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STRONGLY PRIME SUBMODULES AND PSEUDO-VALUATION MODULES

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Abstract. In this paper we introduce strongly prime submodules and pseudo-valuation modules over an integral domain, and obtain some basic results and characterizations.

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1. Introduction

Throughout this paper, R denotes an integral domain with quotient field K, $T = R\backslash\{0\}$ and M is a unitary torsion free R-module. A submodule N of M is called prime if $N \neq M$ and for arbitrary $r \in R$ and $m \in M$, $rm \in N$ implies $m \in N$ or $r \in (N : M)$, where $(N : M) = \{r \in R | rM \subseteq N\}$. It is clear that when N is a prime submodule, $(N : M)$ is a prime ideal of R.

An R-module M is called a multiplication R-module, if for each submodule N of M, there exists an ideal I of R such that $N = IM$. (For more information about multiplication modules, see [1], [3], [4], [10], [13], [14]). An integral domain R is called a valuation ring, if for each $x \in K\backslash\{0\}$, $x \in R$ or $x^{-1} \in R$. (see [5], [6], [10]). An integral domain R is called a pseudo-valuation domain, if whenever a prime ideal P contains the product xy of two elements of K, we have $x \in P$ or $y \in P$. Such a prime ideal P is called a strongly prime ideal. (see [7], [8], [9]). In the first section of this paper, we generalize the notion of strongly prime ideal to a prime submodule of a torsion free R-module and obtain results which characterize it. In the second section, we introduce pseudo-valuation modules and obtain some basic results.

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2. Strongly Prime Submodules

Let R be an integral domain with quotient field K and M be a torsion free R-module. For any submodule N of M, suppose $y = \frac{r}{s} \in K$ and $x = \frac{a}{t} \in M_T$. We say $yx \in N$, if there exists $n \in N$ such that $ra = stn$, where $T = R \backslash \{0\}$ and $M_T = \{\frac{a}{t} | a \in M, t \in T\}$. It is clear that this is a well-defined operation (see [12, p. 399]).

Definition 2.1. Let R be an integral domain with quotient field K and M be a torsion free R-module. A prime submodule P of M is called strongly prime, if for any $y \in K$ and $x \in M_T$, $yx \in P$ gives $x \in P$ or $y \in (P : M)$.

Example 2.2.

i) *Let R be a domain and $P \in Spec(R)$. P is a strongly prime ideal of R if and only if P is a strongly prime R-submodule of R.*

ii) *The zero submodule is a strongly prime submodule of M.*

iii) *For a prime number p, let*

$$R = \{p^n \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, n \in \mathbb{N}^*, (p, a) = (p, b) = 1\},$$

$$M = \{p^n \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, n \in \mathbb{N}, (p, a) = (p, b) = 1\}, \text{ and}$$

$$L = \{p^n \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, n \in \mathbb{N}, n \geq 2, (p, a) = (p, b) = 1\}.$$

Then L is a strongly prime submodule of M.

iv) *Every proper submodule of any vector space, is strongly prime.*

v) *$2\mathbb{Z}$ is a prime, but not a strongly prime submodule of $\mathbb{Z}$-module $\mathbb{Z}$.*

vi) *The unique maximal ideal of a discrete valuation domain R (which is not a field) is a strongly prime ideal, and hence a strongly prime submodule of R (See [7, Proposition 1.1]).*

Following [11], an R-submodule N of M_T is called a fractional submodule of M, if there exists $r \in T$ such that $rN \subseteq M$.

Theorem 2.3. *Let N be a proper submodule of M, then N is strongly prime if and only if for any fractional ideal I of R and any fractional submodule L of M, $IL \subseteq N$, gives $L \subseteq N$ or $I \subseteq (N : M)$.*

Proof. Let N be strongly prime, $x \in L \backslash N$ and $y \in I$. Then $yx \in IL$ and since $IL \subseteq N$, $x \in M_T \backslash N$ and $y \in K$, we have $y \in (N : M)$. So $I \subseteq (N : M)$. Conversely, it is clear that N is a prime submodule of M. Let for $y \in K$ and $x \in M_T$, $yx \in N$. Put $I = Ry$, a fractional ideal of R and $L = Rx$, a fractional submodule of M. $IL \subseteq N$ and so $L = Rx \subseteq N$ or $I = Ry \subseteq (N : M)$. Therefore $x \in N$ or $y \in (N : M)$. Thus N is a strongly prime submodule of M. □

Corollary 2.4. *Let N be a proper submodule of M. Then N is strongly prime if and only if for any $y \in K$ and any fractional submodule L of M, $yL \subseteq N$, gives $L \subseteq N$ or $y \in (N : M)$.*

Theorem 2.5. *Let N be a prime submodule of M. For the following statements we have (i)⇔(ii), (iii)⇔(iv), (iv)⇒(i).*

i) *N is a strongly prime submodule.*
ii) *For any fractional ideal I of R and any fractional submodule L of M, $IL \subseteq N$ gives $L \subseteq N$ or $I \subseteq (N : M)$.*
iii) *N is comparable to each cyclic fractional submodule of M.*
iv) *N is comparable to each fractional submodule of M.*

Proof. (i)⇔(ii) follows from by Theorem 2.3. (iv)⇒(iii) is clear.

(iii)⇒(iv) Let L be a fractional submodule of M such that $L \not\subseteq N$.

So there exists $x \in L\backslash N$. Rx is a cyclic fractional submodule of M and $Rx \not\subseteq N$. So by (iii) $N \subseteq Rx \subseteq L$.

(iv)⇒(i) Suppose that for $y = \frac{r}{s} \in K$, $x = \frac{a}{t} \in M_T$, we have $yx \in N$, $x \notin N$ and $y \notin (N : M)$. Since $x \notin N$ by (iv), $N \subseteq Rx$, $xy \in Rx$. So $y \in R$. On the other hand, since $y \notin (N : M)$, by (iv) $N \subseteq yM$, $xy \in yM$. Therefore $x \in M$. Now for $y \in R$ and $x \in M$, $yx \in N$ where N is prime, we have $x \in N$ or $y \in (N : M)$ which is a contradiction. Thus N is a strongly prime. □

Remark 2.6. *In Theorem 2.5, in general, (i)⇏(iii). For example, let $R = \mathbb{R}$, $M = \mathbb{R} \oplus \mathbb{R}$, $N = \mathbb{R} \oplus (0)$ which is strongly prime, but for $x = (0, 1)$, $Rx \not\subseteq N$ and $N \not\subseteq Rx$.*

Lemma 2.7. *Let P be a strongly prime submodule of M, then $(P : M)$ is a strongly prime ideal of R.*

Proof. It is clear that $(P : M) \in Spec(R)$. Let $ab \in (P : M)$, for $a, b \in K$. Then $a(bM) = abM \subseteq P$. By Corollary 2.4, $a \in (P : M)$ or $bM \subseteq P$. Since $P \subset M$, we have $b \in (P : M)$. So $(P : M)$ is a strongly prime ideal. □

Let $R = \{p^n \frac{a}{b} | a, b \in \mathbb{Z}, b \neq 0, n \in \mathbb{N}^*, (p, a) = (p, b) = 1\}$, $M = R[x]$. Then $P = (x)$ is a prime but not a strongly prime submodule of M, and $(P : M) = (0)$ is a strongly prime ideal of R. So the converse of Lemma 2.7, is not true.

Definition 2.8. Let M be a R-module and P be an ideal of ring R. Define

$$T_P(M) = \{m \in M \mid (1 - p)m = 0, \ for\ some\ p \in P\}.$$

If $M = T_P(M)$, then M is called a P-torsion R-module, and if there exists $m \in M$ and $q \in P$ such that $(1-q)M \subseteq Rm$, then M is called a P-cyclic R-module[1].

Theorem 2.9. *Let M be an R-module. Then M is a multiplication R-module if and only if for every maximal ideal P of R, M is P-cyclic or P-torsion R-module.*

Proof. [1, Theorem1.2]. □

Proposition 2.10. *Let P be a prime ideal of an integral domain R and M be a faithful multiplication R-module. Then P is a strongly prime ideal if and only if PM is a strongly prime submodule.*

Proof. It is enough to show the necessity. It is clear that PM is a prime submodule of M. Suppose that for $y = \frac{r}{s} \in K$, $x = \frac{a}{t} \in M_T$, $yx \in PM$. If $y \notin P$, put $A = \{b \in R | bx \in PM\}$. A is an ideal of R. If $A = R$, then $x \in PM$. Let $A \neq R$, then there exists a maximal ideal Q of R such that $A \subseteq Q$. Since $M \neq T_Q(M)$, M is Q-cyclic. So there exists $m \in M$, $q \in Q$ such that $(1-q)M \subseteq Rm$. Therefore $(1-q)PM \subseteq Pm$. Now there exists $u \in R$, $v \in P$ such that $(1-q)a = um$, $(1-q)ra = stvm$. So $ru = stv$ and $\frac{u}{t} \in P$. Since $(1-q)a = um$, hence $(1-q)x = (1-q)\frac{a}{t} = \frac{u}{t}m \in PM$ and therefore $1-q \in A \subseteqq Q$, which is a contradiction. Thus PM is a strongly prime submodule of M. □

Theorem 2.11. *Let P be a prime submodule of M. Then P is strongly prime if and only if for any $y \in K$, $y^{-1}P \subseteq P$ or $y \in (P : M)$.*

Proof. Let $y \in K\backslash(P : M)$ and $x \in P$. Since $x = yy^{-1}x \in P$ and P is a strongly prime, $y^{-1}x \in P$. So $y^{-1}P \subseteq P$. Conversely, suppose that for $y \in K$, $x \in M_T$, we have $yx \in P$. If $y^{-1}P \subseteq P$, then $x = y^{-1}(yx) \in P$. Otherwise $y \in (P : M)$. So P is strongly prime. □

Lemma 2.12. *Let L be a strongly prime submodule of M and N be a proper submodule of M such that $N_T \cap M = N$. Then $L \cap N = N$ or $L \cap N$ is a strongly prime submodule of N.*

Proof. Let $L \cap N \neq N$. It is clear that $L \cap N$ is a prime submodule of N. Let for $y \in K$, $x \in N_T$, $yx \in L \cap N$. Since $yx \in L$ and L is strongly prime, $y \in (L : M)$ or $x \in L$. Since $N_T \cap M = N$, $y \in (L \cap N : N)$ or $x \in N \cap L$. So $N \cap L$ is a strongly prime submodule of N. □

Lemma 2.13. *Let $N \subseteqq L$ be two submodules of M such that for any $y \in K$, $yN \subseteqq N$. Then L is a strongly prime submodule of M if and only if $\frac{L}{N}$ is a strongly prime submodule of $\frac{M}{N}$.*

Proof. Let L be a strongly prime submodule of M, and $y \in K$. By Theorem 2.11, $y^{-1}L \subseteq L$ or $y \in (L : M)$. So $y^{-1}\frac{L}{N} \subseteqq \frac{L}{N}$ or $y \in (\frac{L}{N} : \frac{M}{N})$.

Conversely, let $y \in K$. Since $\frac{L}{N}$ is strongly prime, $y^{-1}\frac{L}{N} \subseteqq \frac{L}{N}$ or $y \in (\frac{L}{N} : \frac{M}{N})$. So $y^{-1}L \subseteq L$ or $y \in (L : M)$ and by Theorem 2.11, L is a strongly prime submodule of M. □

Remark 2.14. *Let $f : M \to M'$ be an R-epimorphism and N' be a strongly prime submodule of M'. Then in general $N = f^{-1}(N')$ is not a strongly prime submodule of M. Consider*

$$f : \mathbb{Z}[x] \to \mathbb{Z}, \quad p[x] \mapsto p[0]$$

which is clearly a surjective $\mathbb{Z}$-module homomorphism. However the kernel of f which is $f^{-1}(0)$ is not a strongly prime submodule of $\mathbb{Z}[x]$, although $\{0\}$ is strongly prime in $\mathbb{Z}$. To see this, we can take the product $2.\frac{x}{2} = x \in f^{-1}(0)$, in which $2 \notin (f^{-1}(0) : \mathbb{Z}[x]) = 0$ and $\frac{x}{2} \notin f^{-1}(0)$.

Proposition 2.15. *Let Q be a strongly prime submodule of M and P be a prime ideal of R such that $(Q : M) \subseteq P$. Then R_P- module, Q_P is a strongly prime submodule of M_P.*

Proof. Let for $y = \frac{r}{s} \in K$ and $x = \frac{a}{t} \in M_T$, $yx \in Q_P$. Then $ra \in Q$ and since Q is a prime submodule $a \in Q$ or $rM \subseteq Q$. So $x \in Q$ or $y \in (Q_P : M_P)_{R_P}$. □

Following [2], the R-module M is said to be integrally closed whenever $y^n m_n + \cdots + ym_1 + m_0 = 0$, for some $n \in \mathbb{N}$, $y \in K$ and $m_i \in M$, then $ym_n \in M$.

Lemma 2.16. *Let P be a strongly prime submodule of an R-module M. Then P is an integrally closed R-module.*

Proof. Let $y^n x_n + \cdots + yx_1 + x_0 = 0$, for $y \in K$, $x_i \in P$. Since P is strongly prime, $y^{-1}P \subseteqq P$ or $y \in (P : M)$. If $y^{-1}P \subseteqq P$, then $y^{-i}P \subseteqq P$ for all $i \in \mathbb{N}$. So $yx_n = -(x_{n-1} + y^{-1}x_{n-2} + \cdots + y^{-(n-1)}x_0) \in P$. If $y \in (P : M)$, then $yM \subseteqq P$ and so $yx_n \in P$. Thus P is an integrally closed R-module. □

Lemma 2.17. *Let (R, m) be a quasi-local domain and M be an R-module. If M is a finitely generated R-module or $mM \neq M$, where m is a strongly prime ideal of R, then mM is a strongly prime submodule of M.*

Proof. Since $mM \neq M$ and $m \in \max(R)$, hence $mM \in Spec(M)$. Let $y \in K$. If $y \notin R$, then $y^{-1}m \subseteq m$ and so $y^{-1}mM \subseteq mM$. If $y \in R$ and $y \notin m$, then $y^{-1} \in R$ and so $y^{-1}mM \subseteq mM$.

Finally, if $y \in m$, then $y \in (mM : M) = m$. Thus mM is a strongly prime submodule of M. □

3. Pseudo-Valuation Modules

Following [7], an integral domain R is called a pseudo-valuation domain (PVD), if every prime ideal of R is a strongly prime. By [7, Lemma 2.1], any valuation domain is PVD. In this section we generalize this concept to torsion free R-modules and obtain basic results.

Definition 3.1. An R-module M is called a pseudo-valuation module (PVM), if every prime submodule of M is strongly prime.

Example 3.2.
i) *Let R be a domain. R is a PVD if and only if the R-module R is a PVM.*
ii) *The $\mathbb{Z}$-module $\mathbb{Q}$ is a PVM.*
iii) *Any vector space is PVM.*
iv) *The $\mathbb{Z}$-module $\mathbb{Z}$ is not a PVM.*

Lemma 3.3. *Let M be a PVM. Then $\{(P : M)|P \in Spec(M)\}$ is a totally ordered set.*

Proof. Let $P, Q \in Spec(M)$, $a \in (P : M)\backslash(Q : M)$ and $b \in (Q : M)$. If $\frac{a}{b} \in R$, then since $bM \subseteq Q$, we have $aM = \frac{a}{b}bM \subseteqq \frac{a}{b}Q \subseteq Q$. So $a \in (Q : M)$ which is a contradiction. Therefore $\frac{a}{b} \notin R$. By Theorem 2.11, $\frac{b}{a}P \subseteq P$. Now since $a \in (P : M)$, hence $bM = \frac{b}{a}aM \subseteq \frac{b}{a}P \subseteq P$. So $b \in (P : M)$ and $(Q : M) \subseteq (P : M)$. □

Corollary 3.4. *Let M be a multiplication PVM. Then the prime submodules of M are linearly ordered and so M has an unique maximal submodule.*

Remark 3.5. *Let $R = \{p^n\frac{a}{b}|a, b \in \mathbb{Z}, b \neq 0, n \in \mathbb{N}^*, (p, a) = (p, b) = 1\}$ and $M = R[x]$. Then R is a PVD, but M is not a PVM.*

Lemma 3.6. *Let M be a faithful multiplication R-module. Then M is a PVM if and only if R is a PVD.*

Proof. Let M be a PVM and $P \in Spec(R)$. Since M is a multiplication, $PM \in Spec(M)$ and since M is a PVM, PM is a strongly prime submodule of M. By Proposition 2.10, P is a strongly prime ideal of R. So R is a PVD. Conversely, let $N \in Spec(M)$. Since M is a multiplication, $N = PM$, for some prime ideal P of R. Since R is a PVD, P is a strongly prime ideal. Now by Proposition 2.10, N is a strongly prime submodule. So M is a PVM. □

Proposition 3.7. *Let (R, m) be a quasi-local domain and M be an R-module. For the following statements we have (i)⇒(ii)⇒(iii).*

i) *M is a PVM and m is a strongly prime ideal of R.*
ii) *For any two submodules N, L of M, $(N : M) \subseteq (L : M)$ or $m(L : M) \subseteq m(N : M)$.*
iii) *For any two submodules N, L of M, $(N : M) \subseteq (L : M)$ or $m(L : M) \subseteq (N : M)$.*

Proof. (i)⇒(ii) Let N, L be two submodules of M, such that $(N : M) \nsubseteq (L : M)$. So there exists $a \in (N : M)\backslash(L : M)$. Let $b \in (L : M)$, then $\frac{a}{b} \notin R$. Since m is a strongly prime ideal $\frac{b}{a}m \subseteq m$. So $bm \subsetneqq am \subseteq m(N : M)$. Therefore $m(L : M) \subseteq m(N : M)$.

(ii)⇒(iii) This is clear. □

Remark 3.8. *It is easily seen that in the example of Remark 3.5, (iii)⇏(i) .*

Proposition 3.9. *Let M be a finitely generated R-module. Then for the following statements we have (i)⇒(ii)⇔(iii), (i)⇒(iv)⇔(v).*

i) *M is a PVM.*
ii) *For any $y \in K\backslash R$ and $a \in M$, if $M \neq Ra$, then $Ra \subseteq yM$.*
iii) *For any $y \in K\backslash R$ and $a \in M$, if $M \neq Ra$, then $y^{-1}a \in M$.*
iv) *For any $y \in K\backslash R$ and $a \in R$, if $M \neq aM$, then $(y + a)M = yM$.*
v) *For any $y \in K\backslash R$ and $a \in R$, if $M \neq aM$, then $y^{-1}aM \subseteq M$.*

Proof. (i)⇒(ii) Let $y \in K\backslash R$, $M \neq Ra$, for $a \in M$. Since M is finitely generated, there exists a prime submodule P such that $a \in P$. By Proposition 2.10, $y^{-1}P \subseteq P$. So $y^{-1}a \in y^{-1}P \subseteq P \subseteq M$. Therefore $Ra \subseteq yM$.

(ii)⇔(iii) This is clear.

(i)⇒(iv) Let $y \in K\backslash R$, a be a non unit of R. Then $y+a \notin R$, $aM \neq M$. Since M is finitely generated there exists prime submodule P of M such that $aM \subseteq P$. On the other hand, by Lemma 2.7 and Proposition 2.10, $(y + a)^{-1}(P : M) \subseteq (P : M)$. Therefore $(y + a)^{-1}a \in (P : M) \subseteq R$. So $(y + a)^{-1}y = 1 - (y + a)^{-1}a \in R$ and $(y + a)^{-1}yM \subseteq M$. Thus $yM \subseteq (y + a)M$. Conversely, since $y \in K\backslash R$, $y^{-1}(P : M) \subseteq (P : M)$, hence $y^{-1}a \in (P : M) \subseteq R$. Therefore $(y + a)y^{-1} = 1 + y^{-1}a \in R$ and $(y + a)y^{-1}M \subseteq M$. Thus $(y + a)M \subseteq yM$.

(iv)⇒(v) Let $y = \frac{r}{s} \in K\backslash R$ and $x \in M$. So $(y + a)x \in yM$. There exists $u \in M$ such that $(y + a)x = yu$. So $(r + sa)x = ru$ and $y^{-1}ax = u - x \in M$. Therefore $y^{-1}aM \subseteq M$.

(v)⇒(iv) Let $y \in K\backslash R$, a be a non unit of R. Then $y + a \notin R$. By (v), $(y+a)^{-1}aM \subseteq M$. So $y(y+a)^{-1}M = [1-(y+a)^{-1}a]M \subseteq M-(y+a)^{-1}aM \subseteq M$. Therefore $yM \subseteq (y+a)M$. Conversely, by (v), $y^{-1}(y+a)M = (1+y^{-1}a)M \subseteq M$. So $(y + a)M \subseteq yM$. □

Proposition 3.10. *Let M be a free PVM. Then R is a PVD.*

Proof. Let P be a prime ideal of R, then $P \oplus \cdots \oplus P$ is a prime submodule of $R\oplus\cdots\oplus R$. Since M is a PVM, $P\oplus\cdots\oplus P$ is strongly prime. Let $y \in K\backslash R$. Then by Theorem 2.11, $y^{-1}(P \oplus \cdots \oplus P) \subseteq P \oplus \cdots \oplus P$ and so $y^{-1}P \subseteq P$. Therefore P is a strongly prime ideal of R and so R is a PVD. □

Proposition 3.11. *Let M be a finitely generated PVM such that every nonzero prime submodule is maximal. Then R is a PVD.*

Proof. Let P be a nonzero prime ideal of R. By [2, Lemma 3.11], $dimR = 1$. So PM is a prime submodule and hence a strongly prime submodule of M. Now by Lemma 2.7, $P = (PM : M)$ is a strongly prime ideal of R. Therefore R is a PVD. □

Lemma 3.12. *Let M be a Noetherian PVM. Then for any $y \in K\backslash R$, $y^{-1} \in \overline{R}$, where $\overline{R}$ is an integral closure of R.*

Proof. Let $y \in K\backslash R$. There exists a strongly prime submodule of M like P. So by Theorem 2.11, $y^{-1}P \subseteq P$. Since M is Noetherian, P is finitely generated, and we have $y^{-1} \in \overline{R}$. □

Lemma 3.13. *Let M be an R-module and for any $y \in K\backslash R$, $y^{-1} \in \overline{R}$. Then for any prime submodule P of M, $y^{-1}(P : M) \subseteq (P : M)$.*

Proof. Let P be a prime submodule of M. Then $(P : M) \in Spec(R)$ and there exists $q \in Spec(\overline{R})$ such that $q \cap R = (P : M)$. Let $y \in K\backslash R$. Since $y^{-1} \in \overline{R}$, we have $y^{-1}(P : M) \subseteq y^{-1}q \subseteq q$. On the other hand, we can show that $y^{-1}(P : M) \subseteq R$. So $y^{-1}(P : M) \subseteq q \cap R = (P : M)$. □

Lemma 3.14. *Let M be a Noetherian R-module such that for any $y \in K\backslash R$, $y^{-1} \in \overline{R}$. Then R is a PVD.*

Proof. Let $P \in Spec(R)$. There exists a prime submodule N of M such that $(N : M) = P$. By Lemma 3.13, $y^{-1}(N : M) \subseteq (N : M)$. So $y^{-1}P \subseteq P$. By Theorem 2.11, P is a strongly prime ideal and so R is a PVD. □

Theorem 3.15. *Let M be a Noetherian PVM. Then R is a PVD.*

Proof. Take $y \in K \backslash R$ and a prime ideal P of R. There exists a prime submodule N of M such that $(N : M) = P$. Since M is a PVM, N is a strongly prime submodule of M, and so $y^{-1}N \subseteq N$. It follows that $y^{-1}PM \subseteq y^{-1}N \subseteq N$. Since $M \neq N$ and N is strongly prime, we must have $y^{-1}P \subseteq (N : M) = P$. Therefore, it follows from [7, proposition 1.2] that P is a strongly prime ideal of R. □

Theorem 3.16. *Let M be a finitely generated noncyclic PVM which has only one maximal submodule. Then M is an integrally closed R-module.*

Proof. Let $y^n x_n + \cdots + yx_1 + x_0 = 0$, for $x_i \in M$, $y \in K$. Let P be an unique maximal submodule of M. As M is not cyclic and P is the unique maximal submodule of M, we have for any i, $x_i \in P$. Since M is a PVM, P is a strongly prime. So by Theorem 2.11, $y \in (P : M)$ or $y^{-1}P \subsetneqq P$. If $y \in (P : M)$, then $yM \subsetneqq P \subsetneqq M$ and so $yx_n \in M$. If $y^{-1}P \subseteq P$, then for any $i \in \mathbb{N}$, $y^{-i}P \subseteq P$ and so $yx_n \in P \subseteq M$. Therefore M is an integrally closed R-module. □

Lemma 3.17. *Let M be a divisible R-module. Then M is a PVM.*

Proof. Let P be a prime submodule of M, $y = \frac{r}{s} \in K$ and $x \in P$. If $y = 0$, then $y \in (P : M)$. Let $y \neq 0$, so $rM = M$. There exists $u \in M$ such that $x = ru$. Since $x \in P$ and P is a prime submodule $u \in P$ or $r \in (P : M)$.

If $r \in (P : M)$, then $M = rM \subseteq P$ which is a contradiction. So $u \in P$ and $y^{-1}x = \frac{s}{r}x = \frac{s}{r}ru = su \in P$. Therefore $y^{-1}P \subseteq P$ and P is a strongly prime submodule of M. Thus M is a PVM. □

Theorem 3.18. *Let M be an injective R-module. Then M is a PVM.*

Proof. Since any injective R-module is divisible, hence by Lemma 3.17, M is a PVM. □

Following [11], a torsion free R-module M is called a valuation R-module (VM) if for all $y \in K$, $yM \subseteq M$ or $y^{-1}M \subseteq M$.

By [7, Proposition 1.1], every VD is PVD, but by the example in Remark 3.5, any VM is not a PVM. Also by [7], since every PVD is not a VD, hence every PVM is not a VM.

Lemma 3.19. *Let M be a finitely generated, non cyclic, PVM. Then M is a VM.*

Proof. Let $y \in K$. If $y \in R$, then $yM \subsetneqq M$. If $y \notin R$, then by Proposition 3.9, (i⇒iii) for any $a \in M$, $y^{-1}a \in M$ and so $y^{-1}M \subseteq M$. Therefore M is a VM. □

Proposition 3.20. *Let M be a Noetherian, integrally closed, PVM. Then M is a VM.*

Proof. Let $y \in K$. If $y \in R$, then $yM \subsetneqq M$. If $y \notin R$, then by Lemma 3.12, $y^{-1} \in \overline{R}$. Now since M is an integrally closed R-module, M is also an $\overline{R}$-module.

So $y^{-1}M \subseteq M$ and therefore M is a VM. □

Proposition 3.21. *Let an R-module M have an invertible strongly prime submodule. Then M is a VM.*

Proof. Let P be an invertible strongly prime submodule of M, then $P'P = M$. Let $y \in K$. Then by Theorem 2.11, $y^{-1}P \subseteq P$ or $y \in (P : M)$. If $y \in (P : M)$, then $yM \subseteq P \subseteq M$. If $y^{-1}P \subseteq P$, then $y^{-1}M = y^{-1}P'P \subseteq P'P = M$. Therefore M is a VM. □

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References

[1] Z. Abd El-bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, 16(4) (1988), 755–779.

[2] M. Alkan, B. Sarac and Y. Tiras, *Dedekind modules*, Comm. Algebra, 33(5) (2005), 1617–1626.

[3] M. Alkan and Y. Tiras, *On invertible and dense submodules*, Comm. Algebra, 32(10) (2004), 3911–3919.

[4] R. Ameri, *On the prime submodules of multiplication modules*, Int. J. Math. Math. Sci., 27 (2003), 1715–1724.

[5] M. F. Atiyah and I. G. MacDonald, Introduction to Commutative Algebra, Addison-Wesley, 1969.

[6] N. Bourbaki, Commutative Algebra, Addison-Wesley, 1972.

[7] J. R. Hedstrom and Evan G. Houston, *Pseudo-valuation domains*, Pacific J. Math., 75(1) (1978), 137–147.

[8] J. R. Hedstrom and E. G. Houston, *Pseudo-valuation domains (II)*, Houston J. Math., 4(2) (1978), 199–207.

[9] J. A. Huckaba, *Extensions of pseudo-valuations*, Pacific J. Math. 29(2) (1969), 295–302.

[10] M. D. Larsen and P. J. McCarthy, Multiplicative Theory of Ideals, Academic Press, London, 1971.

[11] J. Moghaderi and R. Nekooei, *Valuation, discrete valuation and Dedekind modules*, Int. Electron. J. Algebra, 8 (2010), 18–29.

[12] A. G. Naoum and F. H. Al-Alwan, *Dedekind modules*, Comm. Algebra, 24(2) (1996), 397–412.

[13] R. Nekooei, *On finitely generated multiplication modules*, Czechoslovak Math. J., 55(130)(2005), 503–510.

[14] P. F. Smith, *Some remarks on multiplication modules*, Arch. Math., 50(1988), 223–235.

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