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FINITE GROUPS WITH WEAKLY S-SEMIPERMUTABLY EMBEDDED SUBGROUPS

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Abstract. A subgroup H of G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. This concept was introduced by Kegel in 1962 and has been investigated by many authors. A subgroup H is called S-semipermutable in G if H permutes with every Sylow p-subgroup of G for which $(p, |H|) = 1$. A subgroup H of the group G is said to be c-normal in G if there is a normal subgroup B of G such that $HB = G$ and $H \cap B$ is normal in G. Next, we unify and generalize the above concepts and give the following concept: A subgroup H of the group G is said to be weakly S-semipermutably embedded in G if there is a subnormal subgroup B of G such that $HB = G$ and $H \cap B$ is S-semipermutable or S-quasinormally embedded in G. Groups with certain weakly S-semipermutably embedded subgroups of prime power order are studied.

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1. Introduction

All groups considered in this paper will be finite, the notation and terminology used in this paper are standard, as in [14-16]. In particular, let G be a finite group, we denote $F(G)$ the Fitting subgroup of G, $F^*(G)$ the generalized Fitting subgroup of G, $\Phi(G)$ the Frattini subgroup of G. Given a group G, two subgroups H and K of G are said to permute if $HK = KH$, that is, HK is a subgroup of G. About the generalizing permutability, Foguel in [4] introduced the following concept: For a group G, a subgroup H of G is said to be conjugate permutable if $HH^x = H^xH$ for any $x \in G$. A subgroup H of G is said to be S-quasinormal in G if it permutes

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with every Sylow subgroup of G. This concept was introduced by Kegel in 1962 and has been investigated by many authors, for example, see [1-8, 10-13, 15-18].

In 1998, Ballester-Bolinches and Pedraza-Aguilera extended this concept to S-quasinormally embedded subgroups.

Definition 1.1. A subgroup H of G is *S-quasinormally embedded in* G if for every Sylow subgroup P of H, there is a S-quasinormal subgroup K in G such that P is also a Sylow subgroup of K.

Recently, Chen introduced the following concept.

Definition 1.2. A subgroup H is called *S-semipermutable in* G if H permutes with every Sylow p-subgroup of G for which $(p, |H|) = 1$.

In 1996, Wang introduced the concept of c-normal subgroup.

Definition 1.3. Let G be a group. A subgroup H of the group G is said to be *c-normal in* G if there is a normal subgroup B of G such that $HB = G$ and $H \cap B \leq H_G$.

In this paper, we unify and generalize S-semipermutable, c-normal and S-quasi-normally embedded subgroups, and give the following definition:

Definition 1.4. Let G be a group. A subgroup H of the group G is said to be *weakly S-semipermutably embedded in* G if there is a subnormal subgroup B of G such that $HB = G$ and $H \cap B$ is S-semipermutable or S-quasinormally embedded in G.

Obviously, every S-semipermutable subgroup, c-normal subgroup of G is weakly S-semipermutably embedded. In general, a weakly S-semipermutablly embedded subgroup need not be S-semipermutable subgroup, or c-normal subgroup. For instance, $\langle(34)\rangle$ is a weakly S-semipermutably embedded subgroup of S_4, because $S_4 = \langle(34)\rangle A_4$ and $\langle(34)\rangle \cap A_4 = 1$. However, $\langle(34)\rangle$ is not S-semipermutable subgroup of S_4, because $\langle(34)\rangle\langle(123)\rangle \neq \langle(123)\rangle\langle(34)\rangle$.

Recall that a formation is a class $\mathcal{F}$ of groups satisfying the following conditions: (i) if $G \in \mathcal{F}$ and $N \trianglelefteq G$, then $G/N \in \mathcal{F}$, and (ii) if N_1, $N_2 \trianglelefteq G$ are such that G/N_1, $G/N_2 \in \mathcal{F}$, then $G/(N_1 \cap N_2) \in \mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$.

We study the influence of weakly S-semipermutably embedded subgroups on the structure of group G. The main results are as follows:

Theorem 1.1. *Let p be the smallest prime divisor dividing the order of a group G and P a Sylow p-subgroup of G. Then the following two statements are equivalent:*

(i) *G is p-nilpotent;*

(ii) *P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutably embedded in G.*

Theorem 1.2. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutably embedded in G. Then $G \in \mathcal{F}$.*

2. Preliminaries

Our first result is very useful in proofs using induction arguments. Its proof is a routine checking.

Lemma 2.1. *Suppose that H is weakly S-semipermutable embedded in a group G, $K \leq G$ and N a normal subgroup of G. We have:*

(i) *If $H \leq K$, then H is weakly S-semipermutably embedded in K;*

(ii) *HN/N is weakly S-semipermutably embedded in G/N satisfying $(|H|, |N|) = 1$;*

(iii) *If $N \leq K$ and K/N is weakly S-semipermutably embedded in G/N if and only of K is weakly S-semipermutably embedded in G.*

We will use the following result, which comes from [20, Property 2].

Lemma 2.2. *Suppose that H is an S-semipermutable subgroup of G. Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is S-semipermutable in G/N.*

Lemma 2.3. ([12, Lemma 2.3]) *Suppose that H is S-quasinormal in G, P is a Sylow p-subgroup of H. If $H_G = 1$, then P is S-quasinormal in G.*

Lemma 2.4. *Suppose that H is a p-subgroup for some prime p and H is not S-semipermutable, or S-quasinormally embedded in G. Assume that H is weakly S-semipermutably embedded in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Proof. By the hypothesis, G has a subnormal subgroup T such that $HT = G$ and $T \cap H < H$. Hence G has a proper normal subgroup K such that $T \leq K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$. □

Lemma 2.5. ([11, Lemma 2.2]) *Let H be a p-subgroup of G. Then the following statements are equivalent:*

(i) *H is S-quasinormal in G;*

(ii) *$H \leq O_p(G)$ and H is S-quasinormal embedded in G.*

Lemma 2.6. *Suppose that $H \leq O_p(G)$ and that H is weakly S-semipermutably embedded in G. Then H is weakly S-permutable in G.*

Proof. By the hypothesis, G has a subnormal subgroup B such that $HB = G$ and $H \cap B$ is S-semipermutable or S-quasinormally embedded in G.

Assume that $H \cap B$ is S-semipermutable in G. Note that $H \cap B \leq H \leq O_p(G)$, then $H \cap B$ is S-permutable in G, and thus $H \cap B \leq H_{sG}$. Hence H is weakly S-permutable in G.

Assume that $H \cap B$ is S-quasinormally embedded in G. Note that $H \cap B \leq H \leq O_p(G)$, then by Lemma 2.5 $H \cap B$ is S-quasinormal in G, and thus $H \cap B \leq H_{sG}$. Hence H is weakly S-permutable in G. □

By Lemma 2.11 of [16] and Lemma 2.6, we have the following.

Lemma 2.7. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly S-semipermutably embedded in G. Then some maximal subgroup of N is normal in G.*

By Lemma 2.12 of [16] and Lemma 2.6, we have the following.

Lemma 2.8. *Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and let G be a group with solvable $\mathcal{F}$-residual $P = G^{\mathcal{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathcal{F}$. Then P is a p-group for some prime p. In addition, if every cyclic subgroup of P with prime order or order 4 (if $p = 2$ and P is non-abelian) not having a supersolvable supplement in G is weakly S-semipermutably embedded in G, then $|P/\Phi(P)| = p$.*

Lemma 2.9. ([16, Lemma 2.17]) *Let G be a group and M a subgroup of G.*

(i) *If M is normal in G, then $F^*(M) \leq F^*(G)$.*

(ii) *$F^*(G) \neq 1$ if $G \neq 1$; in fact, $F^*(G)/F(G) = soc(F(G)C_G(F(G))/F(G))$.*

(iii) *$F^*(F^*(G)) = F^*(G) \geq F(G)$; if $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

(iv) *Suppose K is a subgroup of G contained in $Z(G)$, then $F^*(G/K) = F^*(G)/K$. Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly S-semipermutably embedded in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.10. ([8, Theorem 4.10]) *Let A and B be subgroups of G satisfying $G \neq AB$, if $AB^g = B^gA$ holds for all $g \in G$, then A or B is contained in a proper normal subgroups of G.Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly S-semipermutably embedded in G. Then some maximal subgroup of N is normal in G.*

Lemma 2.11. ([3, A, 1.2]) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(i) *$U \cap VW = (U \cap V)(U \cap W)$;*

(ii) *$UV \cap UW = U(V \cap W)$.*

3. Proofs of main Theorems

Theorem 3.1. *Let p be the smallest prime divisor dividing the order of a group G and P a Sylow p-subgroup of G. Then the following two statements are equivalent:*

(i) *G is p-nilpotent;*

(ii) *All maximal subgroups of P are weakly S-semipermutably embedded in G.*

Proof. We only need to prove that (ii) implies (i). Suppose that the theorem is false and that G is a counter-example with minimal order. We will derive a contradiction in several steps.

(1) G has the unique minimal normal subgroup N such that G/N is p-nilpotent and $\Phi(G)$=1.

Let N be a minimal normal subgroup of G. Consider the group G/N, we will show that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. It is easy to see that $M = P_1N$ for some maximal subgroup P_1 of P. It follows that $P \cap N = P_1 \cap N$ is a Sylow subgroup of N. By the hypothesis, there is a subnormal subgroup K_1 of G such that $G = P_1K_1$ and that $P_1 \cap K_1$ is S-semipermutable or S-quasinormally embedded in G. Then $G/N = (M/N)(K_1N/N) = (P_1N/N)(K_1N/N)$. It is easy to see that K_1N/N is a subnormal subgroup of G/N. Since $(|N/P_1 \cap N|, |N/N \cap K_1|) = 1$. $(P_1 \cap N)(K_1 \cap N) =$

$N = N \cap G = N \cap P_1K_1$. By Lemma 2.11, $(P_1N) \cap (K_1N) = (P_1 \cap K_1)N$. It follows from Lemma 2.2 and [1, Lemma 1] that $(P_1N/N) \cap (K_1N/N) = (P_1 \cap K_1)N/N$ is S-semipermutable or S-quasinormally embedded in G/N. Hence M/N is weakly S-semipermutably embedded in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. By the uniqueness of N, $\Phi(G) = 1$.

(2) $O_{p'}(G)$=1.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by (1). Now the p-nilpotency of G/N implies that G is p-nilpotent, a contradiction.

(3) $O_p(G)$=1.

If $O_p(G) \neq 1$, then $N \leq O_p(G)$ by (1). Therefore, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. Since $O_p(G) \cap M$ is normalized by N and M, hence by the uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Since $P \cap M < P$, there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Then $P = NP_1$. By the hypothesis, there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is S-semipermutable or S-quasinormally embedded in G.

Case 1. If $P_1 \cap T$ is S-semipermutable in G, then $(P_1 \cap T)G_q$ is a subgroup, where $q \neq p$ and $G_q \in \mathrm{Sly}_q(G)$. Since $(P_1 \cap T) \cap N = ((P_1 \cap T)G_q) \cap N \trianglelefteq ((P_1 \cap T)G_q)$, $G_q \leq N_G((P_1 \cap T) \cap N)$. On the other hand, Since $N \leq O^p(G) \leq T$, $P_1 \cap N = (P_1 \cap T) \cap N$. Moreover, $P_1 \cap N \trianglelefteq P$. Therefore $P_1 \cap N \trianglelefteq G$. By the uniqueness of N, $P_1 \cap N = 1$ and so $|N| = p$. The p-nilpotenty of M implies that G is p-nilpotent, a contradiction.

Case 2. If $P_1 \cap T$ is S-quasinormally embedded in G, then there is an S-quasinormal subgroup K of G such that $P_1 \cap T \in \mathrm{Syl}_p(K)$. Assume $K_G \neq 1$, then $N \leq K_G \leq K$. It follows that $N \leq P_1 \cap T$, and so $P = NP_1 = P_1$, a contradiction. So $K_G = 1$. Then by Lemma 2.3, $P_1 \cap T$ is S-quasinormal in G. By Case 1, G is p-nilpotent, a contradiction. Thus (3) holds.

(4) G is non-solvable and hence N is a direction production of some non-abelian simple groups.
By (2) and (3).

(5) The final contradiction.

If $N \cap P \leq \Phi(P)$, then N is p-nilpotent by Tate's theorem (Huppert, 1967, Satz 4.7, p. 431), contrary to (4). Consequently, there is a maximal subgroup P_1 of P such that $P = (N \cap P)P_1$. By the hypothesis, P_1 is weakly S-semipermutably embedded in G and so there is a subnormal subgroup T of G such that $G = P_1T$

and $P_1 \cap T$ is S-semipermutable or S-quasinormally embedded in G. If $P_1 \cap T$ is S-quasinormally embedded in G, then there is an S-quasinormal subgroup K of G such that $P_1 \cap T \in \mathrm{Sly}_p(K)$. If $K_G \neq 1$, then $N \leq K_G \leq K$. Since $P_1 \cap T \in \mathrm{Sly}_p(K)$, $(P_1 \cap T) \cap N \in \mathrm{Sly}_p(N)$. Moreover, $P \cap N \in \mathrm{Sly}_p(N)$, so $(P_1 \cap T) \cap N = P_1 \cap N = P \cap N$. Consequently, $P = (P_1 \cap N)P_1 = P_1$, a contradiction. Therefore, $K_G = 1$. Then by Lemma 2.3, $P_1 \cap T$ is S-quasinormal in G. Thus $(P_1 \cap T)G_q$ is a subgroup, where $q \neq p$ and $G_q \in \mathrm{Sly}_q(G)$. Since $(P_1 \cap T) \cap N = ((P_1 \cap T)G_q) \cap N \trianglelefteq ((P_1 \cap T)G_q)$, $G_q \leq N_G((P_1 \cap T) \cap N)$. On the other hand, Since $N \leq O^p(G) \leq T$, $P_1 \cap N = (P_1 \cap T) \cap N$. Note that $P = (N \cap P)P_1$, thus $P_1 \cap N \trianglelefteq P$. Therefore $P_1 \cap N \trianglelefteq G$. By the uniqueness of N, $P_1 \cap N = 1$ and so $|P \cap N| = p$. Recall that $P \cap N \in \mathrm{Sly}_p(N)$, then N is p-nilpotent, contrary to (4). The contradiction completes the proof of the theorem. □

Now we are ready to prove Theorem 1.1.

Proof. Assume that the theorem is not true and let G be a counter-example of minimal order. We prove the theorem by the following several of steps.

(1) $O_{p'}(G) = 1$.

In fact, if $O_{p'}(G) \neq 1$, then we consider the quotient group $G/O_{p'}(G)$. By Lemma 2.1, $G/O_{p'}(G)$ satisfies the hypotheses of the theorem, it follows that $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Hence G is p-nilpotent, a contradiction.

(2) $|D| > p$.

If $|D| = p$, then by Lemma 2.1, G is a minimal non-p-nilpotent group, so $G = [P]Q$, where P, Q are Sylow p-subgroup and Sylow q-subgroup of G, respectively. Set $\Phi = \Phi(P)$ and let X/Φ be subgroup of P/Φ of order p, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then L is order p or 4. L is weakly S-semipermutably embedded in G. Lemma 2.8 implies that $|P/\Phi| = p$, it follows that G is p-nilpotent.

(3) $|P : D| > p$.

By Theorem 3.1.

(4) If $N \leq P$ and N is minimal normal subgroup of G, then $|N| \leq |D|$.

Assume that $|N| > |D|$. It follows by the hypothesis that all subgroups of N of order $|D|$ are weakly S-semipermutably embedded subgroups. Since $N \leq O_p(G)$, N is an elementary abelian group. Then by Lemma 2.7, some maximal subgroup N_1 of N is normal in G. It follows from the minimality of N that $N_1 = 1$, thus $|N| = |D| = p$, a contradiction.

(5) If $N \leq P$ and N is minimal normal subgroup of G, then G/N is p-nilpotent.

If $|N| < |D|$, then it follows by Lemma 2.1 that G/N is p-nilpotent. By (4), we may assume that $|N| = |D|$. Let $N \leq K \leq P$ such that $|K/N| = p$. By (2), N is non-cyclic, so K is also non-cyclic, it follows that K has a maximal subgroup $L \neq N$ and $K = LN$. So L is weakly S-semipermutably embedded in G (note that $|L| = |D|$), it follows that $K/N = LN/N$ is weakly S-semipermutably embedded in G/N. If P/N is abelian, then G/N satisfies hypothesis. Next suppose that that P/N is a non-abelian 2-group. So every subgroup of P of order $2|D|$ is weakly S-semipermutablly embedded in G. In this case one can show as above that every subgroup X of P containing N and such that $|X : N| = 4$ is weakly S-semipermutably embedded in G. Therefore G/N also satisfies the hypothesis.

(6) $O_p(G) = 1$.

If $O_p(G) \neq 1$, then we can find a minimal normal subgroup N of G contained in $O_p(G)$. Note that $N \nleq \Phi(G)$, thus there is a maximal subgroup M of G such that $G = NM$ and $M \cap N = 1$.

By (5), M is p-nilpotent. So $M = M_pO_{p'}(M)$ and $G = NM_pO_{p'}(M)$. Let M_0 be a maximal subgroup of M_p. Then $|G : (NM_0O_{p'}(M))| = p$. Since p is the smallest prime, $NM_0O_{p'}(M) \trianglelefteq G$, and so $P \cap (NM_0O_{p'}(M))$ is a Sylow p-subgroup of $NM_0O_{p'}(M)$. Moreover, $1 < |D| < |P \cap (NM_0O_{p'}(M))|$ by (3). Then the group $NM_0O_{p'}(M)$ also satisfies the hypothesis. Hence by induction, $NM_0O_{p'}(M)$ is p-nilpotent and so $O_{p'}(M) \trianglelefteq G$. Hence G is p-nilpotent. Thus we have (6).

(7) G is non-abelian simple.

If G is not simple, then there exists a minimal normal subgroup L. If $|L_p| > |D|$, then L satisfies the hypothesis. Hence by induction, L is p-nilpotent. By (1), $O_{p'}(G) = 1$, so L is a p-group. (6) implies $L = 1$, this is a contradiction. Therefore, $|L_p| \leq |D|$. So there exists a subgroup P_0 such that $L \cap P \leq P_0 \leq P$ and $|P_0| = p|D|$. Moreover, we have that P_0 is Sylow p-subgroup P_0L. By (3), P_0 is a proper subgroup of P and thus P_0L is also a proper subgroup of G. Note that P_0L also satisfies the hypothesis. Hence by induction, P_0L is p-nilpotent. Hence L is p-nilpotent, a contradiction.

(8) All subgroups of P of order $|D|$ and $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are S-semipermutable or S-quasinormally embedded in G.

Let $H \leq P$ and $|H| = |D|$ or $2|D|$. If H isn't S-semipermutable or S-quasinormally embedded in G, by Lemma 2.4, there is a normal subgroup M of G such that

$|G : M| = p$. By (3), M is p-nilpotent, it follows that G is p-nilpotent, a contradiction.

(9) The final contradiction.

Let H be a subgroup of P of order $|D|$. If H is S-semipermutable, then there exists a Sylow q-subgroup Q of G, such that $HQ^g = Q^gH$, where $q \neq p$ and $g \in G$. Note that G is a non-abelian simple group, then it follows by Lemma 2.10 that $G = HQ$, thus G is solvable, a contradiction. If H is S-quasinormally embedded in G, then there exists a S-quasinormal subgroup R such that H is Sylow p-subgroup of R. Since a S-quasinormal subgroup is subnormal subgroup, it follows by (7) that $R = G$. Hence H is Sylow p-subgroup of G, a contradiction. The contradiction completes the proof. □

Applying Theorem 1.1, we easily get the following three results.

Corollary 3.1. *Let G be a group. If, for every prime p dividing the order of G and $P \in Syl_p(G)$, P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutablly embedded in G, then G has the Sylow tower property of supersolvable type.*

Corollary 3.2. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are S-semipermutable in G, then G is p-nilpotent.*

Corollary 3.3. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are S-permutable in G, then G is p-nilpotent.*

Theorem 3.4. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutably embedded in G. Then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is not true and let G be a counter-example of minimal order. We have the following claims:

(1) Claim that $G/Q \in \mathcal{F}$, where Q is a Sylow q-subgroup of E and q is the largest prime dividing $|E|$.

By Corollary 3.1, E has the Sylow Tower property. Let q be the largest prime dividing $|E|$ and Q a Sylow q-subgroup of E. The fact that E possesses Sylow Tower property implies that Q is normal in E. Now Q is characteristic in E and $E \trianglelefteq G$, so $Q \trianglelefteq G$. Furthermore, $(G/Q)/(E/Q) \cong G/E \in \mathcal{F}$ and Lemma 2.1 shows that G/Q satisfies the conditions of the theorem, thus by the choice of G, $G/Q \in \mathcal{F}$.

(2) Every subgroup H of Q with order $|H| = |D|$ is weakly S-permutable in G.

By lemma 2.6, we have (2).

(3) If $N \leq Q$ and N is minimal normal subgroup of G, then $G/N \in \mathcal{F}$.

If either $|N| < |D|$ or $|Q : D| = q$, then it is clear. So let $|N| = |D|$ and $|Q : D| > q$. Let $N \leq K \leq P$ with $|K/N| = p$. By Lemma 2.7, $|D| > q$, it follows that N is non-cyclic, so K is also non-cyclic. Hence K has a maximal subgroup $L \neq N$ and $K = LN$. So L is weakly S-permutable in G, it follows that $K/N = LN/N$ is weakly S-permutable in G/N. Therefore G/N satisfies the hypothesis, as desired.

(4) Final contradiction.

Let N be a minimal normal subgroup of G contained in Q. Then by (3), N is the only minimal normal subgroup of G contained in Q and so $N = Q$. But by Lemma 2.7 it is impossible, because Q is a minimal normal subgroup of G. This contradiction completes the proof of this theorem. □

By Theorem 1.3 of [16] and Lemma 2.6, we have the following.

Corollary 3.5. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and G a group with a solvable normal subgroup E such that $G/E \in \mathcal{F}$ Suppose that every non-cyclic Sylow subgroup P of $F(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutably embedded in G. Then $G \in \mathcal{F}$.*

Theorem 3.6. *G a group with a normal subgroup E such that G/E is supersolvable, Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$*

and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are weakly S-semipermutably embedded in G. Then G is supersolvable.

Proof. Suppose that the theorem is false and let G be a counterexample of smallest order, then we have:

(1) Every proper normal subgroup of G containing $F^*(E)$ is supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, we have that $N/N \cap E = NE/E$ is supersolvable. By Lemma 2.9, $F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E)$, so $F^*(E \cap N) = F^*(E)$. By Lemma 2.1, $(N, N \cap E)$ satisfy the hypotheses of the theorem, thus the minimal choice of G implies that N is supersolvable.

(2) $E = G$, and $F^*(E) = F(G) < G$.

If $E < G$, then E is supersolvable by (1). In particular, E is solvable, so G is solvable and $F^*(E) = F(E)$, it follows that G is supersolvable by applying Corollary 3.5, a contradiction. If $F^*(G) = G$, then G is supersolvable by Theorem 3.4, a contradiction. Thus $F^*(G) < G$ and $F^*(G)$ is supersolvable by (1), it follows by Lemma 2.9 that $F^*(E) = F^*(G) = F(G)$.

(3) Final contradiction.

Applying Corollary 3.5, G is supersolvable, the final contradiction. □

Proof of Theorem 1.2. By Lemma 2.1, we have that all subgroups of any Sylow subgroup of order $|D|$ of $F^*(E)$ are Weakly S-semipermutably embedded in E, so Theorem 3.6 implies that E is supersolvable. Hence $F^*(E) = F(E)$. Let P be a Sylow p-subgroup of $F(E)$, for some prime p, and let H be an arbitrary subgroup of order $|D|$ of P. Since P is normal in G, it follows that H is subnormal in G. By the hypotheses, H is Weakly S-semipermutably embedded in G. So H is Weakly S-permutable in G by Lemma 2.6. Thus all subgroups of P of order $|D|$ are Weakly S-permutable in G. Applying Corollary 3.5, G belongs to $\mathcal{F}$. □

4. Application

Corollary 4.1. *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If P has a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) are Weakly s-permutable in G, then G is p-nilpotent.*

Corollary 4.2. ([16, Theorem 1.3]) *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$ Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such* *that* $1 < |D| < |P|$ *and all subgroups* H *of* P *with order* $|H| = |D|$ *and with order* $2|D|$ *(if* P *is a non-abelian 2-group and* $|P : D| > 2$*) are weakly s-permutable in* G*. Then* $G \in \mathcal{F}$*.*

Corollary 4.3. ([19, Theorem 4.2]) *Let* $\mathcal{F}$ *be a saturated formation containing all supersolvable groups and* G *a group. Then* $G \in \mathcal{F}$ *if and only if there exists a normal subgroup* E *such that* $G/E \in \mathcal{F}$ *and all maximal subgroups of any Sylow subgroup of* $F(E)$ *are c-normal in* G*.*

Corollary 4.4. ([19, Theorem 4.1]) *Let* $\mathcal{F}$ *be a saturated formation containing all supersolvable groups and* G *a group. Then* $G \in \mathcal{F}$ *if and only if there exists a normal subgroup* E *such that* $G/E \in \mathcal{F}$ *and all maximal subgroups of any Sylow subgroup of* $F^*(E)$ *are c-normal in* G*.*

Corollary 4.5. ([13, Theorem 2.3]) *Let* p *be the smallest prime dividing* $|G|$ *and let* P *be a Sylow* p*-subgroup of* G *of exponent* p^e *where* $e > 1$*. Suppose that all members of the family* $\{H|H < P, H' = 1, Exp(H) = p^e\}$ *are S-quasinormal in* G*. Then* G *has a normal* p*-complement.*

Corollary 4.6. *Let* p *be the smallest prime dividing the order of a group* G *and* P *a Sylow* p*-subgroup of* G*. If* P *has a subgroup* D *such that* $1 < |D| < |P|$ *and all subgroups* H *of* P *with order* $|H| = |D|$ *and with order* $2|D|$ *(if* P *is a non-abelian 2-group and* $|P : D| > 2$*) are SS-quasinormal in* G*, then* G *is* p*-nilpotent.*

Corollary 4.7. ([11, Theorem 3.1]) *Let* G *be a group and* P *a Sylow* p*-subgroup of* G*, where* p *is the minimal prime divisor of* $|G|$*. If every maximal subgroup of* P *is SS-quasinormal in* G*, then* G *is* p*-nilpotent.*

Corollary 4.8. ([11, Theorem 3.2]) *Let* $\mathcal{F}$ *be a saturated formation containing all supersolvable groups and* G *a group. Then* $G \in \mathcal{F}$ *if and only if there exists a normal subgroup* E *such that* $G/E \in \mathcal{F}$ *and all maximal subgroups of any Sylow subgroup of* $F(E)$ *are SS-quasinormal in* G*.*

Corollary 4.9. ([11, Theorem 3.3]) *Let* $\mathcal{F}$ *be a saturated formation containing all supersolvable groups and* G *a group. Then* $G \in \mathcal{F}$ *if and only if there exists a normal subgroup* E *such that* $G/E \in \mathcal{F}$ *and all maximal subgroups of any Sylow subgroup of* $F^*(E)$ *are SS-quasinormal in* G*.*

Corollary 4.10. ([11, Theorem 3.4]) *Let* $\mathcal{F}$ *be a saturated formation containing all supersolvable groups and* G *a group. Then* $G \in \mathcal{F}$ *if and only if there exists a normal subgroup* E *such that* $G/E \in \mathcal{F}$ *and the cyclic subgroups of prime order or order 4 of* $F(E)$ *are SS-quasinormal in* G*.*

Corollary 4.11. ([11, Theorem 3.5]) *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order 4 of $F^*(E)$ are SS-quasinormal in G.*

Corollary 4.12. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order 4 of $F(E)$ are c-normal in G.*

Corollary 4.13. *Let $\mathcal{F}$ be a saturated formation containing all supersolvable groups and G a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup E such that $G/E \in \mathcal{F}$ and the cyclic subgroups of prime order or order 4 of $F^*(E)$ are c-normal in G.*

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