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ON THE ASSOCIATED PRIME IDEALS AND THE DEPTH OF POWERS OF SQUAREFREE PRINCIPAL BOREL IDEALS

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Abstract. We study algebraic properties of powers of squarefree principal Borel ideals I, and show that $\operatorname{astab}(I) = \operatorname{dstab}(I)$. Furthermore, the behaviour of the depth function $\operatorname{depth} S/I^k$ is considered.

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1. Introduction

Borel ideals appear in characteristic zero as generic initial ideals. By applying the stretching operator of Kalai [17] to a Borel ideal, one obtains a squarefree Borel ideal, also called squarefree strongly stable ideal. It has the same graded Betti numbers as the original ideal, see for example [14, Lemma 11.2.6]. This class of ideals was introduced by Hibi and the first author of this paper in [2]. In the sequel, algebraic properties of squarefree strongly stable ideals and their powers have been studied by many authors, see for example [1], [3], [11], [12] and [18].

Among the squarefree strongly stable ideals, the squarefree principal Borel ideals and their powers are best understood. The squarefree principal ideal with Borel generator u is denoted by $B_S(u)$. We use some of the known results from Aslam [3], Francisco, Mermin and Schweig [12] and De Negri [7], to get some additional information about the algebraic and homological properties of powers of $B_S(u)$. These known facts are recalled in Section 3. In this section we also show that $B_S(u)$ is normally torsion free if and only if it is almost normally torsion free, and that this happens if and only if the Borel generator u of the ideal has a specific form, see Corollary 3.4. In Corollary 3.13 we determine the height and bigheight of $B_S(u)$, and characterize those squarefree principal Borel ideals which are Cohen-Macaulay. Corollary 3.12 makes more explicit the set $\operatorname{Ass}^{\infty}(B_S(u))$, as it is described by Aslam in Theorem 3.10.

For our proofs we essentially use monomial localization. Some of the basic facts about monomial localization are recalled in Section 2. We close this section by characterizing in Corollary 2.5 k-strongly stable ideals. This type of ideals play an important role in Section 4, where the socle of the powers of $B_S(u)$ is determined. It turns out, see Corollary 4.2, that the socle of $B_S(u)^k$ is a $(k-1)$-stable set of monomials. By using this fact we exhibit in Corollary 4.5 for each k a monomial w_k with the property that $\operatorname{depth} S/B_S(u)^k = 0$, if and only if $w_k x_n \in B_S(u)^k$. Here u is a squarefree monomial in $S = K[x_1, \dots, x_n]$ with $x_1 \nmid u$ and $x_n \mid u$. These conditions on u are not restrictive, because one can always reduce to this case, see Lemma 5.1 and Proposition 5.2.

The invariant $\operatorname{astab}(u)$ is the smallest number k for which $\operatorname{Ass}(B_S(u)^k$ stabilizes, and $\operatorname{dstab}(B_S(u))$ is the smallest number k for which $\operatorname{depth} S/B_S(u)^k$ stabilizes. As the main result of Section 5 we show in Theorem 5.5 that for any u, $\operatorname{astab}(B_S(u)) = \operatorname{dstab}(B_S(u))$. For general monomial ideals I, the the Ass-stability and the depth stability are usually unrelated.

For the proof of Theorem 5.5 we use the interesting fact, shown in Theorem 5.4, that the depth stability of monomial localizations of $B_S(u)$ are bounded by the depth stability of $B_S(u)$.

In the last section we study the depth of $S/B_S(u)^k$ as a function of k. For short we set $f(k) = \operatorname{depth} S/B_S(u)^k$. Since all powers of $B_S(u)^k$ have a linear resolution it follows from [13, Proposition 2.1] that $f(k) \leq f(k-1)$ for $k \geq 1$. We also know that the depth function $f(k)$ becomes constant for $k \geq \deg u$, see Proposition5.3. By considering many example it seems to be the case that $f(k) < f(k-1)$, before $f(k)$ becomes constant. This is true if $\deg u \leq 3$, and follows from Proposition 6.5, where we compute $f(k)$ explicitly for $\deg u \leq 3$. In Proposition 6.2 we show that $f(1) = \deg u - 1$, and in Proposition 6.3 we determine all squarefree monomials u for which $f(2) = 0$. Finally in Theorem 6.6 we show that $f(2) < f(1)$, as expected, unless $f(1) = f(k)$ for some $k \geq 2$, and in this case $f(k)$ is a constant function.

2. Preliminaries

In this section we introduce some concepts and results which are important in this paper.

Monomial localizations. In this paper, monomial localizations are an important tool. Let K be a field and let $S = K[x_1, \dots, x_n]$ be the polynomial ring in n variables. Let $T \subset [n]$. We define the monomial prime ideal P_T to be the ideal $(x_j \colon\ j \in T^c)$. Here $T^c = [n] \setminus T$ and $[n] = \{1, \dots, n\}$. We also set $[0] = \emptyset$. Notice that $P_{[0]} = (x_1, \dots, x_n)$ and $P_{[n]} = (0)$.

Definition 2.1. Let $I \subset K[x_1, \ldots, x_n]$ be a monomial ideal, and let $T \subset [n]$. The *monomial localization* of I with respect to P_T is the monomial ideal $I(P_T) \subset S(P_T)$, where $S(P_T) = K[x_j\colon j \in T^c]$ and $I(P_T) = \varphi(I)S(P_T)$, where $\varphi\colon S \to S(P_T)$ is the K-algebra homomorphism with $x_j \mapsto x_j$ if $j \in T^c$, and $x_j \mapsto 1$ if $j \in T$.

Monomial localization with respect to P_T and the usual localization with respect to the prime ideal P_T are related as follows:

$$IS_{P_T} = I(P_T)S_{P_T}.$$

This identity justifies the name "monomial localization".

For example, if $I = (x_1x_3, x_1x_4, x_2x_3x_4, x_3x_5x_7x_8) \subset S = K[x_1, \ldots, x_8]$ and $T = \{3, 5\}$, then $S(P_T) = K[x_1, x_2, x_4, x_6, x_7, x_8]$ and $I(P_T) = (x_1, x_2x_4, x_7x_8)$.

If $I = P_T$, then $I(P_T)$ is the graded maximal ideal of $S(P_T)$ which we denote by $\mathfrak{m}_{S(P_T)}$.

To simplify notation, we set $I_{(j)} = I(P_{\{j\}})$ and $S_{(j)}$ for $S(P_{\{j\}})$. Note that $S_{(j)} = K[x_i\colon i \in [n] \setminus \{j\}]$.

Strongly stable ideals. Let $u = x_1^{a_1} \cdots x_n^{a_n}$ be a monomial in $S = K[x_1, \ldots, x_n]$. We set $\nu_i(u) = a_i$ for $i = 1, \ldots, n$. Now let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal. We denote by $G(I)$ the unique set of monomial generators of I. For a given integer $k \geq 1$, we let $I^{\leq k}$ be the ideal generated by all $u \in G(I)$ with $\nu_i(u) \leq k$ for $i = 1, \ldots, n$.

A monomial $u \in S$ be can be written as $u = x_{i_1}x_{i_2} \cdots x_{i_d}$ with $i_1 \leq i_2 \leq \ldots \leq i_d$. The monomial u is called *squarefree* if $i_1 < i_2 < \ldots < i_d$. A monomial ideal I is called a *squarefree monomial ideal* if all monomials in $G(I)$ are squarefree. Note that for any monomial ideal, the ideal $I^{\leq 1}$ is a squarefree monomial ideal. Moreover, the ideal I is squarefree if and only if $I = I^{\leq 1}$.

Definition 2.2. Let $I \subset S = K[x_1, \ldots, x_n]$ be a monomial ideal, and let $k \geq 1$ be an integer, or $k = \infty$. Then I is called *k-strongly stable*, if

(i) $I = I^{\leq k}$;
(ii) for all $u \in G(I)$ and all integers $1 \leq i < j \leq n$ with $\nu_j(u) > 0$ and $\nu_i(u) < k$ it follows that $x_i(u/x_j) \in I$.

The following special cases are of particular interest: let I be a monomial ideal.

(α) If $k = \infty$, then there is no bound on the exponents and ∞-strongly stable is simply called *strongly stable*. In other words, I is strongly stable, if for $u \in G(I)$ and all j such that x_j divides u, it follows that $x_i(u/x_j) \in I$ for all $i \leq j$.

(β) I is called *squarefree strongly stable*, if it is 1-strongly stable.

Let $u_1,\dots,u_m$ be monomials in S with $\nu_i(u_j) \leq k$ for $i = 1,\dots,n$ and $j = 1,\dots,m$. There exists a unique smallest k-strongly stable ideal containing $u_1,\dots,u_m$ which we denote by $B_S^k(u_1,\dots,u_m)$. The monomials $u_1,\dots u_m$ are called *Borel generators* of $B_S^k(u_1,\dots,u_m)$. A monomial ideal I is called *k-principal Borel* if $I = B_S^k(u)$ for some monomial u with $\nu_i(u) \leq k$ for $i = 1,\dots,n$. We call 1-principal Borel ideals also squarefree principal Borel. The k-principal Borel ideals appear as powers of squarefree principal Borel ideals, see Theorem 4.1.

To simplify notation, we write $B_S(u_1,\dots,u_m)$ for $B_S^1(u_1,\dots,u_m)$ when $u_1,\dots,u_m$ are squarefree monomials. The unique strongly stable ideal containing the monomials $u_1,\dots,u_m$ will be denoted by $B_S^\infty(u_1,\dots,u_m)$.

For example, let $I = B_S(x_2x_4, x_1x_3)$. Then $G(I) = \{x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4\}$.

Let u, v be monomials of same degree, and assume that $\nu_i(u) \leq k$ for $i = 1,\dots,n$. Then we write $v \preceq_k u$ if $v \in B_S^k(u)$, and $v \prec_k u$ if $v \in B_S^k(u)$ and $v \neq u$. For $\preceq_\infty$ we simply write $\preceq$.

For each d, $\preceq_k$ defines a partial order on the set of monomials of degree d whose exponents are bounded by k. For example, one has $x_1^2x_2^3x_3 \prec_3 x_2^3x_3^3$. A set $\mathcal{S}$ of monomials of degree d whose exponents are bounded by k is called a *k-stable set*, if for $u \in \mathcal{S}$ and $v \preceq_k u$ it follows that $v \in \mathcal{S}$.

Note that $B_S^k(u_1,\dots,u_m)$ is generated by $\bigcup_{i=1}^m\{v\colon\ v \preceq_k u_i\}$. Therefore,

$$B_S^k(u_1,\dots,u_m) = \sum_{i=1}^m B_S^k(u_i).$$

Let $v, u \in S$ be monomials of degree d, where $v = x_{i_1}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $u = x_{j_1}\cdots x_{j_d}$ with $j_1 \leq j_2 \leq \cdots \leq j_d$. By [14, Lemma 4.2.4] one has

$$v \preceq u \text{ if and only if } i_k \leq j_k \text{ for all } k = 1,\dots,d. \tag{1}$$

Remark 2.3. Let $u, v \in S$ be monomials. It is clear that if $v \preceq_k u$, then $v \preceq u$. On the other hand, if the exponents of u and v are bounded by k and $v \preceq u$, then $v \preceq_k u$. This follows from the next lemma.

Lemma 2.4. *Let $u \in S$ be a monomial with $\nu_i(u) \leq k$ for $i = 1,\dots,n$. Then*

$$B_S^k(u) = B_S^\infty(u)^{\leq k}.$$

Proof. Let $v \in B_S^k(u)$. Then $v \preceq_k u$, and hence $v \preceq u$. Therefore, $v \in B_S^\infty(u)$. Since the exponent of v is bounded by k it follows that $v \in B_S^\infty(u)^{\leq k}$.

Conversely, let $v \in B_S^{\infty}(u)^{\leq k}$, and let $v = x_{i_1}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ and $u = x_{j_1}\cdots x_{j_d}$ with $j_1 \leq j_2 \leq \cdots \leq j_d$.

Let w be a monomial of degree d with $v \preceq w$, and let $w = x_{s_1}\cdots x_{s_d}$ with $s_1 \leq s_2 \leq \cdots \leq s_d$. We define

$$\delta(w,v) = \sum_{l=1}^{d}(s_l - i_l).$$

Since $v \preceq w$, it follows from (1) that $\delta(w,v) \geq 0$, and we have $\delta(w,v) = 0$ if and only if $w = v$.

Let $l+1$ be the smallest integer such that $j_{l+1} > i_{l+1}$. Let $u_1 = x_{j_{l+1}-1}u/x_{j_{l+1}}$. Then the exponents of u_1 are bounded by k, unless $j_{l+1}-1 = j_l = j_{l-1} = \cdots = j_{l-k+1}$. Assume $j_{l+1}-1 = j_l$. By the choice of l it follows that $i_{l-k+1} = \cdots i_l$. Since the exponents of v are bounded by k, it follows that $i_{l+1} > i_l$. Moreover, $j_{l+1} > i_{l+1}$. Then $j_{l+1} \geq i_l + 2 = j_l + 2$, a contradiction. This shows that indeed the exponents of u_1 are bounded by k, and $v \preceq u_1$, and hence $v \in B_S^{\infty}(u_1)^{\leq k}$. Since $\delta(u_1, v) < \delta(u,v)$, we may assume by induction that $v \in B_S^k(u_1) \subset B_S^k(u)$, because $u_1 \in B_S^k(u)$. ☐

An immediate consequence of Lemma 2.4 is

Corollary 2.5. *Let u, v be monomials of degree d and assume that $\nu_i(u), \nu_i(v) \leq k$ for $i = 1, \ldots, n$. Let $u = x_{j_1}\cdots x_{j_d}$ with $j_1 \leq j_2 \leq \ldots \leq j_d$ and $v = x_{i_1}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$. Then $v \prec_k u$ if and only if $i_r \leq j_r$ for $r = 1, \ldots, d$.*

If $L \subset [n]$, $S' = K[x_i\colon\ i \in L]$ and $u_1, \ldots, u_m \in S'$ are monomials with $\nu_i(u_j) \leq k$ for $i \in L$ and $j = 1, \ldots, m$. Then $B_{S'}^k(u_1, \ldots, u_m)$ is defined in a similar way as in the case $S' = S$. For example, if $n = 6$ and $L = \{2, 4, 6\}$, then $S' = K[x_2, x_4, x_6]$. Let $u = x_4^2x_6 \in S'$ and $k = 2$, then $B_{S'}^2(u) = \{x_2^2x_4, x_2^2x_6, x_2x_4x_6, x_2x_4^2, x_4^2x_6\}$.

3. Associated prime ideals of $B_S(u)^k$

In this paper we mainly study squarefree principal Borel ideals. This class of ideals behave well under localization, see [3, Theorem 1.2].

Theorem 3.1. *Let $u \in S$ be a squarefree monomial of degree d, $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 < i_2 < \ldots < i_d$, and let j be an integer with $i_{k-1} < j \leq i_k$, where we set $i_0 = 0$. Then $B_S(u)_{(j)} = B_{S_{(j)}}(v)$, where $v = u/x_{i_k}$.*

Corollary 3.2. *Let $u \in S$ be a squarefree monomial, $I = B_S(u)$ and $P \subset S$ a monomial prime ideal. Then $I(P)$ is a squarefree principal Borel ideal in $S(P)$.*

Proof. Let $P = P_T$. Then $I(P_T)$ is the ideal in $S(P_T)$ which is obtained from I by the substitutions $x_j \mapsto 1$ for $j \in T$. Thus the result follows by repeated application of Theorem 3.1. □

A graded ideal $I \subset S$ is called *normally torsionfree*, if $\operatorname{Ass}(I) = \operatorname{Ass}(I^k)$ for all $k \geq 1$. Moreover, we call I *almost normally torsionfree*, if $\operatorname{Ass}(I^k) \subseteq \operatorname{Ass}(I) \cup \{\mathfrak{m}\}$ for all k, where $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$ is the graded maximal ideal of S.

Corollary 3.3. *Let $u = x_i x_n$, Then $B_S(u)$ is almost normally torsionfree. Moreover, $B_S(u)$ is normally torsionfree, if and only if $i = 1$.*

Corollary 3.4. *Let $u = x_{i_1} \cdots x_{i_{d-1}} x_n$, and let $d \geq 3$. Then the following conditions are equivalent:*

(a) *$B_S(u)$ is normally torsionfree.*
(b) *$B_S(u)$ is almost normally torsionfree.*
(c) *$u = x_1 x_2 \cdots x_{d-1} x_n$.*

In order to prove Corollary 3.3 and Corollary 3.4, we need the following lemmata.

Lemma 3.5. *Let I be normally torsion free and let w be a monomial such that* $\operatorname{Supp}(w) \cap \operatorname{Supp}(u) = \emptyset$ *for all $u \in G(I)$. Then wI is normally torsion free.*

Proof. The assumption imply that

$$\operatorname{Ass}((wI)^k) = \operatorname{Ass}(I^k) \cup \operatorname{Ass}((w)^k) = \operatorname{Ass}(I^k) \cup \operatorname{Ass}((w)). \tag{2}$$

It follows that $\operatorname{Ass}((wI)^k) = \operatorname{Ass}(wI)$ for all $k \geq 1$ if and only if $\operatorname{Ass}(I^k) = \operatorname{Ass}(I)$ for all $k \geq 1$. □

Lemma 3.6. *Let I be a monomial ideal. Then I is almost normally torsionfree if and only if $I_{(j)}$ is normally torsionfree for all j.*

Proof. The proof follows from the fact that, similarly to ordinary localizations, one has

$$P \in \operatorname{Ass}_{S_{(j)}}((I_{(j)})^k) \text{ if and only if } PS \in \operatorname{Ass}_S(I^k) \text{ and } P \subset P_{\{j\}}, \tag{3}$$

see [16, Lemma 1.3]. □

Proof of Corollary 3.3. Note $B_S(u)_{(j)} = B_{S_{(j)}}(v)$, where v is a monomial of degree 1. Therefore, $B_{S_{(j)}}(v)$ is normally torsionfree. By Lemma 3.6, it follows that $B_S(u)$ is almost normally torsionfree.

If $i_1 = 1$. Then Lemma 3.5 implies that $\operatorname{Ass}(B_S(u)^k) = (x_1) \cup (x_2, \ldots, x_n) = \operatorname{Min}(I)$. Therefore, $B_S(u)$ is normaly torsionfree. □

Proof of Corollary 3.4. (a) $\Rightarrow$ (b) is trivial. (b) $\Rightarrow$ (c) We prove the assertion by induction on d. Let $d = 3$ and $u = x_{i_1}x_{i_2}x_n$ with $i_1 < i_2 < n$. The assumption implies that $B_S(u)_{(1)} = B_{S_{(1)}}(v)$ is normally torsionfree, see Lemma 3.6. Here $v = x_{i_2 x_n}$. Now Corollary 3.3 implies that $i_2 = 2$, and therefore and $u = x_1x_2x_n$.

Now let $d > 3$, and let $u = x_{i_1}x_{i_2}\cdots x_{i_{d-1}x_n}$ with $i_1 < i_2 < \ldots < i_{d-1} < n$. Then Theorem 3.1 implies that $B_S(u)_{(1)} = B_{S_{(1)}}(v)$, where $v = x_{i_2}\cdots x_{i_{d-1}}x_n$. By Lemma 3.6 it follows that $B_{S_{(1)}}(v)$ is normally torsionfree. Hence by induction hypothesis, we have $v = x_2x_3\cdots x_{d-1}x_n$. Therefore, $u = x_{i_1}x_2\cdots x_{d-1}x_{i_d}$. This implies that $i_1 = 1$, and proves (c).

(c) $\Rightarrow$ (a) Since $u = x_1x_2\cdots x_{d-1}x_n$, it follows that $B_S(u) = w(x_d,\ldots,x_n)$, where $w = x_1x_2\cdots x_{d-1}$. Therefore, Lemma 3.5 implies that $B_S(u)$ is normally torsionfree. □

Let $u \in S$ be a squarefree monomial. Then $B_S(u)$ does not have any embedded prime ideals, because $B_S(u)$ is a squarefree monomial ideal, and hence a radical ideal. In other words, $\mathrm{Ass}(B_S(u)) = \mathrm{Min}(B_S(u))$. Here, for any ideal $I \subset S$, $\mathrm{Min}(I)$ denotes the set of minimal prime ideals of I.

Let $B_S(u)^\vee$ denote the Alexander dual of $B_S(u)$. Then $v = x_{i_1}\cdots x_{i_k} \in G(B_S(u)^\vee)$, if and only if $(x_{i_1},\ldots,x_{i_k}) \in \mathrm{Min}(B_S(u))$, see for example [14, Theorem 1.4.6]. Therefore, the set $\mathrm{Min}_S(B_S(u))$ is determined, once we know the set $G(B_S(u)^\vee)$. In [12, Theorem 3.18], $G(B_S(u)^\vee)$ has been computed.

Theorem 3.7 (Francisco, Mermin, Schweig)**.** *Let $u \in S$, $u = x_{i_1}\cdots x_{i_d}$ be a monomial with $i_1 < i_2 < \ldots < i_d$. Then $B_S(u)^\vee$ is a squarefree strongly stable ideal with Borel generators $x_r\cdots x_{i_r}$ for $r = 1,\ldots,d$.*

This theorem has been generalized to t-spread principal Borel ideals, see [1, Theorem 1.2].

Let $I \subset S$ be any ideal. Recall that

$$\mathrm{height}(I) = \min\{\mathrm{height}(P)\colon\ P \in \mathrm{Ass}(I)\},$$

and

$$\mathrm{bigheight}(I) = \max\{\mathrm{height}(P)\colon\ P \in \mathrm{Ass}(I)\}.$$

As an immediate consequence of Theorem 3.7 we obtain

Corollary 3.8. *Let $u \in S$, $u = x_{i_1}\cdots x_{i_d}$ be a monomial with $i_1 < i_2 < \ldots < i_d$ and let $I = B_S(u)$. Then*

(a) $\mathrm{height}(I) = i_1$ *and* $\mathrm{bigheight}(I) = i_d - d + 1$.

(b) *The following conditions are equivalent:*

(i) $\operatorname{height}(I) = \operatorname{bigheight}(I)$.
(ii) $i_j = i_1 + j - 1$ *for* $j = 1, \ldots, d$.
(iii) S/I *is Cohen-Macaulay.*

Proof. (i) $\Longleftrightarrow$ (ii) We have

$$i_1 \leq i_2 - 1 \leq \ldots \leq i_j - j + 1 \leq i_{j+1} - (j+1) + 1 \leq \ldots \leq i_d - d + 1.$$

From (a) it follows that $i_1 = i_d - d + 1$. Therefore, $i_j - j + 1 = i_{j+1} - (j+1) + 1$ for all j which implies that $i_{j+1} = i_j$ for all j. This yields the desired conclusion.

(iii) $\Rightarrow$ (i) Since S/I is Cohen-Macaulay, the ideal I is unmixed, see for example [6, Theorem 2.1.6]. Since I is unmixed if and only if $\operatorname{height}(I) = \operatorname{bigheight}(I)$, the assertion follows.

(ii) $\Rightarrow$ (iii) By Theorem 3.7, $I^\vee = B_S(x_d \cdots x_{i_d})$ for ideals satisfying condition (ii). It is known that $B_S(x_d \cdots x_{i_d})$ has a linear resolution, see [1, Proposition 2.4]. Therefore, S/I is Cohen-Macaulay, see for example [14, Theorem 8.1.9]. □

The following results [3, Theorem 2.1 and Corollary 2.3] is also important for this paper. We denote by $\mathfrak{m}$ the graded maximal ideal of $S = K[x_1, \ldots, x_n]$. For a monomial $v \neq 1$ we set $\min(v) = \min\{j : j \in \operatorname{supp}(v)\}$ and $\max(v) = \max\{j : j \in \operatorname{supp}(v)\}$. If $v = 1$, we set $\min(v) = \max(v) = 0$.

Theorem 3.9 (Aslam)**.** *Let* $u \in S = K[x_1, \ldots, x_n]$ *be a squarefree monomial of degree* d *and let* $I = B_S(u)$*. Then* $\mathfrak{m} \in \operatorname{Ass}(I^k)$ *for some* k *if and only if* $\min(u) > 1$ *and* $\max(u) = n$*. If this is the case, then* $\mathfrak{m} \in \operatorname{Ass}(I^k)$ *for all* $k \geq d$*.*

An alternate proof of this fact is given in Proposition 4.6.

By Brodmann ([4] and [5]), for any graded ideal $I \subset S = K[x_1, \ldots, x_n]$, there exists an integer k_0 such that $\operatorname{Ass}(I^k) = \operatorname{Ass}(I^{k_0})$ for all $k \geq k_0$. We set $\operatorname{Ass}^\infty(I) = \operatorname{Ass}(I^{k_0})$.

In our case, $I = B_S(u)$ with $u = x_{i_1} \cdots x_{i_d}$ and $i_1 < i_2 < \ldots < i_d$. Therefore, I is a squarefree monomial ideal, and hence $\operatorname{Ass}(I) = \operatorname{Min}(I)$. Since $\operatorname{Min}(I) \subset \operatorname{Ass}^\infty(I)$, it follows in our case that $\operatorname{Ass}(I) \subset \operatorname{Ass}^\infty(I)$. In Corollary 3.4, we have seen that $\operatorname{Ass}(I) = \operatorname{Ass}^\infty(I)$ if and only if $u = x_1 x_2 \cdots x_{d-1} x_{i_d}$.

Let $T \subset [n]$. By using Theorem 3.1 and induction on $|T|$ it follows that there exists a (unique) monomial $u_T \subset S_T$ of degree $d - |T|$ such that $B_S(u)_T = B_{S_T}(u_T)$.

For example, if $u = x_2 x_5 x_6 x_7 x_9$ and $T = \{3, 5, 9\}$, then

$$u_T = u_{359} = (u_9)_{35} = ((x_2 x_5 x_6 x_7)_5)_3 = (x_2 x_6 x_7)_3 = x_2 x_7.$$

Note that the monomial prime ideal P belongs to $\operatorname{Ass}^\infty(I)$ if and only if $\mathfrak{m}_P \in \operatorname{Ass}_{S(P)}(I(P)^k)$, for all $k \gg 0$. Here $\mathfrak{m}_P = PS(P)$ denotes the graded maximal ideal of $S(P)$. By Corollary 3.2 the monomial localization $I(P)$ of the squarefree principal monomial I is again a squarefree principal monomial ideal.

A graded ideal $I \subset S$ is said to satisfy the *persistence property*, if $\mathrm{Ass}_S(I^k) \subset \mathrm{Ass}_S(I^{k+1})$ for all $k \geq 1$. Since squarefree principal monomial ideals satisfy the persistence property (see [1, Corollary 2.6]), we conclude that P belongs to $\mathrm{Ass}^\infty(I)$ if and only if $\mathfrak{m}_P \in \mathrm{Ass}_{S(P)}(I(P)^k)$ for some k. From this observation together with Theorem 3.9 one obtains [3, Theorem 3.2].

Theorem 3.10 (Aslam)**.** *Let T be any subset of $[n]$ and assume that* $\min(u) > 1$*. Then $P_T \in \mathrm{Ass}^\infty(B_S(u))$ if and only if* $\min(T^c) < \min(u_T)$ *and* $\max(u_T) = \max(T^c)$*.*

For the next result we need

Lemma 3.11. *Suppose* $\min(u) > 1$*. Then* $\min(T^c) < \min(u_T)$ *if and only if* $u_T \neq 1$*.*

Proof. It is enough to show that $\min(T^c) < \min(u_T)$ if $u_T \neq 1$. The other direction is trivial. We use induction on $|T|$. If $|T| = 0$, the assertion is trivial, since $\min(u) > 1$ and $T^c = [n]$. Now let $|T| > 0$. Note that $T \neq [n]$, because we assume that $u_T \neq 1$. This means that T is a non-empty proper subset of $[n]$. Let $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 < i_2 < \ldots < i_d$ and let $T = \{j_1, j_2, \ldots, j_s\}$ with $j_1 < j_2 < \ldots < j_s$. Let r be the biggest number such that $j_i = i$ for $i = 1, \ldots, r-1$. Then $\min(T^c) = r$, and

$$u_T = (x_{i_r}\cdots x_{i_d})_{\{j_r,\ldots,j_s\}}$$

by Theorem 3.1. Therefore, $\min(u_T) \geq i_r \geq r+1$. Thus we see that $\min(T^c) = r < r+1 \leq \min(u_T)$, as desired. □

Corollary 3.12. *Let $u = x_{i_1}\cdots x_{i_d}$ be a monomial in $S = K[x_1, \ldots, x_n]$ with $i_1 < i_2 < \ldots < i_d$, and $T = \{j_1, \ldots, j_s\}$ with $j_1 < j_2 < \ldots < j_s$. Suppose that $i_j = j$ for $j = 1, \ldots, r$ and $i_{r+1} > r+1$. Then*

$$\mathrm{Ass}^\infty(B_S(u)) = \{(x_1), \ldots, (x_r)\} \cup \{P_T\colon [r] \subseteq T, u_T \neq 1, \max(T^c) = \max(u_T)\}.$$

Moreover, $u_T \neq 1$ if and only if $s < d$ or $s \geq d$ and $j_l > i_l$ for some l.

Proof. By assumption we have $u = x_1\cdots x_r u'$, where $u' = x_{i_{r+1}}\cdots x_{i_d}$ with $i_{r+1} > r+1$. Therefore, $B_S(u) = x_1\cdots x_r B_{S'}(u')S$, where $S' = K[x_{r+1}, \ldots, x_n]$. Hence (2) implies that $\mathrm{Ass}^\infty(B_S(u)) = \{(x_1), \ldots, (x_r)\} \cup \mathrm{Ass}^\infty(B_{S'}(u'))S$. Since $\min(u') > r+1$, we may apply Theorem 3.10 to u' in S'. Therefore, the result follows from Theorem 3.10 and Lemma 3.11. □

Corollary 3.13. *Let $u \in S$, $u = x_{i_1} \cdots x_{i_d}$ be a monomial with $i_1 < i_2 < \ldots < i_d$ and let $I = B_S(u)$. Suppose that $i_j = j$ for $j = 1, \ldots, r$ and $i_{r+1} > r + 1$. Then* $\operatorname{bigheight}(I^l) = i_d - r$ *for $l \gg 0$.*

Proof. Since $l \gg 0$, it follows that $\operatorname{bigheight}(I^l) = \max\{\operatorname{height} P_T\colon\ P_T \in \operatorname{Ass}^\infty(I)\}$. The ideal P_T for $T = [r]$ has height $i_d - r$, and this the largest possible height among the ideals P_T. Since $u_T = x_{i_{r+1}} \cdots x_{i_d}$, Corollary 3.12 implies that $P_T \in \operatorname{Ass}^\infty(I)$. □

Example 3.14. Let $u = x_1x_3x_5$ and $I = B_S(u)$. Then

$$\{T\colon [1] \subseteq T, u_T \neq 1, \max(T^c) = \max(u_T)\} = \{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4, 5\}\}.$$

Hence

$$\operatorname{Ass}^\infty(I) = \{(x_1)\} \cup \{(x_2, x_3, x_4, x_5), (x_3, x_4, x_5), (x_2, x_4, x_5), (x_2, x_3)\}.$$

4. The socle of $S/B_S(u)^k$

In this section we determine the powers of $B_S(u)$ and their socle. We first recall a result of De Negri [7, Proposition 3.4], and present a proof it by using sortability. This makes the proof substantially shorter.

Theorem 4.1. *Let $u \in S$ be a squarefree monomial. Then $B_S(u)^k = B_S^k(u^k)$ for all $k \geq 1$. In particular, $B_S(u)^k$ is k-strongly stable.*

Proof. Let $u = x_{l_1} \cdots x_{l_d}$ with $l_1 \leq l_2 \leq \cdots \leq l_d$, and suppose that $v \in B_S^k(u^k)$, and let $v = x_{i_1} \cdots x_{i_{kd}}$ with $i_1 \leq i_2 \leq \cdots \leq i_{kd}$. Since $v \preceq_k u^k$ it follows that $i_{(r-1)k+1}, \ldots, i_{rk} \leq l_r$ for $r = 1, \ldots, d$, see Corollary 2.5.

For $j = 1, \ldots, k$, let $v_j = \prod_{r=1}^d x_{i(r-1)k+j}$. Then $v = v_1 \cdots v_k$ and $v_j \preceq u$. It remains that to be shown that each v_j is squarefree. Suppose v_j is not squarefree. Then there exists r with $1 \leq r < d$ such that $x_{i_{(r-1)k+j}} = x_{i_{rk+j}}$. Hence $i_{(r-1)k+j} = i_{(r-1)k+j+1} = \cdots = i_{rk+j}$. This implies that $\nu_{i(r-1)k+j}(v) > k$, a contradiction.

Conversely, let $v \in B_S(u)^k$. By [1, Proposition 2.4], $G(B_S(u))$ is sortable. Therefore, $v = v_1v_2 \cdots v_k$ with $v_j \in B_S(u)$ and (v_i, v_j) is sorted for all i, j with $1 \leq i < j \leq k$. It follows from [9, (6.3)], that if $v_1 = x_{i_1} \cdots x_{i_d}$ with $i_1 < i_2 < \ldots < i_d$, $v_2 = x_{j_1} \cdots x_{j_d}$ with $j_1 < j_2 < \ldots < j_d$, $\ldots$, $v_k = x_{s_1} \cdots x_{s_d}$ with $s_1 < s_2 < \ldots < s_d$, then $i_1 \leq j_1 \leq \ldots \leq s_1 \leq i_2 \leq j_2 \leq \ldots \leq s_2 \leq \ldots \leq i_k \leq j_k \leq \ldots \leq s_k$. From this it follows that $v \in B_S^\infty(u^k)$. Since each v_j is squarefree it follows that $v \in B_S^k(u^k)$. □

For any monomial ideal we denote by $\mathrm{Soc}(S/I)$ the (finite) set of monomial $v \in S$ with $v \notin I$ and $vx_j \in I$ for $j = 1, \ldots, n$. Note that $\operatorname{depth} S/I = 0$ if and only if $\mathrm{Soc}(S/I) \neq \emptyset$.

Corollary 4.2. *Let $u \in S$ be a squarefree monomial of degree d, and let $I = B_S(u)$. Suppose that* $\operatorname{depth} S/I^k = 0$. *Then* $\mathrm{Soc}(S/I^k)$ *is a $(k-1)$-stable set of monomials generated in degree $kd - 1$.*

Proof. By [1, Proposition 2.4] the ideal I^k has kd-linear resolution. Suppose now that $\operatorname{depth} S/I^k = 0$. Then the graded minimal free resolution of S/I^k is of the form

$$0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to S/I^k \to 0,$$

with $F_0 = S$ and $F_i = S(-kd - i + 1)^{\beta_i}$ for $i = 1, \ldots, n$. So $F_n = S(-kd - n + 1)^{\beta_n}$. This shows that all elements of $\mathrm{Soc}(S/I^k)$ have degree $(kd + n - 1) - n = kd - 1$. Indeed, $F_n/\mathfrak{m}F_n$ and the Koszul homology $H_n(x_1, \ldots, x_n; S/I^k)$ are isomorphic as the graded K-vector spaces. The generators of $H_n(x_1, \ldots, x_n; S/I^k)$ are $ve_1 \wedge \ldots \wedge e_n$ with $v \in \mathrm{Soc}(S/I^k)$. This gives us the above formula for the degree of the socle elements.

By [15, Corollary 1.2], $\nu_i(v) \leq k-1$ for all $v \in \mathrm{Soc}(S/I^k)$ and $i = 1, \ldots, n$. Thus it remains to be shown that if $v \in \mathrm{Soc}(S/I^k)$, $x_j | v$, $i < j$ and $\nu_i(x_i v) \leq k - 1$, then $v_0 := x_i(v/x_j) \in \mathrm{Soc}(S/I^k)$, that is, $x_l v_0 \in I^k$ for $l = 1, \ldots, n$. Indeed, if $l = j$, then $x_l v_0 = x_i v \in I^k$. If $l \neq j$, then $x_l v_0 = x_i(x_l v)/x_j \in I^k$ because $x_l v \in I^k$ and I^k is k-stable, see Theorem 4.1. □

Let $1 < d < n$ be integers and for $k \geq 2$ let c_k, r_k be integers such that

$$kd - 1 = c_k(k-1) + r_k \quad \text{with} \quad 0 \leq r_k < k - 1. \tag{4}$$

Then we define the monomial $w_k = x_1^{k-1} \cdots x_{c_k}^{k-1} x_{c_k+1}^{r}$. Note $w_k \in S$ if and only if $c_k \leq n$ when $r = 0$, and $c_k + 1 \leq n$ if $r \neq 0$.

Corollary 4.3. *Let $u \in S$ be a squarefree monomial of degree d, and let $I = B_S(u)$. Then* $\operatorname{depth} S/I^k = 0$ *if and only if $w_k x_n \preceq u^k$.*

Proof. If $\operatorname{depth} S/I^k = 0$, then $\mathrm{Soc}(S/I^k)$ is a non-empty $(k-1)$-stable set. Let $v \in \mathrm{Soc}(S/I^k)$. Then $\{w\colon\ w \preceq_{k-1} v\} \subseteq \mathrm{Soc}(S/I^k)$. Since $w_k \preceq_{k-1} v$ it follows that $w_k \in \mathrm{Soc}(S/I^k)$. Therefore, $w_k x_n \in I^k = B_S^k(u^k)$. This implies that $w_k x_n \preceq_k u^k$, and hence $w_k x_n \preceq u^k$, see Remark 2.3.

Conversely, suppose that $w_k x_n \preceq u^k$. Therefore, $x_i w_k = x_i(w_k x_n)/x_n \preceq u^k$. Since the exponents of w_k are bounded by $k-1$, the exponents of $x_i w_k$ are bounded by k. Therefore, by Remark 2.3, $x_i w_k \preceq_k u^k$ for all i. This means that $x_i w_k \in I^k$ for all i. Hence $w_k \in \mathrm{Soc}(S/I^k)$, and so $\operatorname{depth} S/I^k = 0$. □

Note that $w_k x_n \preceq u^k$ only if $w_k \in S$.

Remark 4.4. Let $d < n$, $f_d(k) = \lceil (kd-1)/(k-1) \rceil$ and $w_k = x_1^{k-1} \cdots x_{c_k}^{k-1} x_{c_k+1}^r$. Then

(a) $f_d(k) = c_k$ if $r = 0$ and $f_d(k) = c_k + 1$ if $r \neq 0$. Therefore, $w_k \in S$ if and only if $f_d(k) \leq n$.
(b) The function $f_d(k)$ is a non-increasing function with $f_d(k) \leq n$ for $k \geq d$.
(c) Let $k_0 = \min\{k\colon\ f_d(k) \leq n\}$. Then $k_0 \leq d$.

Proof. (a) follows from the definition of c_k, see (4).

(b) It is obvious that $f_d(k)$ is a non-increasing function. Since $f_d(d) = d+1 \leq n$, we also have $f_d(k) \leq n$ for $k \geq d$.

(c) follows from (b). □

Corollary 4.5. *Let k_0 be defined as in Remark 4.5. Then* $\min\{k\colon\ \operatorname{depth} S/I^k = 0\} \geq k_0$. *In particular, this lower bound for* $\min\{k\colon\ \operatorname{depth} S/I^k = 0\}$ *depends only on d and n.*

In general the inequality $\min\{k\colon\ \operatorname{depth} S/I^k = 0\} \geq k_0$ may be strict, as the following example shows: Let $u = x_2x_3x_4x_5x_6 \in S = K[x_1, \ldots, x_6]$. Then $d = 5$ and $n = 6$. Then $f_5(2) = 9$, $f_5(3) = 7$ and $f_5(4) = 6$. Therefore, $k_0 = 4$, but on the other hand 5 is the smallest number k for which $\operatorname{depth} S/I^k = 0$.

We use this characterization of the socle elements to show

Proposition 4.6. *Let $I = B_S(x_{i_1}x_{i_2}\cdots x_{i_d}) \subset S = K[x_1, \ldots, x_n]$ with $1 < i_1 < i_2 < \ldots < i_d = n$. Then $\mathfrak{m} \in \mathrm{Ass}(I^k)$ for $k \geq d$.*

Proof. It is enough to show that $\mathfrak{m} \in \mathrm{Ass}(I^d)$, since I satisfies the persistence property, see [1, Proposition 2.4].

Note that $w_d = x_1^{d-1} \cdots x_{d+1}^{d-1}$. We show that $w_d x_n \in I^d$. From Corollary 4.5 it then follows that $\operatorname{depth}(S/I^d) = 0$ which implies that $\operatorname{dstab}(I) \leq d$. To see that $w_d x_n \in I^d$, we must show that $w_d x_n \prec_d u^d$, where $u = x_{i_1} \cdots x_{i_d}$ with $1 < i_1 < i_2 < \ldots i_d = n$. Since $v = x_2 \cdots x_d x_n \prec_1 u$ it follows $v^d \prec_d u^d$. Therefore, it suffices to show that $w_d x_n \prec_d v^d$.

We write $w_d x_n = x_{k_1} \ldots x_{k_{d^2}}$ with $k_1 \leq k_2 \leq \ldots \leq k_{d^2}$ and $v^d = x_{l_1} \cdots x_{l_{d^2}}$ with $l_1 \leq l_2 \leq \ldots \leq l_{d^2}$. Since $\nu_r(w_d x_n), \nu_r(v^d) \leq d$ for all r, we must show that $k_r \leq l_r$

for $r = 1, \dots, d^2$, see Corollary 2.5. Let r be an integer with $1 \le r \le d^2$. We may assume that $r \le (d-1)d$, because $l_r = n$, for $(d-1)d < r \le d^2$. Let

$$r - 1 = qd + t_1 \text{ with } 0 \le t_1 \le d-1, \text{ and } r-1 = q'(d-1) + t_2 \text{ with } 0 \le t_2 \le d-2.$$

Then $l_r = q+2$ and $k_r = q'+1$. Now $k_r \le l_r$ if and only if $q' - q \le 1$. Indeed, we have $qd + t_1 = q'(d-1) + t_2$. This implies that $d(q'-q) = t_1 - t_2 + q' < 2d$. Therefore, $q' - q < 2$, as desired. □

5. Comparison of $\operatorname{astab}(B_S(u))$ with $\operatorname{dstab}(B_S(u))$

As mentioned in Section 3, for any graded ideal $I \subset S = K[x_1, \dots, x_n]$ there exists an integer k_1 such that $\operatorname{Ass}_S(I^k) = \operatorname{Ass}_S(I^{k_1})$ for all $k \ge k_1$. The smallest integer k_1 with this property is denoted by $\operatorname{astab}(I)$. Similarly there exists an integer k_2 such that $\operatorname{depth} S/I^k = \operatorname{depth} S/I^{k_2}$ for all $k \ge k_2$. The smallest integer k_2 with this property is denoted by $\operatorname{dstab}(I)$. The purpose of the section is to compute $\operatorname{dstab}(I)$ and $\operatorname{astab}(I)$ when I is a squarefree principal Borel ideal.

Let $I = B_S(x_{i_1} x_{i_2} \cdots x_{i_d}) \subset S = K[x_1, \dots, x_n]$ with $i_1 < i_2 < \dots < i_d$. Assume that $i_d = m \le n$, and let $S' = K[x_1, \dots, x_m]$ and $J = B_{S'}(x_{i_1} x_{i_2} \cdots x_{i_d})$. Then, we obviously one gets

Lemma 5.1. *With the assumptions and notation introduced, we have*

$$\operatorname{depth} S/I^k = \operatorname{depth} S'/J^k + n - m \quad \textit{for all} \quad k,$$

and

$$\operatorname{Ass}_S(I^k) = \{PS : P \in \operatorname{Ass}_{S'}(J^k)\} \quad \textit{for all} \quad k.$$

In particular, $\operatorname{dstab}(I) = \operatorname{dstab}(J)$ *and* $\operatorname{astab}(I) = \operatorname{astab}(J)$.

Therefore, for the rest of this section, if not otherwise stated, we may assume that $i_d = n$.

Next we show

Lemma 5.2. *Let $I = B_S(u) \subset S = K[x_1, \dots, x_n]$, where $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ with $1 = i_1 < i_2 < \dots < i_d = n$. Let $J = B_{S'}(u') \subset S' = K[x_2, \dots, x_{n-1}]$, where $u' = u/x_1$. Then for all k,*

$$\operatorname{Ass}_S(I^k) = \{PS\colon\ P \in \operatorname{Ass}_{S'}(J^k)\} \cup (x_1) \quad \textit{and} \quad \operatorname{depth} S/I^k = \operatorname{depth} S'/J^k + 1.$$

In particular, $\operatorname{dstab}(I) = \operatorname{dstab}(J)$ *and* $\operatorname{astab}(I) = \operatorname{astab}(J)$.

Proof. We observe that $I^k = x_1^k \tilde{J}^k$ for all k, where $\tilde{J} = JS$. Therefore, the statement about $\mathrm{Ass}_S(I^k)$ follows from (2). Since $(x_1^k) \cap \tilde{J}^k = x_1^k \tilde{J}^k$ we get the short exact sequence

$$0 \to S/I^k \to S/(x_1^k) \oplus S/\tilde{J}^k \to S/(x_1^k, \tilde{J}^k) \to 0.$$

By the depth lemma (see [8, Corollary 18.6]) we have

$$\begin{aligned} \operatorname{depth} S/I^k &\geq \min\{\operatorname{depth}(S/(x_1^k) \oplus S/\tilde{J}^k), \operatorname{depth} S/(x_1^k, \tilde{J}^k) + 1\} \\ &= \min\{\operatorname{depth} S/(x_1^k), \operatorname{depth} S/\tilde{J}^k, \operatorname{depth} S/(x_1^k, \tilde{J}^k) + 1\}. \end{aligned}$$

Since x_1^k is regular on $S/\tilde{J}^k$, it follows that $\operatorname{depth} S/(x_1^k, \tilde{J}^k) = \operatorname{depth} S/\tilde{J}^k - 1$. Therefore, $\operatorname{depth} S/I^k \geq \operatorname{depth} S/\tilde{J}^k$.

By using again the depth lemma we also get

$$\operatorname{depth} S/\tilde{J}^k - 1 = \operatorname{depth} S/(x_1^k, \tilde{J}^k) \geq \min\{\operatorname{depth} S/(x_1^k), \operatorname{depth} S/\tilde{J}^k, \operatorname{depth} S/I^k - 1\}.$$

This implies that $\operatorname{depth} S/\tilde{J}^k \geq \operatorname{depth} S/I^k$, and hence $\operatorname{depth} S/I^k = \operatorname{depth} S/\tilde{J}^k$. Since $\tilde{J} = JS$, it follows that $\operatorname{depth} S/\tilde{J} = \operatorname{depth} S'/J$. This yields the desired conclusion. □

Because of Lemma 5.2 we may also assume for the rest of the section that $i_1 > 1$.

Combining Lemma 5.2 with Corollary 4.5 we obtain

Proposition 5.3. *Let $I = B_S(u) \subset S = K[x_1, \ldots, x_n]$ with $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $1 < i_1 < i_2 < \ldots < i_d = n$. Then we have:*

(a) $\operatorname{dstab}(I) = \min\{k\colon \ \operatorname{depth} S/I^k = 0\}$.

(b) *Let k_0 be defined as in Corollary 4.5. Then*

$$\operatorname{dstab}(I) = \min\{k \geq k_0\colon \ w_k x_n \in I^k\}.$$

(c) $k_0 \leq \operatorname{dstab}(I) \leq \operatorname{astab}(I) \leq d$.

The following result tells us how dstab behaves under monomial localization.

Theorem 5.4. *Let $I = B_S(u) \subset S = K[x_1, \ldots, x_n]$ with $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ and $1 < i_1 < i_2 < \ldots < i_d = n$. Then*

$$\operatorname{dstab}(I_{(a)}) \leq \operatorname{dstab}(I) \quad \textit{for} \quad a = 1, \ldots, n.$$

Proof. We first consider the case that $a \leq i_{d-1}$. Then $I_{(a)} = B_S(u)_{(a)} = B_{S_{(a)}}(u_a)$ and $\min(u_a) > \min(T^c)$ and $\max(u_a) = \max(T^c) = n$. Then Proposition 4.6 implies that $\operatorname{depth} S/I^k = 0$ for some k and $\operatorname{depth} S_{(a)}/I_{(a)}^l = 0$ for some l. Let k and l be the smallest numbers with this property. Then $k = \operatorname{dstab}(I)$ and $l = \operatorname{dstab}(I_{(a)})$, and we want to show that $l \leq k$. For this it is enough to show that $\operatorname{depth} S_{(a)}/I_{(a)}^k =$

0. Since $\operatorname{depth} S/I^k = 0$, Corollary 4.5 implies that $w_k x_n \in I^k$. By Theorem 4.1, $I^k = B_S^k(u_k)$. Therefore, $w_k x_n \preceq_k u^k$. Let $w_k x_n = x_{t_1(k)} x_{t_2(k)} \ldots x_{t_{kd}(k)}$ with $t_1(k) \leq \ldots \leq t_{kd}(k)$ and $u^k = x_{l_1(k)} x_{l_2(k)} \cdots x_{l_{kd}(k)}$ with $l_1(k) \leq \cdots \leq l_{kd}(k)$. By Corollary 2.5, $w_k x_n \preceq u^k$ if and only if $t_s(k) \leq l_s(k)$ for $s = 1, \ldots, kd$. The numbers $l_s(k)$ and $t_s(k)$ can be computed.

With the notation introduced we have

$$l_s(k) = i_j, \quad \text{where} \quad j = \lceil s/k \rceil,$$

and

$$t_s(k) = \begin{cases} \lceil s/(k-1) \rceil, & \text{if } s \leq (k-1)c; \\ c+1, & \text{if } (k-1)c < s < kd; \\ n, & \text{if } s = kd. \end{cases}$$

Let b be the unique number with $i_{b-1} < a \leq i_b$. Then $u_a = u/i_b \in S_{(a)}$ is of degree $d-1$, see Theorem 3.1. Let $w'_k \in S_{(a)}$ be the socle test element for $I^k_{(a)}$ in $S_{(a)}$. We have $\operatorname{depth} S_{(a)}/I^k_{(a)} = 0$, once we have shown that $w'_k x_m \preceq_k u^k_a$.

Let

$$w'_k x_n = x_{t'_1(k)} x_{t'_2 k)} \ldots x_{t'_{k(d-1)}(k)} \quad \text{with} \quad t'_1(k) \leq \ldots \leq t'_{k(d-1)}(k),$$

and

$$u^k_a = x_{l'_1(k)} x_{l'_2(k)} \cdots x_{l'_{k(d-1)}(k)} \quad \text{with} \quad l'_1(k) \leq \cdots \leq l'_{k(d-1)}(k).$$

It remains to be shown that $t'_s(k) \leq l'_s(k)$ for $s = 1, \ldots, k(d-1)$.

Note that

$$l'_s(k) = \begin{cases} l_s(k), & \text{if } s \leq k(b-1); \\ l_{s+k}(k), & \text{if } k(b-1) < s \leq k(d-1), \end{cases}$$

and

$$t'_s(k) = \begin{cases} t_s(k), & \text{if } s \leq (k-1)(a-1); \\ t_{s+k-1}(k), & \text{if } (k-1)(a-1) < s \leq k(d-1). \end{cases}$$

Note that $b \leq a$, because $b - 1 \leq i_{b-1} < a$. We consider different cases.

Case 1: $s \leq (k-1)(a-1)$: Then, since $b \leq a$, we have

$$\begin{aligned} t'_s(k) &= t_s(k) \leq l_s(k) = \begin{cases} l'_s(k), & \text{if } s \leq k(b-1); \\ l'_{s-k}(k), & \text{if } k(b-1) < s \leq k(d-1), \end{cases} \\ &\leq l'_s(k), \end{aligned}$$

because $l'_s(k)$ is a non-decreasing function.

Case 2: $s > (k-1)(a-1)$: Then

$$t'_s(k) = t_{s+k-1}(k) \leq l_{s+k-1}(k) = \begin{cases} l'_{s+k-1}(k), & \text{if } s \leq k(b-1); \\ l'_{s-1}(k), & \text{if } k(b-1) < s \leq k(d-1), \end{cases}$$

Therefore, if $s > k(b-1)$, then $t'_s(k) \leq l'_s(k)$.

Suppose now that $(k-1)(a-1) < s < k(b-1)$. Then $t_s(b) = \lceil s/(k-1) \rceil \geq a$ and $l_s(k) = i_j$, where $j = \lceil s/k \rceil$. Therefore, $l_s(k) = i_c$ with $c \leq b-1$. Hence,

$$a \leq t_s(k) \leq l_s(k) = i_c \leq i_{b-1},$$

a contradiction. So this case cannot happen.

It remains to treat the case that $a > i_{d-1}$. Then $u_a = x_{i_1} \cdots x_{i_{d-1}}$, and $I_{(a)} = B_{S_{(a)}}(u_a)$. Let $J = B_{S'}(u_a)$, where $S' = k[x_1, \ldots, x_{i_{d-1}}]$. Then, by Lemma 5.1, $\operatorname{dstab}(J) = \operatorname{dstab}(I_{(a)})$, and furthermore by Proposition 5.3 we have $\operatorname{dstab}(J) = \min\{k\colon \ \operatorname{depth} S'/J^k = 0\}$. Therefore, we must show that $\operatorname{depth} S'/J^k = 0$ if $\operatorname{depth} S/I^k = 0$. Since $\operatorname{depth} S/I^k = 0$ it follows $w_k x_n \preceq_k u^k$. Let $w_k = \prod_{s=1}^{kd-1} x_{t_s(k)}$. Then the socle test element for J^k is $w'_k = \prod_{s=1}^{k(d-1)-1} x_{t_s(k)}$. Then it is clear that $w'_k x_{i_{d-1}} \preceq_k u_a^k$. This shows that $\operatorname{depth} S'/J^k = 0$, as desired. □

As the main result of the section we show

Theorem 5.5. *Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in n variables, let $u \in S$ be a squarefree monomial ideal and let $I = B_S(u)$. Then* $\operatorname{astab}(I) = \operatorname{dstab}(I)$.

Proof. Let $u = x_{i_1} \cdots x_{i_d}$ with $i_1 < i_2 < \cdots < i_d$. By the discussion at the beginning of this section and by Lemma 5.2 we may assume that $1 < i_1$ and $i_d = n$. Let $k = \operatorname{dstab}(I)$. By Proposition 5.3, we have $\mathfrak{m} \in \operatorname{Ass}_S(I^k)$. Now let $P \in \operatorname{Ass}_S^\infty(I)$ with $P \neq \mathfrak{m}$. Then there exists j and $P' \in \operatorname{Ass}^\infty_{S_{(j)}}(I_{(j)})$ with $P = P'S$. By induction we may assume that $\operatorname{astab}(I_{(j)}) = \operatorname{dstab}(I_{(j)})$. Thus, if $k' = \operatorname{dstab}(I_{(j)})$, then $P' \in \operatorname{Ass}_{S_{(j)}}(I_{(j)}^{k'})$. Therefore, $P \in \operatorname{Ass}_S(I^{k'})$, by (3). Theorem 5.4 implies that $k' \leq k$. Since squarefree principal monomial ideals satisfy the persistence property, we conclude that $P \in \operatorname{Ass}_S(I^k)$, as desired. This shows that $\operatorname{astab}(I) \leq \operatorname{dstab}(I)$. The other inequality is shown in Proposition 5.3. □

6. On the depth of $S/B_S(u)^k$

We provide some partial results regarding $\operatorname{depth} S/I^k$ for $I = B_S(u)$. Since all powers have a linear resolution it follows that $\operatorname{depth} S/I^{k+1} \leq \operatorname{depth} S/I^k$ for all $k \geq 1$, as mentioned in the proof of Proposition 5.3. Actually all powers of I have linear quotients for a suitable order, as shown in [1, Proposition 2.4]. Here we show

Proposition 6.1. *Let $I = B_S(u)$. Then for all k, the ideal I^k has linear quotients with respect to the lexicographic order induced by $x_1 > x_2 > \cdots > x_n$.*

Proof. Let $v, w \in I^k$ with $w > v$ with respect to the lexicographic order. Let $v = x_{j_1} x_{j_2} \cdots x_{j_{kd}}$ with $j_1 \leq j_2 \leq \ldots \leq j_{kd}$, and $w = x_{l_1} x_{l_2} \cdots x_{l_{kd}}$ with $l_1 \leq l_2 \leq \ldots \leq l_{kd}$. Since $w > v$ there exists an integer r such that $l_s = j_s$ for $s = 1, \ldots, r-1$

and $j_r > l_r$. Then x_{l_r} divides $w/gcd(w,v)$. Let $w' = x_{l_r}v/x_{j_r}$. Since $l_r < j_r$ it follows that $w' > v$ and $w'/\gcd(w',v) = x_{l_r}$ which divides $w/gcd(w,v)$. It remains to be shown that $w' \in I^k$. To see that we use Theorem 4.1 which says $I^k = B_S^k(u^k)$. Since $v \in B_S^k(u^k)$ it follows that $v \prec_k u^k$. We now show that $w' \prec_k v$. Then it follows $w' \prec_k u^k$, and hence belongs to I^k. Indeed, since x_{j_r} in v is replaced by x_{l_r} with $l_r < j_r$ to obtain w' it follows that $w' \prec v$. Let $w' = x_{f_1}x_{f_2}\cdots x_{f_{kd}}$ with $f_1 \leq f_2 \leq \ldots \leq f_{kd}$. Then $f_s = j_s$ for $s \neq r$ and $f_r = l_r$. To have $w' \prec_k v$ we must show that there is no t such that $f_t = f_{t+1} = \cdots = f_{t+k}$, or equivalently there is no t such that $f_t = f_{t+k}$. If $t+k < r$ or $t > r$ we have $f_t = j_t$ and $f_{t+k} = j_{t+k}$. Therefore, $f_t \neq f_{t+k}$ because the exponents of the monomial v are bounded by k. Now let $t \leq r \leq t+k$. Suppose first that $r < t+k$. Then $r+1 \leq t+k$, and $f_{r+1} = j_{r+1} \geq j_r > l_r = f_r$. So not all f_s are the same for s with $t \leq s \leq t+k$. Finally, assume that $r = t+k$ and $f_{r-k} = \cdots = f_r$. For $s < r$ we have $f_s = j_s = l_s$ and $f_r = l_r$. Thus our assumption implies that $l_{r-k} = \cdots = l_r$, a contradiction because $w \in B_S^k(u^k)$. □

In all examples considered, the depth function $f(k) = \operatorname{depth} S/I^k$ is strictly decreasing until it becomes stable. In some special case we show that this is indeed the case.

We first observe

Proposition 6.2. *Let $I = B_S(u)$ with $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ and $i_1 < i_2 < \cdots < i_d = n$. Then* $\operatorname{depth} S/I = d-1$.

Proof. We have to show that $\operatorname{proj\,dim} I = n-d$. Let τ be the inverse of the spreading operator σ. By definition, if $v = x_{k_1}x_{k_2}\cdots x_{k_d}$ with $k_1 < k_2 < \cdots < k_d$, then $\tau(v) = \prod_{j=1}^d x_{k_j-(j-1)}$, and when I is a squarefree monomial ideal with $G(I) = \{v_1,\ldots,v_m\}$, one sets I^τ to be the ideal with $G(I^\tau) = \{\tau(v_1),\ldots,\tau(v_m)\}$.

By [10, Proposition 2.1] we have $B_S(u)^\tau = B_S^\infty(\tau(u))$, where $B_S^\infty(\tau(u))$ is the principal Borel ideal with Borel generator $\tau(u)$. It follows from [10, Theorem 1.11] that $\operatorname{proj\,dim} B_S(u) = \operatorname{proj\,dim} B_S^\infty(\tau(u))$. Note that all generators of $B_S^\infty(\tau(u))$ belong to $T = K[x_1,\ldots,x_{n-d+1}]$. Let $J = G(B_S^\infty(\tau(u)))T$. Then $B_S^\infty(\tau(u)) = JS$. It follows that

$$\operatorname{depth} S/I = \operatorname{depth} S/B_S^\infty(\tau(u)) = \operatorname{depth} T/J + d - 1.$$

It remains to be shown that $\operatorname{depth} T/J = 0$. It is well-known and easy to prove that

$$J = \prod_{j=1}^d (x_1,\ldots,x_{i_j-(j-1)}).$$

Since $i_d = n$ it follows that $J = J_0\mathfrak{n}$, where $J_0 = \prod_{j=1}^{d-1}(x_1, \ldots, x_{i_j-(j-1)})$ and $\mathfrak{n}$ is the maximal ideal of T. This shows that any minimal generator of J_0 defines a socle element of T/J. In particular, $\operatorname{depth} T/J = 0$. $\square$

The next result tells us when $\operatorname{depth} S/I^2 = 0$ for $I = B_S(u)$.

Proposition 6.3. *Let $u = x_{i_1}x_{i_2}\cdots x_{i_d} \subset S$ be a monomial with $1 < i_1 < i_2 < \ldots < i_d = n$, and let $I = B_S(u)$. The following conditions are equivalent:*

(a) $\operatorname{depth} S/I^2 = 0$.

(b) $x_1 \cdots x_{2d-1} \in \operatorname{Soc}(S/I^2)$.

(c) $i_j \geq 2j$ *for* $j = 1, \ldots, d-1$.

Proof. (a) $\Longleftrightarrow$ (b) follows from Corollary 4.3.

(b) $\Longleftrightarrow$ (c) $w_2 = x_1 \cdots x_{2d-1} \in \operatorname{Soc}(S/I^2)$, if and only if $x_1 \cdots x_{2d-1}x_n \in I^2 = B_S(u^2)$, and this is the case if and only if

$$x_1 \cdots x_{2d-1}x_n \prec u^2 = x_{i_1}x_{i_1}x_{i_2}x_{i_2}\cdots x_{i_{d-1}}x_{i_{d-1}}x_nx_n.$$

By Corollary 2.5 this is the case if and only if $i_j \geq 2j$ for $j = 1, \ldots, d-1$, as desired. $\square$

Corollary 6.4. *Let $I = B_S(x_ix_n)$. Then* $\operatorname{depth} S/I = 1$*, and for $k \geq 2$ we have*

$$\operatorname{depth} S/I^k = \begin{cases} 1, & \text{if } i = 1, \\ 0, & \text{if } i > 1. \end{cases}$$

Proof. The fact $\operatorname{depth}(S/I) = 1$ follows from Proposition 6.2. In order to compute $\operatorname{depth}(S/I^k)$, we first suppose that $i = 1$. Then Lemma 5.2 implies that $\operatorname{depth} S/I^k = \operatorname{depth} S'/B_{S'}(x_{n-1})^k + 1$. Since $B_{S'}(x_{n-1}) = (x_1, \ldots, x_{n-1})$, it follows that $\operatorname{depth} S/I^k = 1$ for all k. Finally, for $i > 1$ and $k > 1$, the result follows from Proposition 6.3. $\square$

Next we consider the case $d = 3$.

Proposition 6.5. *Let $I = B_S(x_ix_jx_n)$ with $i < j < n$.*

(a) *If $i = 1$, then* $\operatorname{depth} S/I = 2$*, and for $k \geq 2$ we have*

$$\operatorname{depth} S/I^k = \begin{cases} 2, & \text{if } j = 2, \\ 1, & \text{if } j > 2. \end{cases}$$

(b) *If $i > 1$, then* $\operatorname{depth} S/I = 2$*, and* $\operatorname{depth} S/I^k = 0$ *for $k \geq 3$. Moreover,*

$$\operatorname{depth} S/I^2 = \begin{cases} 1, & \text{if } j = 3, \\ 0, & \text{if } j > 3. \end{cases}$$

Proof. (a) follows from Lemma 5.2 and Proposition 6.4.

(b) Proposition 6.2 shows that $\operatorname{depth} S/I = 2$. Moreover, $\operatorname{depth} S/I^k = 0$ for $k \geq 3$, see Proposition 5.3. Thus it remains to consider the case $k = 2$. If $j > 3$, then $\operatorname{depth} S/I^2 = 0$. Finally assume that $j = 3$. Then $x_i x_j x_n = x_2 x_3 x_n$. Then $\operatorname{depth} S/I^2 > 0$, by Proposition 6.3. On the other hand, $\operatorname{depth} S/I^2 < \operatorname{depth} S/I = 2$, by the next Theorem 6.6. Therefore, $\operatorname{depth} S/I^2 = 1$, as desired. □

The following result supports what we expect, namely that $\operatorname{depth} S/I^k < \operatorname{depth} S/I^{k-1}$ if $k \leq \operatorname{astab}(I)$.

Theorem 6.6. *Let $I = B_S(u)$. Then the following conditions are equivalent:*

(a) $u = x_1 x_2 \cdots x_{d-1} x_n$.
(b) $\operatorname{depth} S/I = \operatorname{depth} S/I^k$ *for all* k.
(c) $\operatorname{depth} S/I = \operatorname{depth} S/I^k$ *for some* $k \geq 2$.
(d) $\operatorname{depth} S/I = \operatorname{depth} S/I^2$.

Moreover, if one of these equivalent conditions fails, then $\operatorname{depth} S/I^2 < \operatorname{depth} S/I$.

For the proof of Theorem 6.6 we use

Lemma 6.7. *Let $I = B_S(u)$, where $u = x_{i_1} \cdots x_{i_d}$ with $1 < i_1 < i_2 < \ldots < i_d = n$. Let $v \in I^2$ and let $J = (w \in G(I^2)\colon\ v <_{\text{lex}} w)$. Assume that x_n divides v. Then*

$$\{x_i\colon\ i < n \text{ and } x_i^2 \nmid v\} \subseteq J : v.$$

Proof. Let i be such that $i < n$ and $x_i^2 \nmid v$, and let $w = v x_i / x_n$. Then $\nu_j(w) \leq 2$ for all j and $w \prec v$. Therefore, $w \in I^2$. Since $v <_{\text{lex}} w$, it follows that $w \in J$. Moreover, $v x_i = w x_n$, and hence $x_i \in J : v$. □

Proof of Theorem 6.6. (a) $\Rightarrow$ (b) Note that $I = x_1 \cdots x_{d-1}(x_d, \ldots, x_n)$. Therefore, $I^k \cong (x_d, \ldots, x_n)^k$, and hence $\operatorname{depth} S/I^k = \operatorname{depth} I^k - 1 = \operatorname{depth}(x_d, \ldots, x_n)^k - 1 = \operatorname{depth} S/(x_d, \ldots, x_n)^k$. Since $\operatorname{depth} K[x_d, \ldots, x_n]/(x_d, \ldots, x_n)^k = 0$, it follows that

$$\operatorname{depth} S/I^k = \operatorname{depth} S/(x_d, \ldots, x_n)^k = d - 1 = \operatorname{depth} S/I.$$

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (d) Since $\operatorname{depth} S/I \geq \operatorname{depth} S/I^2 \geq \operatorname{depth} S/I^k$, it is obvious that (c) implies (d).

(d) $\Rightarrow$ (a) We show that if $u \neq x_1 \cdots x_{d-1} x_n$, then $\operatorname{depth} S/I^2 < \operatorname{depth} S/I$. Since $u \neq x_1 \cdots x_{d-1} x_n$ it follows that $u = x_1 x_2 \cdots x_i v$ with $i < d - 1$ and $v = x_{j_1} x_{j_2} \cdots x_n$ with $j_1 < j_2 < \cdots < n$ and $j_1 > i + 1 \geq 1$. Then $I = B_S(u) = x_1 \cdots x_i J$, where $J = B_S(v)$. From this it follows that $\operatorname{depth} S/I^2 < \operatorname{depth} S/I$ if and only if $\operatorname{depth} S/J^2 < \operatorname{depth} S/J$. Therefore, we may assume that $u = x_{i_1} \cdots x_{i_d}$ with $1 < i_1 < i_2 < \cdots < i_d = n$.

We claim that there exits $v \in G(I^2)$ such that x_n divides v and such that $|\mathcal{S}| \geq n - d + 1$, where $\mathcal{S} = \{x_i\colon\ i < n \text{ and } x_i^2 \nmid v\}|$. By Lemma 6.7, the claim implies that $\operatorname{proj\,dim} I^2 \geq n - d + 1$, and hence $\operatorname{depth} S/I^2 \leq d - 2 < \operatorname{depth} S/I$.

We write $u^2 = x_{l_1} \cdots x_{l_{2d}}$ with $l_1 \leq l_2 \leq \cdots \leq l_{2d}$, and let $v = x_{t_1} x_{t_2} \cdots x_{t_{2d}}$ with $t_1 \leq t_2 \leq \ldots \leq t_{2d}$.

For the proof of the claim we consider two cases.

Case 1: $i_j = i_1 + j - 1$ for $j = 1, \ldots, d$. Then we let $t_j = j$ for $j = 1, \ldots, i_1$ and $t_j = l_j$ for $j = i_1 + 1, \ldots, 2d$. Then $|\mathcal{S}| \geq i_1 = n - d + 1$, because $n = i_1 + d - 1$.

Case 2: There exists k such that $i_{k+1} > i_k + 1$. Then for all j we let $t_j = l_j - 1$ if j is odd, and let $t_j = l_j$ if j is even. Then v is divided by as many squares as we have integers j with $i_{j+1} = i_j + 1$. Therefore, v has at most $d - 1$ squares as factors. This implies that $|\mathcal{S}| \geq n - d + 1$, as desired. □

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