D)$, $V$ is a divisorial fractional ideal of $D$.

**Lemma 2.2.** (1) $\mathrm{F}'(D) = \mathrm{F}(D) \cup \{\mathrm{q}(D)\}$.

(2) *The integral closure $\bar{D}$ of $D$ is a* PVD *with maximal ideal $M$.*

(3) *Let $T$ be an overring of $D$, that is, $T$ is a subring of* $\mathrm{q}(D)$ *containing $D$. Then either $T \supseteq V$ or $T \subseteq V$.*

(4) *Let $\Sigma'_1 = \{\star \in \Sigma'(D) \mid D^\star \supseteq V\}$. Then there is a canonical bijection from $\Sigma'(V)$ onto $\Sigma'_1$.*

(5) *Let $\Sigma'_2 = \{\star \in \Sigma'(D) \mid D^\star \subsetneqq V\}$. Then we have $\Sigma'(D) = \Sigma'_1 \cup \Sigma'_2$.*

(6) *If $\mid \Sigma'(D) \mid < \infty$, then $\dim(D) < \infty$, $V = \bar{D}$, $V$ is a finitely generated $D$-module, and $K$ is a simple extension field of $k$ with degree $[K : k] < \infty$.*

Every star operation on $D$ can be extended uniquely to a semistar operation on $D$, since $\mathrm{F}'(D)\setminus \mathrm{F}(D) = \{\mathrm{q}(D)\}$.

**Lemma 2.3.** *Assume that* $\dim(D) < \infty$, *and let* $\{T_\lambda \mid \lambda \in \Lambda\}$ *be the set of overrings* $T$ *of* $D$ *with* $T \subsetneq V$.

(1) $\mid \Sigma'(V) \mid < \infty$.

(2) $\mid \Sigma'_1 \mid = \mid \Sigma'(V) \mid$.

(3) *There is a canonical bijection from the disjoint union* $\bigcup_\lambda \Sigma(T_\lambda)$ *onto* $\Sigma'_2$.

(4) *If* $|\Sigma'_2| < \infty$, *then* $|\Sigma'(D)| = |\Sigma'_2| + |\Sigma'(V)|$.

Let $T$ be an overring of $D$. Then there is a canonical injective mapping $\delta$ from $\Sigma'(T)$ to $\Sigma'(D)$, and is called the descent mapping from $T$ to $D$.

**Lemma 2.4.** *Assume that* $|\Sigma'(D)| < \infty$, *then* $v(M)$ *has a least element.*

**Proof.** It is well-known that for any integral domain, each overring induces a semistar operation of finite type. Thus the number of overrings is less than the number of semistar operations of finite type. □

**Lemma 2.5.** *Assume that* $|\Sigma'(D)| < \infty$, *and let* $I \in \mathrm{F}(D)$. *If* $\inf v(I)$ *exists in* $\Gamma$, *then it is* $\min v(I)$.

**Proof.** Choose an element $x \in \mathrm{q}(D) \setminus \{0\}$ such that $\inf v(I) = v(x)$. Then $x^{-1}I \subseteq V$ and $\inf v(x^{-1}I) = 0$. Since $v(M)$ has a least element by Lemma 2.4, we have $0 = \min v(x^{-1}I)$, hence $v(x) = \min v(I)$. □

**Lemma 2.6.** *If* $P = M^n$ *for some integer* $n \geq 2$, *then* $v(M)$ *has a least element.*

**Proof.** Suppose the contrary, and let $x \in M\setminus P$. We can take elements $x_1, \cdots, x_n \in M$ such that $v(x) > v(x_1) > \cdots > v(x_n)$. Then we have $x = \dfrac{x}{x_1}\dfrac{x_1}{x_2}\cdots\dfrac{x_{n-1}}{x_n}x_n \in M^n = P$; a contradiction. □

**Lemma 2.7.** *Let* $Q$ *be an ideal of* $V$ *with* $M \supsetneq Q \supseteq P$, *and set* $D + Q = T$. *Then* $T$ *is an* APVD *which is not a* PVD, $Q$ *is the maximal ideal of* $T$, *and* $V = (Q : Q)$.

**Proof.** We rely on [1, Theorem 3.4]. Then $P$ is strongry primary, $P$ is an $M$-primary ideal of $V$, and so is $Q$. Clearly, $Q$ is the unique maximal ideal of $T = D+Q$, hence $T$ is an APVD, and $W = (Q : Q)$ is a valuation domain with $Q$ primary to the maximal ideal $N$ of $W$. Since $(Q : Q) \supseteq V$, $N$ is a prime ideal of $V$, hence $N = M$, and $W = V$. Finally, $T$ is not a PVD, because $Q$ is not strongly prime. □

**Lemma 2.8.** *Let* $\star$ *be a star operation (resp., a semistar operation) on* $D$.

(1) *Let* $T$ *be an overring of* $D$. *Then* $T^\star$ *is an overring of* $D$.

(2) *Both* $D^\star$ *and* $V^\star$ *are overrings of* $D$.

**Proof.** Because $T^\star = (TT)^\star = (T^\star T^\star)^\star \supseteq T^\star T^\star$. □

**Lemma 2.9.** *If* $\min v(M)$ *exists, then we may assume that* $\boldsymbol{Z}$ *is the rank one convex subgroup of* $\Gamma$*, and* $\min v(M) = 1 \in \boldsymbol{Z} \subseteq \Gamma$.

**Proof.** The rank one convex subgroup of $\Gamma$ is isomorphic with the ordered group $\boldsymbol{Z}$. Therefore there is an isomorphism compatible with orders from $\Gamma$ onto an ordered group $\Gamma'$ the rank one convex subgroup of which is $\boldsymbol{Z}$. □

**Lemma 2.10.** *To prove our* Theorem*, we may assume that* $v(M)$ *has a least element and* $\min v(M) = 1 \in \boldsymbol{Z} \subseteq \Gamma$.

The proof follows from Lemmas 2.4, 2.6 and 2.9.

## 3. The case where $K = k$ and $\min v(M)$ exists

In this section, let $D$ be an APVD which is not a PVD, $P$ be the maximal ideal of $D$, $V = (P : P)$, $M$ be the maximal ideal of $V$, $v$ be a valuation belonging to the valuation domain $V$, $\Gamma$ be the value group of $v$, assume that $K = \dfrac{V}{M} = \dfrac{D}{P}$, and $\min v(M)$ exists with $\min v(M) = v(\pi) = 1 \in \boldsymbol{Z} \subseteq \Gamma$ for some element $\pi \in M$, and let $\{\alpha_i \mid i \in \mathcal{I}\} = \mathcal{K}$ be a complete system of representatives of $V$ modulo $M$ with $\{0, 1\} \subseteq \mathcal{K} \subseteq D$.

**Lemma 3.1.** *Let* $x \in \mathrm{q}(D) \setminus \{0\}$ *with* $v(x) \in \boldsymbol{Z}$*, and let* $k$ *be a positive integer with* $k > v(x)$*. Then* $x$ *can be expressed uniquely as* $x = \alpha_l \pi^l + \alpha_{l+1}\pi^{l+1} + \cdots + \alpha_{k-1}\pi^{k-1} + a\pi^k$*, where* $l = v(x)$ *and each* $\alpha_i \in \mathcal{K}$ *with* $\alpha_l \neq 0$ *and* $a \in V$.

**Proof.** Since $\dfrac{x}{\pi^l}$ is a unit of $V$, we have $\dfrac{x}{\pi^l} \equiv \alpha_l \pmod{M}$ for a unique element $\alpha_l \in \mathcal{K} \setminus \{0\}$. Inductively, there are required elements $\alpha_{l+1}, \cdots, \alpha_{k-1} \in \mathcal{K}$ and $a \in V$. □

In Lemma 3.1, we may say that $\alpha_i$ is the coefficient of $\pi^i$ in $x$ (or, $\alpha_i$ is the coefficient of degree $i$ in $x$).

**Lemma 3.2.** *There is a unique integer* $n \geq 2$ *such that* $P = M^n$.

**Proof.** Set $\min \{v(x) \mid x \in P\} = n$, and let $x \in P$ such that $v(x) = n$. There is a unit $u$ of $V$ such that $\pi^n = xu$. Since $P$ is an ideal of $V$, we have $\pi^n \in P$, and hence $P = M^n$. Since $P \subsetneqq M$, we have $n \geq 2$. □

For every subset $X$ of $\mathrm{q}(D)$, the $D$-submodule of $\mathrm{q}(D)$ generated by $X$ is denoted by $(X)$. If $P = M^n$, then we have $P = (\pi^n, \pi^{n+1}, \cdots, \pi^{2n-2}, \pi^{2n-1})$ and $V = (1, \pi, \cdots, \pi^{n-1})$.

If $a_1, \cdots, a_n$ is a finite ordered set, and not only a finite set, we denote it by $< a_1, \cdots, a_n >$ if necessary. That is, $< a_1, \cdots, a_n > = < b_1, \cdots, b_m >$ if and only if $n = m$ and $a_i = b_i$ for each $i$.

**Lemma 3.3.** *Let $I \in \mathrm{F}(D)$.*

(1) *If* $\inf v(I)$ *exists, then it is* $\min v(I)$*.*

(2) *If* $\inf v(I)$ *does not exist, then we have $I = I^{\mathrm{v}}$.*

**Proof.** (1) Then $\min v(M)$ exists by the assumption, and the proof is similar to that of Lemma 2.5.

(2) By Lemma 3.2, there is an integer $n \geq 2$ such that $P = M^n$. Since $dI \subseteq D$ for some element $d \in D \setminus \{0\}$, $v(I)$ is bounded below. Let $\{v(x_\lambda) \mid \lambda \in \Lambda\}$ be the lower bound of $v(I)$, and let $x \in \bigcap_\lambda (x_\lambda)$. Suppose that $v(x)$ is in the lower bound of $v(I)$. Then $v(x) < v(x_\lambda)$ for some element $\lambda \in \Lambda$, hence $x \notin (x_\lambda)$; a contradiction. Therefore there are elements $a_1, a_2, \cdots, a_n \in I$ such that $v(a_n) < \cdots < v(a_2) < v(a_1) < v(x)$. Then $x = \dfrac{x}{a_1}\dfrac{a_1}{a_2}\cdots\dfrac{a_{n-1}}{a_n}a_n \in M^n a_n \subseteq I$. Hence we have $\bigcap_\lambda (x_\lambda) \subseteq I$. On the other hand, obviously we have $I \subseteq (x_\lambda)$ for every $\lambda$. It follows that $I = \bigcap_\lambda (x_\lambda)$, and hence $I = I^{\mathrm{v}}$ by [4, Theorem (34.1)]. □

**Example 3.4.** (1) *Assume that $P = M^2$, then we have*

$\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{(1), (1, \pi)\}$.

(2) *Assume that $P = M^3$. Set $(1) = I_0, (1, \pi^2) = I_{0,2}, (1, \pi, \pi^2) = I_{0,1,2}$, and set $(1, \pi + \alpha\pi^2) = I_{0,1}^{\alpha}$ for every $\alpha \in \mathcal{K}$. Then we have*

$\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{I_0, I_{0,2}, I_{0,1,2}\} \cup \{I_{0,1}^{\alpha} \mid \alpha \in \mathcal{K}\}$.

*If $I_{0,1}^{\alpha} = I_{0,1}^{\beta}$ for an element $\beta \in \mathcal{K}$, then $\alpha = \beta$.*

(3) *Assume that $P = M^4$. For elements $\alpha_1, \alpha_2 \in \mathcal{K}$, set*

$(1) = I_0$,

$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3) = I_{0,1}^{\alpha_1,\alpha_2}$,

$(1, \pi^2 + \alpha_1\pi^3) = I_{0,2}^{\alpha_1}$,

$(1, \pi^3) = I_{0,3}$,

$(1, \pi + \alpha_1\pi^3, \pi^2 + \alpha_2\pi^3) = I_{0,1,2}^{\alpha_1,\alpha_2}$,

$(1, \pi + \alpha_1\pi^2, \pi^3) = I_{0,1,3}^{\alpha_1}$,

$(1, \pi^2, \pi^3) = I_{0,2,3}$,

$(1, \pi, \pi^2, \pi^3) = I_{0,1,2,3}$.

*Then we have*

$\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{I_0, I_{0,1}^{\alpha_1,\alpha_2}, I_{0,2}^{\alpha_1}, I_{0,3}, I_{0,1,2}^{\alpha_1,\alpha_2}, I_{0,1,3}^{\alpha_1}, I_{0,2,3}, I_{0,1,2,3} \mid \alpha_1, \alpha_2 \in \mathcal{K}\}$.

*For elements $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathcal{K}$, if $I_{0,2}^{\alpha_1} = I_{0,2}^{\beta_1}$, then $\alpha_1 = \beta_1$; if $I_{0,1,3}^{\alpha_1} = I_{0,1,3}^{\beta_1}$, then $\alpha_1 = \beta_1$; if $I_{0,1}^{\alpha_1,\alpha_2} = I_{0,1}^{\beta_1,\beta_2}$, then the ordered set $< \alpha_1, \alpha_2 > = < \beta_1, \beta_2 >$; if $I_{0,1,2}^{\alpha_1,\alpha_2} = I_{0,1,2}^{\beta_1,\beta_2}$, then $< \alpha_1, \alpha_2 > = < \beta_1, \beta_2 >$.*

(4) *Assume that $P = M^5$. For elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{K}$, set*

$(1) = I_0$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3 + \alpha_3\pi^4) = I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi^2 + \alpha_1\pi^3 + \alpha_2\pi^4) = I_{0,2}^{\alpha_1,\alpha_2}$,
$(1, \pi^3 + \alpha_1\pi^4) = I_{0,3}^{\alpha_1}$,
$(1, \pi^4) = I_{0,4}$,
$(1, \pi + \alpha_1\pi^3 + \alpha_2\pi^4, \pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4) = I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^4, \pi^3 + \alpha_3\pi^4) = I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi + \alpha_1\pi^2 + \alpha_2\pi^3, \pi^4) = I_{0,1,4}^{\alpha_1,\alpha_2}$,
$(1, \pi^2 + \alpha_1\pi^4, \pi^3 + \alpha_2\pi^4) = I_{0,2,3}^{\alpha_1,\alpha_2}$,
$(1, \pi^2 + \alpha_1\pi^3, \pi^4) = I_{0,2,4}^{\alpha_1}$,
$(1, \pi^3, \pi^4) = I_{0,3,4}$,
$(1, \pi + \alpha_1\pi^4, \pi^2 + \alpha_2\pi^4, \pi^3 + \alpha_3\pi^4) = I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3}$,
$(1, \pi + \alpha_1\pi^3, \pi^2 + \alpha_2\pi^3, \pi^4) = I_{0,1,2,4}^{\alpha_1,\alpha_2}$,
$(1, \pi + \alpha_1\pi^2, \pi^3, \pi^4) = I_{0,1,3,4}^{\alpha_1}$,
$(1, \pi^2, \pi^3, \pi^4) = I_{0,2,3,4}$,
$(1, \pi, \pi^2, \pi^3, \pi^4) = I_{0,1,2,3,4}$.

*Then we have*

$\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{I_0, I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,2}^{\alpha_1,\alpha_2}, I_{0,3}^{\alpha_1}, I_{0,4}, I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4}, I_{0,1,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,4}^{\alpha_1,\alpha_2}, I_{0,2,3}^{\alpha_1,\alpha_2}, I_{0,2,4}^{\alpha_1}, I_{0,3,4}, I_{0,1,2,3}^{\alpha_1,\alpha_2,\alpha_3}, I_{0,1,2,4}^{\alpha_1,\alpha_2}, I_{0,1,3,4}^{\alpha_1}, I_{0,2,3,4}, I_{0,1,2,3,4} \mid$ *each* $\alpha_i \in \mathcal{K}\}$.

*For elements $\alpha_1, \cdots, \alpha_4, \beta_1, \cdots, \beta_4 \in \mathcal{K}$, if $I_{0,3}^{\alpha_1} = I_{0,3}^{\beta_1}$, then $\alpha_1 = \beta_1$; if $I_{0,2}^{\alpha_1,\alpha_2} = I_{0,2}^{\beta_1,\beta_2}$, then $< \alpha_1, \alpha_1 > = < \beta_1, \beta_2 >$; if $I_{0,1}^{\alpha_1,\alpha_2,\alpha_3} = I_{0,1}^{\beta_1,\beta_2,\beta_3}$, then $< \alpha_1, \alpha_2, \alpha_3 > = < \beta_1, \beta_2, \beta_3 >$; etc.*

**Proof.** (4) Let $I$ be a fractional ideal of $D$ such that $D \subseteq I \subseteq V$. Let $\tau = \{v(x) \mid x \in I \setminus P\}$, and let, for instance, $\tau = \{0, 1, 3\}$. Then $I$ contains elements $a, b$ of the form $a = \pi + \alpha_2\pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4$ and $b = \pi^3 + \beta\pi^4$, where $\alpha_2, \alpha_3, \alpha_4, \beta \in \mathcal{K}$. Exchanging $a$ by $a - \alpha_3 b$, we may assume that $\alpha_3 = 0$. Let $x = \beta_0 + \beta_1\pi + \beta_2\pi^2 + \beta_3\pi^3 + \beta_4\pi^4 + p \in I$, where each $\beta_i \in \mathcal{K}$ and $p \in P$. We have $x = \beta_0 + \beta_1 a + \beta_3 b + \beta_1'\pi^2 + \beta_2'\pi^4 + p'$ for some elements $\beta_i' \in \mathcal{K}$ and $p' \in P$. Since $\tau = \{0, 1, 3\}$, we have $\beta_1' = \beta_2' = 0$, hence $I = (1, a, b)$.

For the second assertion, say $I_{0,2,3}^{\alpha_1,\alpha_2} = I_{0,2,3}^{\beta_1,\beta_2}$. Then $\pi^2 + \beta_1\pi^4 = d_0 + d_1(\pi^2 + \alpha_1\pi^4) + d_2(\pi^3 + \alpha_2\pi^4)$ for some elements $d_0, d_1, d_2 \in D$. Comparing coefficients of $1, \pi^2$, $\pi^3$ in both sides, we have $d_0 \equiv 0(P), d_1 \equiv 1(P)$ and $d_2 \equiv 0(P)$. Then $\pi^2 + \beta_1\pi^4 = \pi^2 + \alpha_1\pi^4 + p$ for some element $p \in P$, hence $\beta_1 = \alpha_1$.

Similarly, we have $\pi^3 + \beta_2\pi^4 = d_0 + d_1(\pi^2 + \alpha_1\pi^4) + d_2(\pi^3 + \alpha_2\pi^4)$ for some elements $d_0, d_1, d_2 \in D$. Comparing coefficients of $1, \pi^2, \pi^3$ in both sides, we have $d_0 \equiv 0, d_1 \equiv 0$ and $d_2 \equiv 1$. Then $\pi^3 + \beta_2\pi^4 = \pi^3 + \alpha_2\pi^4 + p$ for some element $p \in P$. Hence $\beta_2 = \alpha_2$, and hence $< \alpha_1, \alpha_2 > = < \beta_1, \beta_2 >$.

The proofs for (1), (2) and (3) are similar and simpler. □

**Lemma 3.5.** *Assume that $P = M^n$ with $n \geq 2$, and let $I \in \mathrm{F}(D)$ with $D \subseteq I \subseteq V$. Then there is a set of generators $f_0, f_1, \cdots, f_m$ for $I$ satisfying the following conditions:*

(1) *Each $f_i$ has the following form: $f_0 = 1$, and*

$$f_i = \pi^{k_i} + \sum_{j=1}^{l(i)} \alpha_{i,j}\pi^{e_{i,j}} \text{ for each } 1 \leq i \leq m, \text{ where } \alpha_{i,j} \in \mathcal{K} \text{ for each } i,\ j.$$

(2) *In (1), the set $\{0, k_1, \cdots, k_m\}$ is a subset of $\{0, 1, 2, \cdots, n-1\}$ with $0 < k_1 < \cdots < k_m$.*

(3) *$\{k_i + 1, k_i + 2, \cdots, n-1\} \setminus \{k_{i+1}, \cdots, k_m\} = \{e_{i,1}, \cdots, e_{i,\ l(i)}\}$ with $e_{i,1} < e_{i,2} < \cdots < e_{i,\ l(i)}$ for each $1 \leq i \leq m$.*

**Proof.** We have $\{v(x) \mid x \in I \setminus P\} = \{1, k_1, \cdots, k_m\}$, where $1 < k_1 < \cdots < k_m \leq n-1$. By Lemma 3.1, there are elements $f_0, f_1, \cdots, f_m \in I$ which have the following form: $f_0 = 1$, and

$$f_i = \pi^{k_i} + \sum_{j=1}^{n-1-k_i} \beta_{i,k_i+j}\pi^{k_i+j} \text{ for each } 1 \leq i \leq m, \text{ where } \beta_{i,j} \in \mathcal{K} \text{ for each } i, j.$$

For each $1 \leq i \leq m$, exchanging $f_i$ by $f_i - \beta_{i,k_j}f_j$ for each $j > i$, we may assume that $\beta_{i,k_{i+1}} = \beta_{i,k_{i+2}} = \cdots = \beta_{i,k_m} = 0$. Then $f_0, f_1, \cdots, f_m$ satisfy the conditions (1), (2) and (3).

Suppose that $(f_0, f_1, \cdots, f_m) \subsetneqq I$, and let $x \in I \setminus (f_0, f_1, \cdots, f_m)$. Then $v(x) \in \{1, k_1, \cdots, k_m\}$. Let $k_i = \max\ \{v(x) \mid x \in I \setminus (f_0, f_1, \cdots, f_m)\}$, where we put $1 = k_0$, and let $y \in I \setminus (f_0, f_1, \cdots, f_m)$ such that $v(y) = k_i$. Then there is an element $\alpha \in \mathcal{K}$ such that $v(y - \alpha f_i) > k_i$. It follows that $y - \alpha f_i \in (f_0, f_1, \cdots, f_m)$, and hence $y \in (f_0, f_1, \cdots, f_m)$; a contradiction. The proof is complete. □

**Lemma 3.6.** *Assume that $P = M^n$ with $n \geq 2$, and let $I \in \mathrm{F}(D)$ with $D \subseteq I \subseteq V$. Then the system of generators $f_0, f_1, \cdots, f_m$ for $I$ satisfying the conditions in* Lemma 3.5 *is determined uniquely.*

**Proof.** Let $f'_0, \cdots, f'_{m'}$ be generators for $I$ satisfying the conditions in Lemma 3.5. Then each $f'_i$ has the following form: $f'_0 = 1$, and

$f'_i = \pi^{k'_i} + \sum_{j=1}^{l'(i)} \alpha'_{i,j}\pi^{e'_{i,j}}$ for each $1 \leq i \leq m'$, where $\alpha'_{i,j} \in \mathcal{K}$ for each $i$ and $j$, $\{0, k'_1, \cdots, k'_{m'}\}$ is a subset of $\{0, 1, 2, \cdots, n-1\}$ with $0 < k'_1 < \cdots < k'_{m'}$, and

$\{k'_i+1, k'_i+2, \cdots, n-1\} \setminus \{k'_{i+1}, \cdots, k'_m\} = \{e'_{i,1}, \cdots, e'_{i,l'(i)}\}$ with $e'_{i,1} < e'_{i,2} < \cdots < e'_{i,l'(i)}$ for each $1 \leq i \leq m'$.

Suppose that $k_i = k'_i$ for each $i < j$ and $k'_j < k_j$ for some $j$. Then $f'_j \notin (f_0, f_1, \cdots, f_m)$; a contradiction.

It follows that $m = m'$, $k_i = k'_i$ for each $i$, $l(i) = l'(i)$ for each $i$, and $e_{i,j} = e'_{i,j}$ for each $i, j$.

Suppose that $f_i = f'_i$ for each $i < j$ and that $f_j \neq f'_j$. We have $f'_j = f_j + d_{j+1}f_{j+1} + \cdots + d_m f_m + p$ for some elements $d_{j+1}, \cdots, d_m \in D$ and $p \in P$. If $d_{j+1}, \cdots, d_m \in P$, there is a contradiction to the uniqueness in Lemma 3.1. Otherwise, there is an integer $k > j$ and an element $\alpha \in \mathcal{K} \setminus \{0\}$ such that $f'_j = f_j + \alpha f_k + d'_{k+1}f_{k+1} + \cdots + d'_m f_m + p'$ for some elements $d'_{k+1}, \cdots, d'_m \in D$ and for some element $p' \in P$. The coefficient of $\pi^k$ in the left side $f'_j$ is zero and that in the right side is $\alpha \neq 0$; a contradiction. The proof is complete. □

Assume that $P = M^n$ for an integer $n \geq 2$. Let $\{0, k_1, \cdots, k_m\}$ be a subset of $\{0, 1, 2, \cdots, n-1\}$ containing 0 with $0 < k_1 < \cdots < k_m$. Then the ordered set $< 0, k_1, \cdots, k_m >$ with order $0 < k_1 < \cdots < k_m$ is called a type on $D$. There are $2^{n-1}$ types on $D$. Let $\tau = < 0, k_1, \cdots, k_m >$ be a type on $D$. Set

$\{k_i+1, k_i+2, \cdots, n-1\} \setminus \{k_{i+1}, \cdots, k_m\} = \{e_{i,1}, \cdots, e_{i,l(i)}\}$ with $e_{i,1} < e_{i,2} < \cdots < e_{i,l(i)}$ for each $1 \leq i \leq m$.

Then an ordered set $\bar{p} = < \alpha_{1,1}, \cdots, \alpha_{1,\ l(1)}, \cdots, \alpha_{m,1}, \cdots, \alpha_{m,\ l(m)} >$ of elements in $\mathcal{K}$ is called a system of parameters on $D$ belonging to $\tau$. The ordered set $\sigma = < 0, k_1, \cdots, k_m, \alpha_{1,1}, \cdots, \alpha_{1,\ l(1)}, \cdots, \alpha_{m,1}, \cdots, \alpha_{m,\ l(m)} >$ is called a data on $D$ belonging to $\tau$. We denote the data by $< 0, k_1, \cdots, k_m; \alpha_{1,1}, \cdots, \alpha_{1,\ l(1)}, \cdots, \alpha_{m,1}, \cdots, \alpha_{m,\ l(m)} >$. $\tau$ (resp., $\bar{p}$) is said to belong to $\sigma$, and is denoted by $\tau(\sigma)$ (resp., $\bar{p}(\sigma)$). A system of parameters belonging to $\tau$ may be empty. In this case, the data belonging to $\tau$ is $\tau$ itself. Set $f_0^\sigma = 1$, and

$f_i^\sigma = \pi^{k_i} + \sum_{j=1}^{l(i)} \alpha_{i,j}\pi^{e_{i,j}}$ for each $1 \leq i \leq m$.

Then $< f_0^\sigma, f_1^\sigma, \cdots, f_m^\sigma >$ is called a canonical system of generators on $D$ belonging to $\sigma$. And the fractional ideal $(f_0^\sigma, f_1^\sigma, f_2^\sigma, \cdots, f_m^\sigma)$ is said to be associated to $\sigma$, and is denoted by $I_\tau^{\bar{p}}$ or, by $I(\sigma)$.

Let $I$ be a fractional ideal of $D$ with $D \subseteq I \subseteq V$. Lemmas 3.5 and 3.6 show that there are a type $\tau$, a system of parameters $\bar{p}$, a data $\sigma$ uniquely such that $I = I(\sigma)$

on $D$. Then $\tau$ (resp., $\bar{p}, \sigma$) is called the type (resp., the system of parameters, the data) of $I$. The system of generators $< f_0^\sigma, f_1^\sigma, \cdots, f_m^\sigma >$ for $I$ is called the canonical system of generators for $I$.

**Lemma 3.7.** *Assume that $P = M^n$ with $n \geq 2$. Then we have $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{I(\sigma) \mid \sigma$ is a data on $D\}$.*

Let $I, J \in \mathrm{F}(D)$. If there is an element $x \in \mathrm{q}(D) \setminus \{0\}$ such that $xJ = I$, then $I$ and $J$ are said similar, and is denoted by $I \sim J$.

**Lemma 3.8.** *Assume that $P = M^n$ with $n \geq 2$. Let $\sigma, \sigma'$ be two datas on $D$ such that $\tau(\sigma) \neq \tau(\sigma')$. Then $I(\sigma)$ is not similar to $I(\sigma')$.*

**Proof.** Suppose that $xI(\sigma) = I(\sigma')$ for some element $x \in \mathrm{q}(D) \setminus \{0\}$. Then $v(x) = 0$. Let $\tau(\sigma) = \{0, k_1, k_2, \cdots, k_m\}$ with $0 < k_1 < k_2 < \cdots < k_m$, and let $\tau(\sigma') = \{0, k'_1, k'_2, \cdots, k'_{m'}\}$ with $0 < k'_1 < k'_2 < \cdots < k'_{m'}$. We may assume that $k_i = k'_i$ for each $i < j$ and $k_j < k'_j$ for some positive integer $j$. Then we have $xf_j^\sigma \notin I(\sigma')$, and hence $xI(\sigma) \not\subseteq I(\sigma')$; a contradiction. □

**Lemma 3.9.** *Assume that $K$ is a finite field. Then $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\}$ is a finite set.*

The proof follows from Lemma 3.7.

**Lemma 3.10.** *Assume that $K$ is a finite field, and let $l$ be a negative integer. Then $\{I \in \mathrm{F}(D) \mid I$ has $\min v(I)$, and $l \leq \min v(I) \leq 0\}$ is a finite set.*

**Proof.** Let $P = M^n$. By Lemma 3.9, the set $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = X$ is a finite set. Let $I$ be a fractional ideal of $D$ such that $\min v(I) = l_0$ exists with $l \leq l_0 \leq 0$. We have $v(a_0) = l_0$ for some element $a_0 \in I$. We may assume that $a_0 = \pi^{l_0}(1 + \alpha_1\pi + \alpha_2\pi^2 + \cdots + \alpha_{n-1}\pi^{n-1} + p)$ for some element $p \in P$. Since $D \subseteq \dfrac{1}{a_0}I \subseteq V$, we have $\dfrac{1}{a_0}I \in X$, completing the proof. □

**Lemma 3.11.** *Assume that $K$ is a finite field. Then $\{T \mid T$ is an overring of $D$ with $D \subseteq T \subseteq V\}$ is a finite set.*

**Proof.** Because each overring $T$ with $T \subseteq V$ has some type, and each type has only a finite number of systems of parameters. □

**Lemma 3.12.** *Assume that $K$ is a finite field. Let $T$ be an overring of $D$ with $T \subseteq V$, and let $l$ be a negative integer.*

(1) *$\{I \in \mathrm{F}(T) \mid T \subseteq I \subseteq V\}$ is a finite set.*

(2) *$\{I \in \mathrm{F}(T) \mid \min v(I)$ exists, and $l \leq \min v(I) \leq 0\}$ is a finite set.*

**Proof.** Since $\mathrm{F}(T) \subseteq \mathrm{F}(D)$, the proof follows from Lemmas 3.9 and 3.10. $\square$

## 4. The case where $K = k$ and $P = M^2$ or $P = M^3$

In this section, let $D, P, V, M, K, v, \Gamma, \pi$ and $\mathcal{K}$ be as in Section 3. We will prove the following,

**Proposition 4.1.** (1) *If $K$ is a finite field, then $|\Sigma(D)| < \infty$.*

(2) *If $P = M^2$, then $|\Sigma(D)| = 1$.*

(3) *If $P = M^2$, and if $\dim(D) < \infty$, then $|\Sigma'(D)| = 1 + |\Sigma'(V)|$.*

(4) *If $P = M^3$, then $|\Sigma(D)| = 3$.*

(5) *If $P = M^3$, and if $\dim(D) < \infty$, then $|\Sigma'(D)| = 4 + |\Sigma'(V)|$.*

We note that if $\dim(D) = \infty$, then $|\Sigma'(D)| = |\Sigma'(V)| = \infty$. For, $\mathrm{Spec}(D) = \{P_\lambda \mid \lambda \in \Lambda\}$ is an infinite set. And, for every $\lambda$, there is a semistar operation $I \longmapsto ID_{P_\lambda}$. Furthermore, if we have an infinite number of overrings of $D$, then $|\Sigma'(D)| = \infty$. For, for every overring $T$, there is a semistar operation $I \longmapsto IT$.

**Lemma 4.2.** *If $K$ is a finite field, then we have $|\Sigma(D)| < \infty$.*

**Proof.** Then $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = X$ is a finite set by Lemma 3.9. Let $\star$ be a star operation on $D$, and let $I \in X$. Since $V$ is a divisorial fractional ideal of $D$, we have $D \subseteq I^\star \subseteq V^\star \subseteq V^{\mathrm{v}} = V$, and hence $I^\star \in X$.

If we set $I^\star = g_\star(I)$, then the element $\star \in \Sigma(D)$ gives an element $g_\star \in X^X$, where $X^X$ is the set of mappings from $X$ to $X$. And the mapping $g : \star \longmapsto g_\star$ from $\Sigma(D)$ to $X^X$ is injective by the definition. $\square$

**Lemma 4.3.** *Assume that $P = M^2$. Then $\{T \mid T$ is an overring of $D$ with $T \subsetneqq V\} = \{D\}$.*

**Proof.** Because $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{(1), (1, \pi)\}$ by Example 3.4 (1). $\square$

**Lemma 4.4.** *Assume that $P = M^2$. Then we have $|\Sigma(D)| = 1$, and if $\dim(D) < \infty$, then $|\Sigma'(D)| = 1 + |\Sigma'(V)|$.*

**Proof.** If $\inf v(I)$ does not exist, then $I = I^{\mathrm{v}}$ by Lemma 3.3. Hence every member $I \in \mathrm{F}(D)$ is divisorial. It follows that $|\Sigma(D)| = 1$, and Lemma 2.3 completes the proof. $\square$

A mapping $\star$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ is said to satisfy condition (C) if it satisfies the following three conditions: (1) $D^\star = D$ and $V^\star = V$; (2) $(xI)^\star = xI^\star$ for every element $x \in \mathrm{q}(D) \setminus \{0\}$ and $I \in \mathrm{F}(D)$; (3) If $\inf v(I)$ does not exist, then $I^\star = I$. Obviously, every star operation satisfies the condition (C).

Throughout the rest of this section, assume that $P = M^3$.

**Lemma 4.5.** *We have* $\{T \mid T$ *is an overring of* $D$ *with* $T \subsetneqq V\} = \{D, D + M^2\}$.

**Proof.** We have that $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = \{I_0, I_{0,2}, I_{0,1,2}\} \cup \{I_{0,1}^{\alpha} \mid \alpha \in \mathcal{K}\}$ by Example 3.4 (2), and that $I_0 = D, I_{0,2} = D + M^2, I_{0,1,2} = V$, and $I_{0,1}^{\alpha}$ is not a subring of $\mathrm{q}(D)$ for every $\alpha \in \mathcal{K}$. □

**Lemma 4.6.** (1) *For elements* $\alpha, \beta \in \mathcal{K}$, *we have* $I_{0,1}^{\alpha} \subseteq I_{0,1}^{\beta}$ *if and only if* $\alpha = \beta$.

(2) $I_{0,2}$ *and* $I_{0,1}^{\alpha}$ *are not comparable for every* $\alpha \in \mathcal{K}$.

(3) $I_{0,1}^{\alpha}$ *and* $I_{0,1}^{\beta}$ *are similar for every* $\alpha, \beta \in \mathcal{K}$.

**Proof.** (3) Set $1 + \alpha\pi + \alpha^2\pi^2 = x$. Then we have $x(1, \pi) = (1, \pi + \alpha\pi^2)$.

The proofs for (1) and (2) are similar. □

**Lemma 4.7.** *Let* $\star$ *be a star operation on* $D$. *Then* $(I_{0,2})^{\star}$ *is either* $I_{0,2}$ *or* $V$, *and* $(I_{0,1}^{0})^{\star}$ *is either* $I_{0,1}^{0}$ *or* $V$.

**Proof.** Since $V$ is a divisorial fractional ideal of $D$, we have $(I_{0,2})^{\star} \subseteq V$ and $(I_{0,1}^{0})^{\star} \subseteq V$. Then the assertion follows from Lemma 4.6. □

**Lemma 4.8.** (1) *Set* $I_{0,2} = (I_{0,2})^{\star}$ *and* $I_{0,1}^{0} = (I_{0,1}^{0})^{\star}$. *Then* $\star$ *can be extended uniquely to a mapping* $\star_1$ *from* $\mathrm{F}(D)$ *to* $\mathrm{F}(D)$ *with condition* (C).

(2) *Set* $I_{0,2} = (I_{0,2})^{\star}$ *and* $V = (I_{0,1}^{0})^{\star}$. *Then* $\star$ *can be extended uniquely to a mapping* $\star_2$ *from* $\mathrm{F}(D)$ *to* $\mathrm{F}(D)$ *with condition* (C).

(3) *Set* $V = (I_{0,2})^{\star}$ *and* $I_{0,1}^{0} = (I_{0,1}^{0})^{\star}$. *Then* $\star$ *can be extended uniquely to a mapping* $\star_3$ *from* $\mathrm{F}(D)$ *to* $\mathrm{F}(D)$ *with condition* (C).

(4) *Set* $V = (I_{0,2})^{\star}$ *and* $V = (I_{0,1}^{0})^{\star}$. *Then* $\star$ *can be extended uniquely to a mapping* $\star_4$ *from* $\mathrm{F}(D)$ *to* $\mathrm{F}(D)$ *with condition* (C).

**Proof.** We confer Example 3.4 (2) and Lemma 3.3. Let $I \in \mathrm{F}(D)$, then Lemma 3.8 implies that either $I$ is similar to one and only one in $\{I_0, I_{0,2}, I_{0,1,2}, I_{0,1}^{0}\}$, or $\inf v(I)$ does not exist. If $\inf v(I)$ does not exist, then we set $I = I^{\star_i}$ for each $i$. □

**Lemma 4.9.** *In* Lemma 4.8, *we have the following:*

(1) $\star_1$ *is a star operation on* $D$, *and* $\star_1 = \mathrm{d}$.

(2) $\star_2$ *is a star operation on* $D$.

(3) $\star_3$ *is not a star operation on* $D$.

(4) $\star_4$ *is a star operation on* $D$, *and* $\star_4 = \mathrm{v}$.

**Proof.** We confer Lemma 4.6.

(2) For elements $x \in \mathrm{q}(D) \setminus \{0\}$ and $I \in \mathrm{F}(D)$, we have $(x)^{\star_2} = (x)$, $(xI)^{\star_2} = xI^{\star_2}$, $I \subseteq I^{\star_2}$, and $(I^{\star_2})^{\star_2} = I^{\star_2}$.

Let $I_1, I_2 \in \mathrm{F}(D)$ with $I_1 \subseteq I_2$. The proof for $I_1^{\star_2} \subseteq I_2^{\star_2}$ follows from the following two facts:

(i) Let $(1, \pi) \subseteq I \in \mathrm{F}(D)$ such that $\inf v(I)$ does not exist. Then $V \subseteq I$.

(ii) For elements $x \in \mathrm{q}(D) \setminus \{0\}$ and $I \in \{I_0, I_{0,2}\}$, if $xI_{0,1}^0 \subseteq I$, then $xV \subseteq I$.

(3) Set $\pi + \pi^2 = x$. Then $x(1, \pi^2) \subseteq (1, \pi + \pi^2)$ and $xV \not\subseteq (1, \pi + \pi^2)$.

The proofs for (1) and (4) are similar. $\square$

**Lemma 4.10.** *Assume that $P = M^3$. Then $|\Sigma(D)| = 3$, and, if $\dim(D) < \infty$, then $|\Sigma'(D)| = 4 + |\Sigma'(V)|$.*

**Proof.** By Lemma 4.9, $\Sigma(D) = \{\mathrm{d}, \mathrm{v}, \star_2\}$, and hence $|\Sigma(D)| = 3$.

Assume that $\dim(D) < \infty$. By Lemma 2.7, we can apply Lemma 4.4 for $D' = D + M^2$. Then, in Lemma 2.3, we have $|\Sigma_2'| = |\Sigma(D)| + |\Sigma(D + M^2)| = 3 + 1 = 4$. It follows that $|\Sigma'(D)| = |\Sigma_1'| + |\Sigma_2'| = 4 + |\Sigma'(V)|$. $\square$

The proof for Proposition 4.1 is complete.

## 5. The case where $K = k$ and $P = M^n$ with $n \geq 4$

In this section, let $D, P, V, M, K, v, \Gamma, \pi$ and $\mathcal{K}$ be as in Section 3. We will prove the following,

**Proposition 5.1.** (1) *Assume that $K$ is an infinite field and $P = M^n$ with $n \geq 4$. Then $|\Sigma(D)| = \infty$.*

(2) *Assume that $K$ is a finite field and $\dim(D) < \infty$. Then $|\Sigma'(D)| < \infty$.*

**Lemma 5.2.** *Let $T$ be an overring of $D$ with $T \subseteq V$, and let $I \in \mathrm{F}(T)$.*

(1) *If $\inf v(I)$ exists, then it is $\min v(I)$.*

(2) *If $\inf v(I)$ does not exist, then $I$ is a divisorial fractional ideal of $T$.*

The proof is similar to that of Lemma 3.3.

**Lemma 5.3.** *Assume that $K$ is a finite field, and let $T$ be an overring of $D$ with $T \subseteq V$. Then $|\Sigma(T)| < \infty$.*

**Proof.** Let $P = M^n$. Set $\{I \in \mathrm{F}(T) \mid T \subseteq I \subseteq V\} = X$, and set $\{I \in \mathrm{F}(T) \mid \min v(I)$ exists, and $-n \leq \min v(I) \leq 0\} = Y$. Then $X$ and $Y$ are finite sets by Lemma 3.12. Let $I \in \mathrm{F}(T)$. Then either $\min v(I)$ exists or $\inf v(I)$ does not exist, and, if $\inf v(I)$ does not exist, then $I$ is a divisorial fractional ideal of $T$ by Lemma 5.2.

Let $\star$ be a star operation on $T$, and let $I \in X$. Since $\pi^n I \subseteq T$, we have $\pi^n I^\star \subseteq T$. Hence min $v(I^\star)$ exists, and $-n \leq$ min $v(I^\star) \leq 0$, that is, $I^\star \in Y$. If we set $I^\star = g_\star(I)$, there is a canonical mapping $g : \Sigma(T) \longrightarrow Y^X$, where $Y^X$ is the set of mappings from $X$ to $Y$. Moreover, $g$ is injective by the definition, and hence $|\Sigma(T)| < \infty$. □

**Lemma 5.4.** *Assume that $K$ is a finite field and* $\dim(D) < \infty$*. Then* $|\Sigma'(D)| < \infty$*.*

**Proof.** By Lemmas 3.11 and 5.3, we have $|\Sigma'_2| < \infty$ and $|\Sigma'(D)| < \infty$ in Lemma 2.3. □

**Lemma 5.5.** *Let $< \tau; \alpha_1, \cdots, \alpha_k >$, $< \tau; \beta_1, \cdots, \beta_k >$ be two datas on $D$ with the same type $\tau$ and with $k \geq 1$. Then $I(\tau; \alpha_1, \cdots, \alpha_k) \subseteq I(\tau; \beta_1, \cdots, \beta_k)$ if and only if $\alpha_i = \beta_i$ for each $i$.*

**Proof.** For instance, assume that $P = M^5$ and that $I_{0,1,2}^{\alpha_1,\alpha_2,\alpha_3,\alpha_4} \subseteq I_{0,1,2}^{\beta_1,\beta_2,\beta_3,\beta_4}$. Then we have $\pi + \alpha_1\pi^3 + \alpha_2\pi^4 = (\pi + \beta_1\pi^3 + \beta_2\pi^4) + (\pi^2 + \beta_3\pi^3 + \beta_4\pi^4)p_1 + p_2$ for some elements $p_1, p_2 \in P$. Hence $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Similarly, we have $\pi^2 + \alpha_3\pi^3 + \alpha_4\pi^4 = (\pi^2 + \beta_3\pi^3 + \beta_4\pi^4) + p_3$ for some element $p_3 \in P$. Hence $\alpha_3 = \beta_3$ and $\alpha_4 = \beta_4$. □

**Lemma 5.6.** *Assume that $P = M^n$ with $n \geq 4$ and that $K$ is an infinite field.*

(1) *The set $\{T \mid T$ is an overring of $D$ with $T \subseteq V\}$ is an infinite set.*

(2) $|\Sigma'(D)| = \infty$.

**Proof.** (1) $I_{0,n-2}^{\alpha}$ is an overring of $D$ with $I_{0,n-2}^{\alpha} \subseteq V$ for every $\alpha \in \mathcal{K}$. Since $|\mathcal{K}| = \infty$, the assertion holds by Lemma 5.5.

(2) follows from (1). □

**Lemma 5.7.** *Assume that $P = M^n$ with $n \geq 3$. Let $I \in$* F$(D)$ *such that $D \subseteq I \subseteq V$ with type $\tau$, let $J \in$* F$(D)$*, and let $x \in$* q$(D) \setminus \{0\}$*.*

(1) *If $I \subseteq J$, and if* inf $v(J)$ *does not exist, then $V \subseteq J$.*

(2) *If $xI \subseteq I_0$, and if $\tau \notin \{< 0 >, < 0, n-1 >\}$, then $xV \subseteq I_0$.*

(3) *If $xI \subseteq I_{0,n-1}$, and if $\tau \notin \{< 0 >, < 0, n-1 >\}$, then $xV \subseteq I_{0,n-1}$.*

(4) *If $xI \subseteq I_{0,1}^{\alpha_1,\cdots,\alpha_{n-2}}$, and if $\tau \notin \{< 0 >, < 0, 1 >, < 0, n-1 >\}$, then $xV \subseteq I_{0,1}^{\alpha_1,\cdots,\alpha_{n-2}}$.*

**Proof.** (3) Suppose that $v(x) = 0$. Since $\tau \notin \{< 0 >, < 0, n-1 >\}$, $I$ contains an element $a$ such that $0 < v(a) < n-1$. We have $xa \in I_{0,n-1}$ and $0 < v(xa) < n-1$; a contradiction.

(4) We have $v(xI) \subseteq \{0, 1, n, n+1, \cdots\}$. Since $x \in I_{0,1}^{\alpha_1, \cdots, \alpha_{n-2}}$, we have $v(x) \in \{0, 1, n, n+1, \cdots\}$.

If $v(x) = 0$, then $v(I) \subseteq \{0, 1, n, n+1, \cdots\}$. Hence $\tau$ is either $< 0 >$ or $< 0, 1 >$; a contradiction.

If $v(x) = 1$, then $v(I) \subseteq \{0, n-1, n, \cdots\}$. Hence $\tau$ is either $< 0 >$ or $< 0, n-1 >$; a contradiction.

Finally, if $v(x) \geq n$, then $xV \subseteq I_{0,1}^{\alpha_1, \cdots, \alpha_{n-2}}$.

The proofs for (1) and (2) are similar. □

**Lemma 5.8.** *Assume that $P = M^n$ with $n \geq 4$. Then $I(0, 1; 0, \cdots, 0, \alpha) \sim I(0, 1; 0, \cdots, 0, \beta)$ if and only if $\alpha = \beta$.*

**Proof.** The necessity: There is an element $x \in \mathrm{q}(D) \setminus \{0\}$ such that $x(1, \pi + \alpha\pi^{n-1}) = (1, \pi + \beta\pi^{n-1})$. We may assume that $x = 1 + (\pi + \beta\pi^{n-1})\alpha'$ for some element $\alpha' \in \mathcal{K}$. Since $x(\pi + \alpha\pi^{n-1}) \in (1, \pi + \beta\pi^{n-1})$, we have $\alpha = \beta$. □

**Example 5.9.** *Assume that $P = M^5$. In the following, let $\alpha_i, \beta_i, \alpha_{(i)} \in \mathcal{K}$ for each $i$.*

(1) *$I_{0,1}^{\alpha_1,\alpha_2,\alpha_3} \sim I_{0,1}^{\beta_1,\beta_2,\beta_3}$ if and only if $\alpha_2 - \beta_2 \equiv (\alpha_1 - \beta_1)(\alpha_1 + \beta_1)$ (mod$P$) and $(\alpha_3 - \beta_3) \equiv (\alpha_1 - \beta_1)(\alpha_2 + \alpha_1\beta_1 + \beta_2)$ (mod$P$).*

(2) *Let $x \in \mathrm{q}(D) \setminus \{0\}$. If $xI_{0,1}^{\alpha_{(1)},\alpha_{(2)},\alpha_{(3)}} \subseteq I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$, and if $I_{0,1}^{\alpha_{(1)},\alpha_{(2)},\alpha_{(3)}} \not\sim I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$, then $xV \subseteq I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$.*

(3) *Fix a data $< 0, 1; \alpha_{(1)}, \alpha_{(2)}, \alpha_{(3)} >$ on $D$. Let $I \in \mathrm{F}(D)$ with $D \subseteq I \subseteq V$. If $I$ is either $I_0$ or $I_{0,4}$ or $I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$ with $I_{0,1}^{\alpha_1,\alpha_2,\alpha_3} \not\sim I_{0,1}^{\alpha_{(1)},\alpha_{(2)},\alpha_{(3)}}$, set $I = I^{\star_0}$, and otherwise set $V = I^{\star_0}$. Then $\star_0$ determines uniquely a star operation $\star$ on $D$.*

(4) *If $K$ is an infinite field, then $|\Sigma(D)| = \infty$.*

**Proof.** We confer Example 3.4 (4).

(1) Set $\pi + \alpha_1\pi^2 + \alpha_2\pi^3 + \alpha_3\pi^4 = A$ and set $\pi + \beta_1\pi^2 + \beta_2\pi^3 + \beta_3\pi^4 = B$.

The necessity: There is an element $x \in \mathrm{q}(D) \setminus \{0\}$ such that $xI_{0,1}^{\alpha_1,\alpha_2,\alpha_3} = I_{0,1}^{\beta_1,\beta_2,\beta_3}$. Then we have $v(x) = 0$. We may assume that $x = 1 + B\alpha$ for some element $\alpha \in \mathcal{K}$. Since $xA \in (1, B)$, we have $\alpha \equiv \beta_1 - \alpha_1, \beta_2 - \alpha_2 \equiv \alpha(\alpha_1 + \beta_1)$ and $\beta_3 - \alpha_3 \equiv \alpha(\alpha_2 + \alpha_1\beta_1 + \beta_2)$.

The sufficiency: Let $\beta_1 - \alpha_1 \equiv \alpha$ with $\alpha \in \mathcal{K}$, and set $1 + B\alpha = x$. We have that $A + AB\alpha = B + p_1$ for some element $p_1 \in P$, and hence $x(1, A) \subseteq (1, B)$. Similarly, let $\alpha_1 - \beta_1 \equiv \beta$ with $\beta \in \mathcal{K}$, $1 + A\beta = y$, and $B + AB\beta = A + p_2$ for some element $p_2 \in P$. Then $y(1, B) \subseteq (1, A)$. On the other hand, since $xy$ is a unit of $D$, it follows that $x(1, A) = (1, B)$ and $y(1, B) = (1, A)$.

(2) Suppose that $v(x) = 0$. Then we may assume that $x = 1+(\pi+\alpha_1\pi^2+\alpha_2\pi^3+\alpha_3\pi^4)\alpha$ for some element $\alpha \in \mathcal{K}$. Then $x(\pi + \alpha_{(1)}\pi^2 + \alpha_{(2)}\pi^3 + \alpha_{(3)}\pi^4) \in I_{0,1}^{\alpha_1,\alpha_2,\alpha_3}$ implies that $\alpha_{(2)} - \alpha_2 \equiv (\alpha_{(1)} - \alpha_1)(\alpha_{(1)} + \alpha_1)$ and $\alpha_{(3)} - \alpha_3 \equiv (\alpha_{(1)} - \alpha_1)(\alpha_{(2)} + \alpha_{(1)}\alpha_1 + \alpha_2)$; a contradiction.

(3) We introduced the condition (C) in Section 4. Then $\star_0$ can be extended uniquely to a mapping $\star$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ with condition (C). Let $I_1, I_2 \in \mathrm{F}(D)$ with $I_1 \subseteq I_2$, then we have $I_1^\star \subseteq I_2^\star$ by Lemma 5.7 and (2).

(4) Let $\star_{\alpha_{(1)},\alpha_{(2)},\alpha_{(3)}}$ be the star operation on $D$ determined in (3). If $I_{0,1}^{\alpha_1,\alpha_2,\alpha_3} \not\sim I_{0,1}^{\beta_1,\beta_2,\beta_3}$, then $\star_{\alpha_1,\alpha_2,\alpha_3} \neq \star_{\beta_1,\beta_2,\beta_3}$. By Lemma 5.8, we have $|\Sigma(D)| = \infty$. □

**Lemma 5.10.** *Assume that $P = M^n$ with $n \geq 4$.*

(1) *Then $I(0,1;\alpha_1,\cdots,\alpha_{n-2}) \sim I(0,1;\beta_1,\cdots,\beta_{n-2})$ if and only if $\alpha_k - \beta_k \equiv (\alpha_1 - \beta_1)(\sum_0^{k-1}\beta_i\alpha_{k-1-i})$ $(\mathrm{mod} P)$ for each $2 \leq k \leq n-2$.*

(2) *Let $x \in \mathrm{q}(D) \setminus \{0\}$. If $xI(0,1;\alpha_1,\cdots,\alpha_{n-2}) \subseteq I(0,1;\beta_1,\cdots,\beta_{n-2})$ with $I(0,1;\alpha_1,\cdots,\alpha_{n-2}) \not\sim I(0,1;\beta_1,\cdots,\beta_{n-2})$, then $xV \subseteq I(0,1;\beta_1,\cdots,\beta_{n-2})$.*

**Proof.** We confer Lemma 5.9, where $n = 5$.

(1) Set $\pi+\alpha_1\pi^2+\cdots+\alpha_{n-2}\pi^{n-1} = A$, and set $\pi+\beta_1\pi^2+\cdots+\beta_{n-2}\pi^{n-1} = B$.

The necessity: There is an element $x \in \mathrm{q}(D) \setminus \{0\}$ such that $xI_{0,1}^{\alpha_1,\alpha_2,\cdots,\alpha_{n-2}} = I_{0,1}^{\beta_1,\beta_2,\cdots,\beta_{n-2}}$. Since $v(x) = 0$, we may assume that $x = 1 + B\alpha$ for some element $\alpha \in \mathcal{K}$. Since $xA \in (1, B)$, we have $\alpha \equiv \beta_1 - \alpha_1$ and $\beta_k - \alpha_k \equiv \alpha(\sum_0^{k-1}\beta_i\alpha_{k-1-i})$ for each $2 \leq k \leq n-2$.

The sufficiency is similar to the proof for Lemma 5.9 (1).

(2) Suppose that $v(x) = 0$. Then we may assume that $x = 1 + (\pi + \beta_1\pi^2 + \cdots + \beta_{n-2}\pi^{n-1})\alpha$ for some element $\alpha \in \mathcal{K}$. Then $x(\pi + \alpha_1\pi^2 + \cdots + \alpha_{n-2}\pi^{n-1}) \in I_{0,1}^{\beta_1,\cdots,\beta_{n-2}}$ implies that $\beta_k - \alpha_k \equiv (\beta_1 - \alpha_1)(\sum_0^{k-1}\alpha_i\beta_{k-1-i})$ for each $2 \leq k \leq n-2$; a contradiction. □

**Lemma 5.11.** *Assume that $P = M^n$ with $n \geq 4$. Fix a data $< 0,1;\alpha_{(1)},\alpha_{(2)},\cdots,\alpha_{(n-2)} >$ on $D$, and let $I \in \mathrm{F}(D)$ with $D \subseteq I \subseteq V$. If $I$ is either $I_0$ or $I_{0,n-1}$ or $I(0,1;\alpha_1,\alpha_2,\cdots,\alpha_{n-2})$ with $I(0,1;\alpha_1,\alpha_2,\cdots,\alpha_{n-2}) \not\sim I(0,1;\alpha_{(1)},\alpha_{(2)},\cdots,\alpha_{(n-2)})$, set $I = I^{\star_0}$, and otherwise set $V = I^{\star_0}$. Then $\star_0$ determines uniquely a star operation $\star$ on $D$.*

**Proof.** We confer Lemma 5.9 (3). Then $\star_0$ can be extended uniquely to a mapping $\star$ from $\mathrm{F}(D)$ to $\mathrm{F}(D)$ with condition (C). Let $I_1, I_2 \in \mathrm{F}(D)$ with $I_1 \subseteq I_2$. Then, by Lemma 5.7 and Lemma 5.10 (2), we have $I_1^\star \subseteq I_2^\star$. □

**Lemma 5.12.** *Assume that $K$ is an infinite field and $P = M^n$ with $n \geq 4$. Then $|\Sigma(D)| = \infty$.*

**Proof.** Let $\star_{\alpha_{(1)},\alpha_{(2)},\cdots,\alpha_{(n-2)}}$ be the star operation on $D$ determined in Lemma 5.11. If $I_{0,1}^{\alpha_1,\alpha_2,\cdots,\alpha_{n-2}} \not\sim I_{0,1}^{\beta_1,\beta_2,\cdots,\beta_{n-2}}$, then $\star_{\alpha_1,\alpha_2,\cdots,\alpha_{n-2}} \neq \star_{\beta_1,\beta_2,\cdots,\beta_{n-2}}$. By Lemma 5.8, we have $|\Sigma(D)| = \infty$. □

The proof for Proposition 5.1 is complete, and the proof for the case where $K = k$ in our Theorem is complete.

## 6. The case where $K \supsetneqq k$

In this final section, let $D$ be an APVD which is not a PVD, $P$ be the maximal ideal of $D$, $V = (P : P)$, $M$ be the maximal ideal of $V$, $K = \dfrac{V}{M}$, $k = \dfrac{D}{P}$, $v$ be a valuation belonging to $V$, $\Gamma$ be the value group of $v$, $\{\alpha_i \mid i \in \mathcal{I}\} = \mathcal{K}$ be a complete system of representatives of $V$ modulo $M$ with $\{0, 1\} \subseteq \mathcal{K}$, and assume that $K \supsetneqq k$, and that min $v(M)$ exists with min $v(M) = v(\pi) = 1 \in \boldsymbol{Z} \subseteq \Gamma$ for some element $\pi \in M$. We will prove the following,

**Proposition 6.1.** *The following conditions are equivalent.*

(1) $|\Sigma'(D)| < \infty$.

(2) *$K$ is a finite field,* $\dim(D) < \infty$*, and* $P = M^n$ *for some* $n \geq 2$.

**Lemma 6.2.** (1) *Let* $x \in \mathrm{q}(D)\setminus\{0\}$ *with* $v(x) \in \boldsymbol{Z}$*, and let $k$ be a positive integer with* $k > v(x)$*. Then $x$ can be expressed uniquely as* $x = \alpha_l \pi^l + \alpha_{l+1}\pi^{l+1} + \cdots + \alpha_{k-1}\pi^{k-1} + a\pi^k$*, where* $l = v(x)$ *and each* $\alpha_i \in \mathcal{K}$ *with* $\alpha_l \neq 0$ *and* $a \in V$.

(2) *There is a unique integer* $n \geq 2$ *such that* $P = M^n$.

(3) *Let* $I \in \mathrm{F}(D)$ *such that* inf $v(I)$ *exists. Then* inf $v(I) = \min v(I)$.

(4) *Let* $I \in \mathrm{F}(D)$ *such that* inf $v(I)$ *does not exist. Then* $I = I^{\mathrm{v}}$.

The proofs are similar to those for Lemmas 3.1, 3.2 and 3.3.

**Lemma 6.3.** *Assume that* $P = M^n$ *for some* $n \geq 2$*. Let $T$ be an overring of $D$ with* $T \subseteq V$ *and let* $I \in \mathrm{F}(T)$.

(1) *If* inf $v(I)$ *exists, then it is* min $v(I)$.

(2) *If* inf $v(I)$ *does not exist, then $I$ is a divisorial fractional ideal of $T$.*

The proof is similar to that for Lemma 3.3.

**Lemma 6.4.** *Assume that $K$ is a finite field and* $P = M^n$ *for some* $n \geq 2$.

(1) *The set* $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\}$ *is a finite set.*

(2) *Let $l$ be a negative integer. Then the set* $\{I \in \mathrm{F}(D) \mid \min v(I)$ *exists, and* $l \leq \min v(I) \leq 0\}$ *is a finite set.*

(3) *The set* $\{T \mid T$ *is an overring of $D$ with* $D \subseteq T \subseteq V\}$ *is a finite set.*

(4) *The set* $\{I \in \mathrm{F}(T) \mid T \subseteq I \subseteq V\}$ *is a finite set.*

(5) *Let* $T$ *be an overring of* $D$ *with* $T \subseteq V$*, and let* $l$ *be a negative integer. Then the set* $\{I \in \mathrm{F}(T) \mid \min v(I)$ *exists, and* $l \leq \min v(I) \leq 0\}$ *is a finite set.*

The proofs are similar to those for Lemmas 3.9, 3.10, 3.11 and 3.12.

**Lemma 6.5.** *Assume that* $k$ *is an infinite field and* $P = M^n$ *for some* $n \geq 2$*. Then there is an infinite number of intermediate rings between* $D$ *and* $V$*.*

**Proof.** Let $u \in V$ such that $\bar{u} = u + M \in K \setminus k$. Let $a \in D \setminus P$, and set $(1, (1+au)\pi^{n-1}) = D_a$. Then $D_a$ is an overring of $D$ with $D_a \subseteq V$.

Let $a, b \in D \setminus P$ such that $D_a = D_b$. Then we have $\bar{a} = \bar{b}$. For, we have $(1+au)\pi^{n-1} = (1+bu)\pi^{n-1}d + p$ for some elements $d \in D$ and $p \in P$. It follows that $1 - d = (bd - a)u + m$ for some element $m \in M$. If $bd - a \equiv 0$, then $1 - d \equiv 0$, hence $\bar{b} = \bar{b}\bar{d} = \bar{a}$. Suppose that $\overline{bd - a} \neq \bar{0}$. Since $\overline{1 - d} = \overline{bd - a}\ \bar{u}$, we have $\bar{u} \in k$; a contradiction. It follows that $\{D_a \mid a \in D \setminus P\}$ is an infinite set, since $k$ is an infinite field. The proof is complete. □

*Proof* for Proposition 6.1. (1) $\Longrightarrow$ (2): By Lemma 2.2 (6), we have $\dim(D) < \infty$ and $[K : k] < \infty$. We may apply Lemma 6.2. Then we have $P = M^n$ for some $n \geq 2$. Suppose that $K$ is an infinite field. Since $[K : k] < \infty$, $k$ is an infinite field. By Lemma 6.5, there is an infinite number of intermediate rings between $D$ and $V$. It follows that $|\Sigma'(D)| = \infty$; a contradiction.

(2) $\Longrightarrow$ (1): We can apply Lemma 6.4. The set $\{I \in \mathrm{F}(D) \mid D \subseteq I \subseteq V\} = X$ is a finite set. Let $\star$ be a star operation on $D$, and let $I \in X$. We note that $V$ is a divisorial fractional ideal of $D$. Since $D \subseteq I^\star \subseteq V$, we have $I^\star \in X$.

If we set $I^\star = g_\star(I)$, then the element $\star \in \Sigma(D)$ gives an element $g_\star \in X^X$. By Lemma 6.2 (3), the mapping $g : \star \longmapsto g_\star$ from $\Sigma(D)$ to $X^X$ is an injection. It follows that $|\Sigma(D)| < \infty$.

Let $T$ be an overring of $D$ with $T \subseteq V$. Set $\{I \in \mathrm{F}(T) \mid T \subseteq I \subseteq V\} = X$, and set $\{I \in \mathrm{F}(T) \mid \min v(I)$ exists, and $-n \leq \min v(I) \leq 0\} = Y$. Then $X$ and $Y$ are finite sets. For every $I \in \mathrm{F}(T)$, either $\min v(I)$ exists or $\inf v(I)$ does not exist by Lemma 6.3 (1). Let $\star$ be a star operation on $T$, and let $I \in X$. Since $\pi^n I \subseteq T$, we have $\pi^n I^\star \subseteq T$. Hence $\min v(I^\star)$ exists, and $-n \leq \min v(I^\star) \leq 0$, that is, $I^\star \in Y$. If we set $I^\star = g_\star(I)$, there is a canonical mapping $g : \Sigma(T) \longrightarrow Y^X$. Lemma 6.3 implies that $g$ is an injection, hence $|\Sigma(T)| < \infty$. By Lemma 6.4 (3) and Lemma 2.3, we have $|\Sigma_2'| < \infty$, and $|\Sigma'(D)| < \infty$.

The proof for our Theorem is complete by Propositions 5.1 and 6.1.

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