International Electronic Journal of Algebra
Volume 19 (2016) 110-118

CONSTRUCTION OF HOMOTOPY EQUIVALENCE OF TRUNCATED COMPLEXES

Yanping Rao and Gang Yang

Received: 10 June 2015; Revised: 2 August 2015
Communicated by Abdullah Harmancı

Abstract. For a ring R, given two truncated proper left $\mathcal{C}$-resolutions of equal length for the same module, where $\mathcal{C}$ is a subcategory of R-modules, we obtain a pair of complexes of the same homotopy type and give some examples.

Mathematics Subject Classification (2010): 16E05
Keywords: Proper resolution, homotopy equivalent, precover, precovering class

1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary modules. Let R be a ring. We denote by R-Mod the category of left R-modules.

Truncated projective resolutions are of interest in both algebraic geometry and algebraic topology. The final modules of two truncated projective resolutions of the same module may be stabilized to produce homotopy equivalent complexes.

In 2007, Mannan in [3] considered two truncated projective resolutions of equal length for the same module and obtained a pair of complexes of the same homotopy type. This paper generalizes projective resolutions to proper left $\mathcal{C}$-resolutions and similar results are obtained, where $\mathcal{C}$ is a subcategory of R-modules. Moreover, some examples are given.

2. Main results

Let $\mathcal{C}$ be a subcategory of R-modules. Recall that a complex of modules

$$X = \cdots \to X_1 \to X_0 \to X^0 \to X^1 \to \cdots$$

in R-Mod is called $\mathrm{Hom}_R(\mathcal{C}, -)$-exact (respectively, $\mathrm{Hom}_R(-, \mathcal{C})$-exact) if it remains exact after applying the functor $\mathrm{Hom}_R(C, -)$ (respectively, $\mathrm{Hom}_R(-, C)$) for any object $C \in \mathcal{C}$.

This work was partly supported by NSF of China (Grant No. 11561039), and NSF of Gansu Province of China (No. 145RJZA079).

Definition 2.1. (see [1]) Let $\mathcal{C}$ be a subcategory of R-modules and M a module in R-Mod. A homomorphism $f : C \to M$ with $C \in \mathcal{C}$ is called a $\mathcal{C}$-precover of M if the abelian group homomorphism $\mathrm{Hom}_R(C', f) : \mathrm{Hom}_R(C', C) \to \mathrm{Hom}_R(C', M)$ is surjective for any object $C' \in \mathcal{C}$. If every R-module has a $\mathcal{C}$-precover, we say that $\mathcal{C}$ is a precovering class. Dually, we have the definitions of a $\mathcal{C}$-preenvelope and a preenveloping class.

Definition 2.2. Let $\mathcal{C}$ be a subcategory of R-modules and M a module in R-Mod. A complex $\cdots \to C_1 \to C_0 \to M \to 0$ is called a proper left $\mathcal{C}$-resolution of M if each object $C_i \in \mathcal{C}$ and if $C_0 \to M$, $C_{i+1} \to Ker(C_i \to C_{i-1})$ for $i \geq 0$ are all $\mathcal{C}$-precovers (equivalently, the complex X is $\mathrm{Hom}_R(\mathcal{C}, -)$-exact), where $C_{-1} = M$. Dually, a proper right $\mathcal{C}$-resolution $0 \to M \to C^0 \to C^1 \to \cdots$ of M can be defined. A proper $\mathcal{C}$-resolution of M is a complex $\cdots \to C_1 \to C_0 \to C^0 \to C^1 \to \cdots$, where $M = \mathrm{Im}(C_0 \longrightarrow C^0)$, complex $\cdots \to C_1 \to C_0 \to M \to 0$ is a proper left $\mathcal{C}$-resolution of M and complex $0 \to M \to C^0 \to C^1 \to \cdots$ is a proper right $\mathcal{C}$-resolution of M.

Let

$$X = \cdots \longrightarrow X_{i+1} \xrightarrow{\alpha_{i+1}} X_i \xrightarrow{\alpha_i} X_{i-1} \xrightarrow{\alpha_{i-1}} \cdots$$

and

$$Y = \cdots \longrightarrow Y_{i+1} \xrightarrow{\beta_{i+1}} Y_i \xrightarrow{\beta_i} Y_{i-1} \xrightarrow{\beta_{i-1}} \cdots$$

be two complexes. Recall that a chain map $f : X \to Y$ is a family of morphisms $\{f_i\}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_{i+1} & \xrightarrow{\alpha_{i+1}} & X_i & \xrightarrow{\alpha_i} & X_{i-1} & \xrightarrow{\alpha_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle f_{i+1}} & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\
\cdots & \longrightarrow & Y_{i+1} & \xrightarrow{\beta_{i+1}} & Y_i & \xrightarrow{\beta_i} & Y_{i-1} & \xrightarrow{\beta_{i-1}} & \cdots .
\end{array}$$

Recall that two chain maps $f, g : X \to Y$ are said to be chain homotopic, denoted by $f \sim g$, if there exists a family of morphisms $\{s_i\}$ with each $s_i : X_i \to Y_{i+1}$ a morphism such that

$$f_i - g_i = \beta_{i+1} s_i + s_{i-1} \alpha_i.$$

The complexes X and Y are said to be chain homotopy equivalent, if there exist chain maps $f : X \to Y$ and $g : Y \to X$, such that $fg \sim I_Y$ and $gf \sim I_X$, where I_Y, I_X are identity maps.

Lemma 2.3. *Let* $F = \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$ *be a* $Hom_R(\mathcal{C}, -)$*-exact sequence of* R*-modules. By adding* $\cdots \to 0 \to E \xrightarrow{id} E \to 0 \to \cdots$ *to* F*,* *then the sequence* $G = \cdots \to F_1 \to F_0 \oplus E \to F^0 \oplus E \to F^1 \to \cdots$ *is* $Hom_R(\mathcal{C}, -)$-*exact. Furthermore, the sequences* F *and* G *are chain homotopy equivalent.*

Proof. For any $C \in \mathcal{C}$, applying the functor $\mathrm{Hom}_R(C, -)$ to G, we have the following commutative diagram

$$\begin{array}{ccccccccc} \cdots \to & \mathrm{Hom}_R(C, A) & \to & \mathrm{Hom}_R(C, F_0 \oplus E) & \to & \mathrm{Hom}_R(C, F^0 \oplus E) & \to & \mathrm{Hom}_R(C, F^1) & \to \cdots \\ & \| & & \downarrow \varphi & & \downarrow \psi & & \| & \\ \cdots \to & \mathrm{Hom}_R(C, A) & \longrightarrow & H & \longrightarrow & L & \longrightarrow & \mathrm{Hom}_R(C, F^1) & \to \cdots \end{array}$$

where $H = \mathrm{Hom}_R(C, F_0) \oplus \mathrm{Hom}_R(C, E)$, $L = \mathrm{Hom}_R(C, F^0) \oplus \mathrm{Hom}_R(C, E)$, φ and ψ are isomorphisms. It is easy to see that the lower sequence in the above diagram is exact, so is the upper sequence. Thus the sequence G is $\mathrm{Hom}_R(\mathcal{C}, -)$-exact. That sequences F and G are chain homotopy equivalent is simple. This completes the proof. $\square$

The following result plays a crucial role in this paper.

Theorem 2.4. *Let* $\mathcal{C}$ *be a subcategory of* R*-modules such that* $\mathcal{C}$ *is closed under finite direct sums, and* M *a module in* R*-Mod. Suppose that we have two proper left* $\mathcal{C}$*-resolutions of* M*:*

$$C_n \xrightarrow{\alpha_n} C_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \xrightarrow{\sigma} M \longrightarrow 0 \qquad (\natural)$$

and

$$C'_n \xrightarrow{\alpha'_n} C'_{n-1} \xrightarrow{\alpha'_{n-1}} \cdots \xrightarrow{\alpha'_2} C'_1 \xrightarrow{\alpha'_1} C'_0 \xrightarrow{\sigma'} M \longrightarrow 0 \ . \qquad (\sharp)$$

Then the complexes

$$C_n \oplus S_n \xrightarrow{\alpha_n \oplus 0} C_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_2} C_1 \xrightarrow{\alpha_1} C_0 \qquad (\natural')$$

and

$$C'_n \oplus T_n \xrightarrow{\alpha'_n \oplus 0} C'_{n-1} \xrightarrow{\alpha'_{n-1}} \cdots \xrightarrow{\alpha'_2} C'_1 \xrightarrow{\alpha'_1} C'_0 \qquad (\sharp')$$

are chain homotopy equivalent, where the modules T_i*,* S_i *are defined inductively by* $T_0 \cong C_0$*,* $S_0 \cong C'_0$*, and for* $i = 1, 2, \cdots, n$*,* $T_i \cong C_i \oplus S_{i-1}$*,* $S_i \cong C'_i \oplus T_{i-1}$*.*

Proof. We follow the proof of [3, Theorem 1.1]. For each $i = 1, 2, \cdots, n$, we have natural inclusions of summands: $\lambda_i : C_i \to T_i$, $\lambda'_i : C'_i \to S_i$. Let $\lambda_0 : C_0 \to T_0$ and $\lambda_0 : C'_0 \to S_0$ both be the identity maps. We define $\rho_i : T_i \to T_{i-1} \oplus S_{i-1}$, and $\rho'_i : S_i \to S_{i-1} \oplus T_{i-1}$ by

$$\rho_i = \begin{pmatrix} \lambda_{i-1}\alpha_i & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho'_i = \begin{pmatrix} \lambda'_{i-1}\alpha'_i & 0 \\ 0 & 1 \end{pmatrix}.$$

For $r = 0, 1, \cdots, n-1$, let $\mathcal{C}_r$ denote the chain complex

$$C_n \oplus S_n \xrightarrow{\alpha_n \oplus 0} \cdots \xrightarrow{\alpha_{r+2}} C_{r+1} \xrightarrow{\lambda_r \alpha_{r+1}} T_r \xrightarrow{\rho_r} T_{r-1} \oplus S_{r-1} \xrightarrow{\rho_{r-1} \oplus 0} \cdots \xrightarrow{\rho_1 \oplus 0} T_0 \oplus S_0 \ .$$

Also let $\mathcal{C}_n$ denote chain complex

$$T_n \oplus S_n \xrightarrow{\rho_n \oplus 0} T_{n-1} \oplus S_{n-1} \xrightarrow{\rho_{n-1} \oplus 0} \cdots \xrightarrow{\rho_2 \oplus 0} T_1 \oplus S_1 \xrightarrow{\rho_1 \oplus 0} T_0 \oplus S_0 \ .$$

Clearly, $\mathcal{C}_0$ is the chain complex ($\natural'$). For $r = 0, 1, \cdots, n-1$, the chain complex $\mathcal{C}_{r+1}$ is obtained from $\mathcal{C}_r$ by replacing $\xrightarrow{\alpha_{r+2}} C_{r+1} \xrightarrow{\lambda_r \alpha_{r+1}} T_r \xrightarrow{\delta_r}$ with

$$\xrightarrow{\lambda_{r+1} \alpha_{r+2}} C_{r+1} \oplus S_r \xrightarrow{\rho_{r+1}} T_r \oplus S_r \xrightarrow{\rho_r \oplus 0} \ .$$

Similarly, for $r = 0, 1, \cdots, n-1$, let $\mathcal{D}_r$ denote the chain complex

$$C'_n \oplus T_n \xrightarrow{\alpha'_n \oplus 0} \cdots \xrightarrow{\alpha'_{r+2}} C'_{r+1} \xrightarrow{\lambda'_r \alpha'_{r+1}} S_r \xrightarrow{\rho'_r} S_{r-1} \oplus T_{r-1} \xrightarrow{\rho'_{r-1} \oplus 0} \cdots \xrightarrow{\rho'_1 \oplus 0} S_0 \oplus T_0 \ .$$

Again let $\mathcal{D}_n$ denote chain complex

$$S_n \oplus T_n \xrightarrow{\rho'_n \oplus 0} S_{n-1} \oplus T_{n-1} \xrightarrow{\rho'_{n-1} \oplus 0} \cdots \xrightarrow{\rho'_2 \oplus 0} S_1 \oplus T_1 \xrightarrow{\rho'_1 \oplus 0} S_0 \oplus T_0 \ .$$

Clearly, $\mathcal{D}_0$ is the chain complex ($\sharp'$).

By Lemma 2.3, for $i = 0, 1, \cdots, n$, $\mathcal{C}_i$ and $\mathcal{D}_i$ are $\mathrm{Hom}_R(\mathcal{C}, -)$-exact. For $r = 0, 1, \cdots, n-1$, $\mathcal{C}_{r+1}$ (respectively, $\mathcal{D}_{r+1}$) is chain homotopy equivalent to $\mathcal{C}_r$ (respectively, $\mathcal{D}_r$). Hence ($\natural'$) (respectively, ($\sharp'$)) is chain homotopy equivalent to $\mathcal{C}_n$ (respectively, $\mathcal{D}_n$).

To prove that ($\natural'$) and ($\sharp'$) are chain homotopy equivalent, it suffices to show that $\mathcal{C}_n$ is chain isomorphic to $\mathcal{D}_n$, that is, there exist isomorphisms h_i, k_i making the following diagram commute:

$$\begin{array}{ccccccccccccc}
T_n \oplus S_n & \xrightarrow{\rho_n \oplus 0} & T_{n-1} \oplus S_{n-1} & \xrightarrow{\rho_{n-1} \oplus 0} & \cdots & \xrightarrow{\rho_2 \oplus 0} & T_1 \oplus S_1 & \xrightarrow{\rho_0 \oplus 0} & T_0 \oplus S_0 & \xrightarrow{\sigma \oplus 0} & M & \longrightarrow & 0 \\
\downarrow h_n & & \downarrow h_{n-1} & & & & \downarrow h_1 & & \downarrow h_0 & & \downarrow 1 & & \\
S_n \oplus T_n & \xrightarrow{\rho'_n \oplus 0} & S_{n-1} \oplus T_{n-1} & \xrightarrow{\rho'_{n-1} \oplus 0} & \cdots & \xrightarrow{\rho'_2 \oplus 0} & S_1 \oplus T_1 & \xrightarrow{\rho'_0 \oplus 0} & S_0 \oplus T_0 & \xrightarrow{\sigma' \oplus 0} & M & \longrightarrow & 0 \\
\downarrow k_n & & \downarrow k_{n-1} & & & & \downarrow k_1 & & \downarrow k_0 & & \downarrow 1 & & \\
T_n \oplus S_n & \xrightarrow{\rho_n \oplus 0} & T_{n-1} \oplus S_{n-1} & \xrightarrow{\rho_{n-1} \oplus 0} & \cdots & \xrightarrow{\rho_2 \oplus 0} & T_1 \oplus S_1 & \xrightarrow{\rho_0 \oplus 0} & T_0 \oplus S_0 & \xrightarrow{\sigma \oplus 0} & M & \longrightarrow & 0
\end{array}$$

For $i = 0, 1, \cdots, n$, T_i, $S_i \in \mathcal{C}$ since $\mathcal{C}$ is closed under finite direct sums.

We proceed by induction on n. For $n = 0$, as the sequences $T_0 \xrightarrow{\sigma} M \to 0$ and $S_0 \xrightarrow{\sigma'} M \to 0$ are $\mathrm{Hom}_R(\mathcal{C}, -)$-exact, there exist f_0, g_0 such that the following diagrams commute:

$$\begin{array}{ccccc} T_0 & \xrightarrow{\sigma} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle 1} & & \\ S_0 & \xrightarrow{\sigma'} & M & \longrightarrow & 0 \end{array} \qquad \begin{array}{ccccc} S_0 & \xrightarrow{\sigma'} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle 1} & & \\ T_0 & \xrightarrow{\sigma} & M & \longrightarrow & 0. \end{array} \qquad (\flat)$$

Define $h_0 : T_0 \oplus S_0 \to S_0 \oplus T_0$ and $k_0 : S_0 \oplus T_0 \to T_0 \oplus S_0$ by

$$h_0 = \begin{pmatrix} f_0 & 1 - f_0 g_0 \\ 1 & -g_0 \end{pmatrix}, \ k_0 = \begin{pmatrix} g_0 & 1 - g_0 f_0 \\ 1 & -f_0 \end{pmatrix}.$$

Then $h_0 k_0 = 1$ and $k_0 h_0 = 1$. From commutativity of $(\flat)$, we deduce:

$$(\sigma', \ 0)h_0 = (\sigma, \ 0), \ (\sigma, \ 0)k_0 = (\sigma', \ 0).$$

Hence we get the following commutative diagrams:

$$\begin{array}{ccc} T_0 \oplus S_0 & \xrightarrow{\sigma \oplus 0} & M \\ \downarrow{\scriptstyle h_0} & & \downarrow{\scriptstyle 1} \\ S_0 \oplus T_0 & \xrightarrow{\sigma' \oplus 0} & M \end{array} \qquad \begin{array}{ccc} S_0 \oplus T_0 & \xrightarrow{\sigma' \oplus 0} & M \\ \downarrow{\scriptstyle k_0} & & \downarrow{\scriptstyle 1} \\ T_0 \oplus S_0 & \xrightarrow{\sigma \oplus 0} & M. \end{array}$$

Now suppose that for some $j < i \leq n$, we have defined $h_j : T_j \oplus S_j \to S_j \oplus T_j$ and $k_j : S_j \oplus T_j \to T_j \oplus S_j$ for $j = 0, 1, \cdots, i-1$, so that for each j, we have $h_j k_j = 1$ and $k_j h_j = 1$.

Since the sequences $T_i \xrightarrow{\rho_i} \mathrm{Ker}(\rho_{i-1} \oplus 0)$ and $S_i \xrightarrow{\rho'_i} \mathrm{Ker}(\rho'_{i-1} \oplus 0)$ are $\mathrm{Hom}_R(\mathcal{C}, -)$-exact, and $T_i, \ S_i \in \mathcal{C}$, there exist $f_i, \ g_i$ such that the following diagrams commute:

$$\begin{array}{ccc} T_i & \xrightarrow{\rho_i} & T_{i-1} \oplus S_{i-1} \\ \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle h_{i-1}} \\ S_i & \xrightarrow{\rho'_i} & S_{i-1} \oplus T_{i-1} \end{array} \qquad \begin{array}{ccc} S_i & \xrightarrow{\rho'_i} & S_{i-1} \oplus T_{i-1} \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle k_{i-1}} \\ T_i & \xrightarrow{\rho_i} & T_{i-1} \oplus S_{i-1}. \end{array} \qquad (\flat')$$

Define $h_i : T_i \oplus S_i \to S_i \oplus T_i$ and $k_i : S_i \oplus T_i \to T_i \oplus S_i$ by

$$h_i = \begin{pmatrix} f_i & 1 - f_i g_i \\ 1 & -g_i \end{pmatrix} \qquad k_i = \begin{pmatrix} g_i & 1 - g_i f_i \\ 1 & -f_i \end{pmatrix}.$$

Then $h_i k_i = 1$ and $k_i h_i = 1$. Recall $h_{i-1} k_{i-1} = 1$ and $k_{i-1} h_{i-1} = 1$. From commutativity of $(\flat')$, we deduce:

$$(\rho'_i, \ 0)h_i = h_{i-1}(\rho_i, \ 0), \ (\rho_i, \ 0)k_i = k_{i-1}(\rho'_i, \ 0).$$

Hence we get the following commutative diagrams:

$$\begin{array}{ccc} T_i \oplus S_i & \xrightarrow{\rho_i \oplus 0} & T_{i-1} \oplus S_{i-1} \\ \downarrow{h_i} & & \downarrow{h_{i-1}} \\ S_i \oplus T_i & \xrightarrow{\rho'_i \oplus 0} & S_{i-1} \oplus T_{i-1} \end{array} \qquad \begin{array}{ccc} S_i \oplus T_i & \xrightarrow{\rho'_i \oplus 0} & S_{i-1} \oplus T_{i-1} \\ \downarrow{k_i} & & \downarrow{k_{i-1}} \\ T_i \oplus S_i & \xrightarrow{\rho_i \oplus 0} & T_{i-1} \oplus S_{i-1}. \end{array}$$

So $\mathcal{C}_n$ is chain isomorphic to $\mathcal{D}_n$, $(\natural')$ and $(\sharp')$ are chain homotopy equivalent. This completes the proof. ☐

As applications of Theorem 2.4, we will give some examples. Firstly, the following result follows immediately from Theorem 2.4 since the class of projective modules is closed under direct sums.

Corollary 2.5. ([3, Theorem 1.1]) *Let R be a ring and M a module in R-Mod. Suppose we have exact sequences:*

$$P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

and

$$Q_n \xrightarrow{\partial'_n} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0 \xrightarrow{\epsilon'} M \longrightarrow 0$$

with the P_i and Q_i all projective modules in R-Mod. Then the complexes

$$P_n \oplus S_n \xrightarrow{\partial_n \oplus 0} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0$$

and

$$Q_n \oplus T_n \xrightarrow{\partial'_n \oplus 0} Q_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \xrightarrow{\partial'_2} Q_1 \xrightarrow{\partial'_1} Q_0$$

are chain homotopy equivalent, where the modules T_i, S_i are defined inductively by $T_0 \cong P_0$, $S_0 \cong Q_0$, and for $i = 1, 2, \cdots, n$, $T_i \cong P_i \oplus S_{i-1}$, $S_i \cong Q_i \oplus T_{i-1}$.

Recall from [5] that a module M is called FP-injective if $Ext^1_R(F, M) = 0$ for any finitely presented module F. Recently, Pinzon in [4] shows that every module in R-Mod has an FP-injective cover if R is a left coherent ring. So every module M in R-Mod has a proper left FP-injective resolution if R is coherent.

Example 2.6. *Let R be a left coherent ring and M a module in R-Mod. Suppose the complexes $(\natural)$ and $(\sharp)$ in Theorem 2.4 are two proper left FP-injective resolutions of M, then the complexes $(\natural')$ and $(\sharp')$ in Theorem 2.4 are chain homotopy equivalent.*

Proof. Note that the class of FP-injective modules in R-Mod is closed under direct sums. Then the result follows from Theorem 2.4. ☐

In the following, we denote the class of all projective, flat and injective R-modules, respectively, by $\mathcal{P}(R)$, $\mathcal{F}(R)$ and $\mathcal{I}(R)$.

Definition 2.7. (see [2]) An R-module M is called Gorenstein projective if there exists an exact sequence of projective modules $P = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that $M \cong \mathrm{Im}(P_0 \to P^0)$ and P is $\mathrm{Hom}_R(-, \mathcal{P}(R))$-exact. In this case, we say P is a complete projective resolution of M. Gorenstein injective modules are defined dually.

Holm in [2] shows that, every module M in R-Mod with finite Gorenstein dimension admits a proper left Gorenstein projective resolution.

Example 2.8. *Let M be a module in R-Mod. Suppose the complexes ($\natural$) and ($\sharp$) in Theorem 2.4 are two proper left Gorenstein projective resolutions of M, then the complexes ($\natural'$) and ($\sharp'$) in Theorem 2.4 are chain homotopy equivalent.*

Proof. Note that the class of Gorenstein projective modules in R-Mod is closed under finite direct sums. Then the result follows from Theorem 2.4. □

Example 2.9. *Let M be a Gorenstein injective module. Then M admits a proper left $\mathcal{I}(R)$-resolution by the dual of* [2, Proposion 2.3]. *Suppose the complexes ($\natural$) and ($\sharp$) in Theorem 2.4 are two proper left $\mathcal{I}(R)$-resolutions of M, then the complexes ($\natural'$) and ($\sharp'$) in Theorem 2.4 are chain homotopy equivalent.*

Theorem 2.10. *Let $\mathcal{C}$ be a subcategory of R-modules such that $\mathcal{C}$ is closed under finite direct sums and M a module in R-Mod. Suppose we have two proper right $\mathcal{C}$-resolutions of M:*

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{m-1} \longrightarrow I^m \qquad (\dagger)$$

and

$$0 \longrightarrow M \longrightarrow J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^{m-1} \longrightarrow J^m \ . \qquad (\ddagger)$$

Then the complexes

$$I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^{m-1} \longrightarrow I^m \oplus S^m \qquad (\dagger')$$

and

$$J^0 \longrightarrow J^1 \longrightarrow \cdots \longrightarrow J^{m-1} \longrightarrow J^m \oplus T^m \qquad (\ddagger')$$

are chain homotopy equivalent, where the modules T^i, S^i are defined inductively by $T^0 \cong I^0$, $S^0 \cong J^0$, and for $i = 1, 2, \cdots, m$, $T^i \cong I^i \oplus S^{i-1}$, $S^i \cong J^i \oplus T^{i-1}$.

Proof. The proof is dual to that of Theorem 2.4. □

It was showed in [1, Proposion 6.5.1.] that, a ring R is right coherent if and only if the class of flat left modules in R-Mod is preenveloping. So every module M in R-Mod has a proper right flat resolution if R is coherent.

Example 2.11. *Let R be a right coherent ring and M a module in R-Mod. Suppose the complexes (†) and (‡) in Theorem 2.10 are two proper right $\mathcal{F}(R)$-resolutions of M. Then the complexes (†′) and (‡′) in Theorem 2.10 are chain homotopy equivalent.*

Example 2.12. *Let M be a Gorenstein projective module. Then M admits a proper right $\mathcal{P}(R)$-resolution by* [2, Proposion 2.3]. *Suppose the complexes (†) and (‡) in Theorem 2.10 are two proper right $\mathcal{P}(R)$-resolutions of M, then the complexes (†′) and (‡′) in Theorem 2.10 are chain homotopy equivalent.*

Proposition 2.13. *Let $\mathcal{C}$ be a subcategory of R-modules such that $\mathcal{C}$ is closed under finite direct sums and M a module in R-Mod. Suppose we have two proper $\mathcal{C}$-resolutions of M:*

$$P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots \to P^{m-1} \to P^m \tag{1}$$

and

$$Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to Q^0 \to Q^1 \to \cdots \to Q^{m-1} \to Q^m. \tag{2}$$

Then the complexes

$$P_n \oplus S_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} P^0 \xrightarrow{\partial^0} P^1 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{m-2}} P^{m-1} \xrightarrow{\partial^{m-1}} P^m \oplus S^m \tag{3}$$

and

$$Q_n \oplus T_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to Q^0 \to Q^1 \to \cdots \to Q^{m-1} \to Q^m \oplus T^m \tag{4}$$

are chain homotopy equivalent, where the modules T_i, S_i, T^i, S^i are defined inductively by $T_0 \cong P_0$, $S_0 \cong Q_0$, $T^0 \cong P^0$, $S^0 \cong Q^0$ and for $i = 1, 2, \cdots, n$, $T_i \cong P_i \oplus S_{i-1}$, $S_i \cong Q_i \oplus T_{i-1}$, $T^i \cong P^i \oplus S^{i-1}$, $S^i \cong Q^i \oplus T^{i-1}$.

Proof. For $0 \leq i \leq n$, there exist $f_i : P_i \to Q_i$, $g_i : Q_i \to P_i$ and $s_i : P_i \to P_{i+1}$ such that $g_{i+1}f_{i+1} - I_{P_{i+1}} = \partial_{i+2}s_{i+1} + s_i\partial_{i+1}$ by Theorem 2.4. For $0 \leq i \leq m$, there exist $f^i : P^i \to Q^i$, $g^i : Q^i \to P^i$ and $s^{i+1} : P^{i+1} \to P^i$ such that $g^{i+1}f^{i+1} - I^{P_{i+1}} = \partial_i s_{i+1} + s_{i+2}\partial_{i+1}$ by Theorem 2.10. Let $s^0 : P^0 \to P_0$ and $s^0 = 0$. Clearly, we have $g_0f_0 - I_{P_0} = \partial_1 s_0 + s^0\partial_0$ and $g^0f^0 - I^{P_0} = \partial_0 s^0 + s^1\partial^0$. Thus the complexes (3) and (4) are chain homotopy equivalent. □

Proposition 2.14. *Let M be a Gorenstein projective R-module. Suppose the complexes* (1) *and* (2) *are two complete projective resolutions of M, then the complexes* (3) *and* (4) *are chain homotopy equivalent, where the modules T_i, S_i, T^i, S^i are defined as in Proposition 2.13 for $i = 0, 1, \cdots, n$.*

Proof. By [3, Theorem 1.1] and Example 2.12. □

Acknowledgment. The authors would like to thank the referee for the valuable suggestions and comments.

References

[1] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, De Gruyter Expositions in Mathematics, 30, Walter De Gruyter, Berlin, 2000.

[2] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra, 189(1-3) (2004), 167-193.

[3] W. H. Mannan, *Homotopy types of truncated projective resolutions*, Homology, Homotopy Appl., 9(2) (2007), 445-449.

[4] K. Pinzon, *Absolutely pure covers*, Comm. Algebra, 36(6) (2008), 2186-2194.

[5] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc., 2(2) (1970), 323-329.

Yanping Rao and **Gang Yang**
Department of Mathematics
Lanzhou Jiaotong University
Lanzhou 730070, China
e-mails: yprao0213@gmail.com (Y. Rao)
yanggang10@gmail.com (G. Yang)