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EXAMPLES OF (NON-)BRAIDED TENSOR CATEGORIES

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Abstract. Six examples of non-braidable tensor categories which are extensions of the category $Comod(H)$, for H a supergroup algebra; and two examples of braided categories where the only possible braiding is the trivial braiding are introduced.

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1. Introduction

Braided categories were introduced by Joyal and Street [5]. They are related to knot invariants, topology and quantum groups, since they can express symmetries. Some examples of braided categories are:

- graded modules over a commutative ring,
- (co)modules over a (co)quasi-triangular Hopf algebra,
- the Braid category, [5, Section 2.2],
- the center of a tensor category.

In the last example, we begin with a tensor category and construct a braided one. In a general scenario, a natural question is it is possible to construct braidings starting with tensor categories. In particular, if G is a finite group, can a G-extension of a tensor category be braided? In this work we show that this can be done in very few cases. Then, an extension of a braided category is not necessarily braided, so it is really complicated to extend that property.

However, constructing examples of non-braided categories is also important. A big family of these come from the category of (co)modules of a Hopf algebra without a (co)quasi-triangular structure, see [9, T 10.4.2]. Masuoka in [6] and [7] constructs explicit examples of non-Quasi-triangular or non-CoQuasi-triangular Hopf algebras. In particular these Hopf algebras can not be obtained from any group algebra by twist (or cocycle) deformation. Other examples were constructed in [4].

In the literature there are a few explicit examples of tensor categories, for this reason we construct in [8] eight tensor categories, following the description introduced in [3] of Crossed Products. These categories extend the module category over certain quantum groups, called *supergroup algebras*. In a few words, a *crossed product tensor category* is, as Abelian category, the direct sum of copies of a fixed tensor category, and the tensor product comes from certain data. Then founding all possible data, we explicitly construct tensor categories.

In the same work [3], the author also describes all possible braidings over a crossed product. Following this, three conditions were introduced to decide if a G-crossed product is braidable:

(1) the base category has to be braided,
(2) G has to be Abelian, and the biGalois objects associated to each crossed product have to be trivial,
(3) the 3-cocycle associated to each crossed product over an specific supergroup algebra has to be trivial, if G is the cyclic group of order 2.

The goal in the present paper is to obtain all possible braidings over the categories introduced in [8]. With this, only two categories of the eight found in [8] are braided with the trivial braiding only, and the other 6 are not braidable.

In [8, Theorem 6.3], using the Frobenius-Perron dimension, we proved that these eight categories are the module category of a quasi-Hopf algebra. Although we do not know how to explicitly compute these algebras, as a corollary of this work, we know that six of these algebras are non-Quasi-triangular and two are Quasi-triangular only. In particular, we are obtaining information about certain quasi-Hopf algebras without knowing them explicitly; showing how useful it is to work in the category world. In a future, when we can explicitly describe these quasi-Hopf algebras, we will already know how their Quasi-triangular structures are.

2. Preliminaries and notation

Throughout this paper we shall work over an algebraically closed field $\Bbbk$ of characteristic zero. For basic knowledge of Hopf algebras see [9]. Let H be a finite-dimensional Hopf algebra and A be a left H-comodule. Then A is also a right H-comodule with right coaction $a \mapsto a_0 \otimes S(a_{-1})$, see [1, Proposition 2.2.1(iii)]. A *left H-Galois extension of* $A^{co(H)}$ is a left H-comodule algebra (A, ρ) such that $A \otimes_{A^{co(H)}} A \to H \otimes A$, $a \otimes b \mapsto (1 \otimes a)\rho(b)$ is bijective. Similarly, we define right H-Galois extension.

Consider L another finite-dimensional Hopf algebra. An *(H,L)-biGalois object* [10] is an algebra A that is a left H-Galois extension and a right L-Galois extension of the base field $\Bbbk$ such that the two comodule structures make it an (H, L)-bicomodule. Two biGalois objects are *isomorphic* if there exists a bijective bicomodule morphism that is also an algebra map. For A an (H, L)-biGalois object, define the tensor functor

$$\mathcal{F}_A : \mathrm{Comod}(L) \to \mathrm{Comod}(H), \quad \mathcal{F}_A = A\square_L - .$$

By [10], every tensor functor between comodule categories is one of these, and $\mathcal{F}_A \simeq \mathcal{F}_B$ as tensor functors if and only if $A \simeq B$ as biGalois objects.

If $A = H$, then every natural monoidal equivalence $\beta : \mathcal{F}_H \to \mathcal{F}_H$ is given by

$$f \otimes \mathrm{id}_X : H\square_H X \to H\square_H X, \quad (X, \rho_X) \in \mathrm{Comod}(H),$$

where $f : H \to H$ is a bicomodule algebra isomorphism.

Lemma 2.1. *Every natural monoidal equivalence* $\mathrm{id}_{\mathrm{Comod}(H)} \to \mathrm{id}_{\mathrm{Comod}(H)}$ *is given by* $(\varepsilon f \otimes \mathrm{id}_X)\rho_X$.

Proof. For $X \in \mathrm{Comod}(H)$, the coaction induces an isomorphism $X \simeq H\square_H X$ with inverse induced by ε, the counit. Then $\mathrm{id}_{\mathrm{Comod}(H)} \simeq \mathcal{F}_H$ as tensor functors. Since all natural monoidal autoequivalences of $\mathcal{F}_H$ are given by $f \otimes \mathrm{id}_X$ then all natural monoidal autoequivalences of $\mathrm{id}_{\mathrm{Comod}(H)}$ are given by $(\varepsilon f \otimes \mathrm{id}_X)\rho_X$. $\square$

Definition 2.2. [9, Definition 10.1.5] (H, R) is a *Quasi-triangular* (or QT) Hopf algebra if H is a Hopf algebra and there exists $R \in H \otimes H$, called the *R-matrix*, invertible such that

$$(\Delta \otimes \mathrm{id})R = R^{13}R^{23}, \quad (\mathrm{id} \otimes \Delta)R = R^{13}R^{12}, \quad \Delta^{op}(h) = R\Delta(h)R^{-1}, h \in H.$$

Dualizing we can define, (H, r) is a *CoQuasi-triangular* (or CQT) Hopf algebra if H is a Hopf algebra and $r : H \otimes H \to \Bbbk$, called the *r-form*, is a linear functional which is invertible with respect to the convolution multiplication and satisfies for arbitrary $a, b, c \in H$

$$r(c \otimes ab) = r(c_1 \otimes b)r(c_2 \otimes a), \quad r(ab \otimes c) = r(a \otimes c_1)r(b \otimes c_2),$$

$$r(a_1 \otimes b_1)a_2b_2 = r(a_2 \otimes b_2)b_1a_1.$$

Remark 2.3. Drinfeld defined a *quantum group* as a non-commutative, non-cocommutative Hopf algebra. Examples of these are the QT Hopf algebras. The importance of quantum groups lies in they allow to construct solutions for the quantum Yang-Baxter equation in statistical mechanics (the R-matrix is a solution of this equation). An example of quantum group are the supergroup algebras.

A *supergroup algebra* is a supercocommutative Hopf algebra of the form $\Bbbk[G] \ltimes \wedge V$, where G is a finite group and V is a finite-dimensional G-module. They appear and have an interesting role in the classification of triangular algebras, see [2, Theorem 4.3].

Example 2.4. Consider $H = \Bbbk C_2 \ltimes \Bbbk V$, for V a 2-dimensional vector space and C_2 the 2-cyclic group generated by u with $u \cdot v = -v$ for $v \in V$. As an algebra, it is generated by elements $v \in V, g \in C_2$ subject to relations $vw + wv = 0; gv = (g \cdot v)g$ for all $v, w \in V, g \in C_2$. The coproduct and antipode are determined by

$$\Delta(v) = v \otimes 1 + u \otimes v; \Delta(g) = g \otimes g; S(v) = -uv; S(g) = g^{-1}, \quad v \in V, g \in C_2.$$

Taking $R = \frac{1}{2}(1 \otimes 1 + 1 \otimes u + \otimes 1 - u \otimes u)$, (H, R) is a QT-Hopf algebra. We can construct a CoQuasi-triangular structure taking $r = R^*$ since H is auto-dual. Then (H, R^*) is a CQT-Hopf algebra.

Definition 2.5. A *finite tensor category* is a locally finite, $\daleth$-linear, rigid, monoidal Abelian category $\mathcal{D}$ with $\mathrm{End}_{\mathcal{D}}(\mathbf{1}) \cong \daleth$. Given a finite group Γ, a (faithful) Γ-*grading* on a finite tensor category $\mathcal{D}$ is a decomposition $\mathcal{D} = \oplus_{g \in \Gamma} \mathcal{D}_g$, where $\mathcal{D}_g$ are full Abelian subcategories of $\mathcal{D}$ such that

- $\mathcal{D}_g \neq 0$;
- $\otimes : \mathcal{D}_g \times \mathcal{D}_h \to \mathcal{D}_{gh}$ for all $g, h \in \Gamma$.

We have that $\mathcal{C} := \mathcal{D}_e$ is a tensor subcategory of $\mathcal{D}$. The category $\mathcal{D}$ is call a Γ-*extension* of $\mathcal{C}$. Denote by $[V, g]$ the homogeneous elements in $\mathcal{D}$, for $V \in \mathcal{D}_g$, $g \in \Gamma$.

A *braided tensor category* is a tensor category $\mathcal{C}$ with natural isomorphisms $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that

$$\alpha_{V,W,U} c_{U,V \otimes W} \alpha_{U,V,W} = (\mathrm{id} \otimes c_{U,W}) \alpha_{V,U,W} (c_{U,V} \otimes \mathrm{id}), \tag{1}$$

$$\alpha^{-1}_{W,U,V} c_{U \otimes V,W} \alpha^{-1}_{U,V,W} = (c_{U,W} \otimes \mathrm{id}) \alpha^{-1}_{U,W,V} (\mathrm{id} \otimes c_{V,W}). \tag{2}$$

If (H, r) is a CQT-Hopf algebra then $\mathrm{Comod}(H)$ is a braided tensor category with braiding given by $c_{V \otimes W}(x \otimes y) = r(y_{-1} \otimes x_{-1}) y_0 \otimes x_0$, for all $V, W \in \mathrm{Comod}(H)$.

The following theorem gives us the first condition to know when an extension can be braided.

Theorem 2.6. *Let $\mathcal{D} = \oplus_{g\in G}\mathcal{D}_g$ be a Γ-extension of $\mathcal{C}$. If $\mathcal{D}$ is a braided tensor category then $\mathcal{C}$ is a braided tensor category.*

Proof. Let c be the braiding of $\mathcal{D}$, then $c_{[V,e],[W,e]} : [V\otimes W, e] \to [W \otimes V, e]$ and $c_{[V,e],[W,e]} = [\overline{c}_{V,W}, e]$ for some natural isomorphism $\overline{c}_{V,W} : V \otimes W \to W \otimes V$, for V, W objects in $\mathcal{C}$. Since the associativity isomorphism satisfies $a_{[V,e],[W,e],[U,e]} = [\overline{a}_{V,W,U}, e]$, where $\overline{a}$ is the associativity morphism for $\mathcal{C}$; then $\overline{c}$ is a braiding for $\mathcal{C}$. $\square$

In [3], the author describes and classifies a family of such extensions and calls it crossed product tensor category. Fix H a finite-dimensional Hopf algebra. In the case when $\mathcal{C} = \mathrm{Comod}(H)$, in [8], we described crossed products in terms of Hopf-algebraic datum. A continuation they are introduced.

If $g \in G(H)$ and L is a (H,H)-biGalois object then the cotensor product $L\Box_H \Bbbk_g$ is one-dimensional. Let $\phi(L,g) \in \Gamma$ be the group-like element such that $L\Box_H\Bbbk_g \simeq \Bbbk_{\phi(L,g)}$ as left H-comodules. Assume that A is an H-biGalois object with left H-comodule structure $\lambda : A \to H \otimes_{\Bbbk} A$. If $g \in G(H)$ is a group-like element we can define a new H-biGalois object A^g on the same underlying algebra A with unchanged right comodule structure and a new left H-comodule structure given by $\lambda^g : A^g \to H \otimes_{\Bbbk} A^g$, $\lambda^g(a) = g^{-1}a_{-1}g \otimes a_0$ for all $a \in A$.

Theorem 2.7. [8, Lemma 5.7, Theorem 5.4] *Let $\Upsilon = (L_a, (g(a,b), f^{a,b}), \gamma)_{a,b\in\Gamma}$ be a collection where*

- *L_a is a (H,H)-biGalois object;*
- *$g(a,b) \in G(H)$;*
- *$f^{a,b} : (L_a\Box_H L_b)^{g(a,b)} \to L_{ab}$ are bicomodule algebra isomorphisms;*
- *$\gamma \in Z^3(G(H), \Bbbk^\times)$ normalized,*

such that for all $a, b, c \in \Gamma$:

$$L_e = H, \quad (g(e,a), f^{e,a}) = (e, \mathrm{id}_{L_a}) = (g(a,e), f^{a,e}); \tag{3}$$

$$\phi(L_a, g(b,c))g(a,bc) = g(a,b)g(ab,c); \tag{4}$$

$$f^{ab,c}(f^{a,b} \otimes \mathrm{id}_{L_c}) = f^{a,bc}(\mathrm{id}_{L_a} \otimes f^{b,c}). \tag{5}$$

Then $\mathrm{Comod}(H)(\Upsilon) := \oplus_{g\in\Gamma}\mathrm{Comod}(H)$ *as a structure of tensor category.*

Proof. We give an sketch of the proof. Let Υ be a collection as in the Theorem. For $V, W \in \mathrm{Comod}(H)$, $a, b \in \Gamma$, define

$$\begin{aligned}[V,a]\otimes[W,b] &:= [V\otimes(L_a\square_H W)\otimes \Bbbk_{g(a,b)}, ab],\\ [V,1]^* &:= [V^*,1],\\ [\Bbbk,a]^* &:= [\Bbbk_{g(a,a^{-1})}, a^{-1}].\end{aligned}$$

Using [8, Eq (5.8)], we obtain the pentagon diagram and therefore $\mathrm{Comod}(H)(\Upsilon)$ is a monoidal category. Since $\mathrm{Comod}(H)$ is finite tensor category, then $\mathrm{Comod}(H)(\Upsilon)$ is also finite tensor category. □

The following theorem gives us a second condition to decided if our extensions can be braided.

Theorem 2.8. *If* $\mathrm{Comod}(H)(\Upsilon)$ *is braided with braiding* c *then the following conditions have to hold*

(1) $L_a \simeq H$ *for all* $a \in \Gamma$,
(2) Γ *is Abelian,*
(3) Υ *comes from a data* $(g, f^{a,b}, \gamma)_{a,b\in\Gamma}$ *with*
- $g \in Z^2(\Gamma, G(H))$ *normalized,*
- $f^{a,b} : H^{g(a,b)} \to H$ *a bicomodule algebra isomorphism with* $f^{ab,c}f^{a,b} = f^{a,bc}f^{b,c}$,
- $\gamma \in Z^3(G(H), \Bbbk^\times)$ *normalized.*

Proof. (1) Take, for any $V \in \mathrm{Comod}(H)$, $c_{[V,e][\mathbf{1},a]} : [V,a] \to [L_a\square_H V, a]$, this defines a natural isomorphism $\bar{c}_a : \mathrm{id}_{\mathcal{C}} \to L_a\square_H -$ which is monoidal since c is a braiding. Then $L_a \simeq H$ as bicomodule algebras for all $a \in \Gamma$.

(2) Consider $c_{[\mathbf{1},a][\mathbf{1},b]} : [\Bbbk_{g(a,b)}, ab] \to [\Bbbk_{g(b,a)}, ba]$ then $ab = ba$ for all $a, b \in \Gamma$ and Γ is Abelian.

(3) Since L_a is trivial, then Equation (4) of Theorem 2.7 is equivalent to $g \in Z^2(\Gamma, G(H))$ and it is normalized by Equation (3) of Theorem 2.7. Moreover $f^{a,b} : H^{g(a,b)} \to H$ is a bicomodule algebra isomorphism that satisfies $f^{ab,c}f^{a,b} = f^{a,bc}f^{b,c}$ which is equivalent to Equation (5) of Theorem 2.7. □

Remark 2.9. By definition of bicomodule morphism, $f^{a,b} : H \to H$ has to be an algebra isomorphism such that $f^{a,b}(h)_1 \otimes f^{a,b}(h)_2 = g^{-1}h_1 g \otimes f^{a,b}(h_2)$ and $f^{a,b}(h)_1 \otimes f^{a,b}(h)_2 = f^{a,b}(h_1) \otimes h_2$, then $g^{-1}h_1 g \otimes f^{a,b}(h_2) = f^{a,b}(h_1) \otimes h_2$.

In the case when $H = \wedge V\#\Bbbk C_2$, as Example 2.4, using the previous Theorem we obtained eight tensor categories non-equivalent pairwise, [8, Section 6.3], named $\mathcal{C}_0(1,\mathrm{id},\pm1)$, $\mathcal{C}_0(u,\iota,\pm1)$, $\mathcal{D}(1,\mathrm{id},\pm1)$, $\mathcal{D}(u,\iota,\pm1)$.

In all cases, the underlying Abelian category is $\mathrm{Comod}(H)\oplus\mathrm{Comod}(H)$ and for $V,W,Z\in\mathrm{Comod}(H)$ they are defined in the following way:

- The tensor product, dual objects and associativity in $\mathcal{C}_0(1,\mathrm{id},\pm1)$ are given by

$$[V,e][W,g]=[V\otimes W,g], \qquad [V,u][W,g]=[V\otimes \mathbf{U}_0\square_H W,ug],$$
$$[V,e]^*=[V^*,e], \qquad [\mathbf{1},u]^*=[\Bbbk,u],$$

$\alpha_{[V,u],[W,u],[Z,u]}$ is not trivial, and $\mathbf{U}_0$ is certain BiGalois object, see [8, Section 4].

- The tensor product, dual objects and associativity in $\mathcal{C}_0(u,\iota,\pm1)$ are given by

$$[V,e][W,e]=[V\otimes W,1], \qquad [V,u][W,u]=[V\otimes \mathbf{U}_0\square_H W\otimes\Bbbk_u,e],$$
$$[V,e][W,u]=[V\otimes W,u], \qquad [V,u][W,e]=[V\otimes \mathbf{U}_0\square_H W,u],$$
$$[V,e]^*=[V^*,e], \qquad [\mathbf{1},u]^*=[\Bbbk_u,u],$$

$\alpha_{[V,u],[W,u],[Z,u]}$ is not trivial.

- The tensor product, dual objects and associativity in $\mathcal{D}(1,\mathrm{id},\pm1)$ are given by

$$[V,e][W,g]=[V\otimes W,g], \qquad [V,u][W,g]=[V\otimes W,ug],$$
$$[V,e]^*=[V^*,e], \qquad [\mathbf{1},u]^*=[\Bbbk,u],$$

$\alpha_{[V,u],[W,u],[Z,u]}=[\pm\,\mathrm{id}_{V\otimes W\otimes Z},u]$ and the others are trivial.

- The tensor product, dual objects and associativity in $\mathcal{D}(u,\iota,\pm1)$ are given by

$$[V,e][W,e]=[V\otimes W,e], \qquad [V,u][W,u]=[V\otimes W\otimes\Bbbk_u,e],$$
$$[V,e][W,u]=[V\otimes W,u], \qquad [V,u][W,e]=[V\otimes W,u],$$
$$[V,e]^*=[V^*,e], \qquad [\mathbf{1},u]^*=[\Bbbk_u,u],$$

$\alpha_{[V,u],[W,u],[Z,u]}=[\pm\,\mathrm{id}_{V\otimes W}\otimes\tau(\varepsilon\iota\rho_Z\otimes\mathrm{id}_{Z\otimes\Bbbk_u}),u]$, where $\iota: H^u\to H$ is the unique bicomodule algebra isomorphism which satisfies $\iota(u)=-u$ and $\iota(x)=-x$ for $x\in V$; and $\tau: X\otimes Y\to Y\otimes X$, $\tau(z\otimes k)=k\otimes z$ for all $X,Y\in\mathrm{Comod}(H)$, see [8, Remark 2.2].

Remark 2.10. By Lemma 2.8(1), we obtain that only the categories $\mathcal{D}(1,\mathrm{id},\pm1)$ and $\mathcal{D}(u,\iota,\pm1)$ could be braided, since the BiGalois objects have to be trivial.

By direct calculation on Equation (1), $\mathcal{D}(u,\iota,-1)$ is not braided with trivial braiding. So, in this case, we want to know if there exist another possible braidings.

3. Braided crossed product

Let Γ be an Abelian group. In [8], following the ideas developed in [3], we described all Γ-crossed product tensor categories which are extensions of $\mathrm{Comod}(H)$ for H a Hopf algebra in terms of certain Hopf-algebraic datum. Fix (H, r) a CQT-Hopf algebra. In the first Lemma of this Section, we do the same for the braiding of crossed products that are Γ-extensions of $\mathrm{Comod}(H)$.

Remark 3.1. If $v : H \to H$ is a left H-comodule morphism, since the coaction is the coproduct, v satisfies $v(x)_1 \otimes v(x)_2 = x_1 \otimes v(x_2)$, for all $x \in H$. In particular, v is not a coalgebra morphism and if $g \in G(H)$, $v(g) = g\varepsilon(v(g))$.

Lemma 3.2. *Fix a datum $(g, f^{a,b}, \gamma)_{a,b\in\Gamma}$, as in Lemma 2.8, and let $\mathcal{C}$ be the associated tensor category. Consider a pair $(v^a, w^a)_{a\in\Gamma}$ where $v^a, w^a : H \to H$ are left H-comodule algebra isomorphisms. Let $W^a = \varepsilon w^a$, $V^a = \varepsilon v^a$ and $F^{a,b} = \varepsilon f^{a,b}$. If for all $a, b, c \in \Gamma$ and $X \in \mathrm{Comod}(H)$ we have*

$$v^1 = w^1 = \mathrm{id}_H, \tag{6}$$

$$(g(a, b), f^{a,b}) = (g(b, a), f^{b,a}), \tag{7}$$

$$W^b(x_{-3})W^a(x_{-2})(W^{ab})^{-1}(x_{-1})x_0 = F^{a,b}(x_{-2})r(x_{-1} \otimes g(a, b))x_0, \quad x \in X, \tag{8}$$

$$V^b(x_{-3})V^a(x_{-2})(V^{ab})^{-1}(x_{-1})x_0 = r(x_{-2} \otimes g(a, b))F^{a,b}(x_{-1})x_0, \quad x \in X, \tag{9}$$

$$V^a(g(b, c)) = (\gamma_{a,b,c}\gamma_{b,c,a})^{-1}\gamma_{b,a,c}, \tag{10}$$

$$W^b(g(c, a)) = \gamma_{c,a,b}\gamma_{b,c,a}\gamma^{-1}_{c,b,a}; \tag{11}$$

then we obtain a braiding over $\mathcal{C}$ given by

$$\mathbf{c}_{[V,a],[W,b]} = c_{V,W}((V^a \otimes \mathrm{id})\rho_V \otimes (W^a \otimes \mathrm{id})\rho_W) \otimes \mathrm{id}, \quad V, W \in \mathrm{Comod}(H), a, b \in \Gamma.$$

All braidings over $\mathcal{C}$ come from a pair $(v^a, w^a)_{a\in\Gamma}$ which satisfies (7) to (11).

Proof. By [3, Definition 5.3], a datum $(g, f^{a,b}, \gamma)_{a,b\in\Gamma}$ has associated a braiding if there exist a triple $(\theta^a, \tau^a, t_{a,b})_{a,b\in G}$ where

- $\theta^a, \tau^a : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ are monoidal natural isomorphisms,
- for all $a, b \in G$, $t_{a,b} : (U_{a,b}, \sigma^{a,b}) \to (U_{b,a}, \sigma^{b,a})$ are isomorphisms in $\mathcal{Z}(\mathcal{C})$, where $\sigma^{a,b}_X = \tau(\varepsilon f^{a,b} \otimes \mathrm{id}_X)\rho_X$, for $(X, \rho_X) \in \mathrm{Comod}(H)$, and $U_{a,b} = \Bbbk_{g(a,b)}$,

such that for all $a, b, c \in \Gamma$ and $X \in \mathcal{C}$, the following conditions hold

$$\theta^1 = \tau^1 = \mathrm{id}, \quad \theta^a_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}} = \tau^a_{\mathbf{1}}, \quad t_{a,1} = t_{1,a} = \mathrm{id}_{\mathbf{1}}, \tag{12}$$

$$c_{U_{a,b},X}\sigma^{a,b}_X = ((\tau^{ab}_X)^{-1}\tau^a_X\tau^b_X) \otimes \mathrm{id}_{U_{a,b}}, \tag{13}$$

$$\sigma_X^{a,b} c_{U_{a,b},X} = \mathrm{id}_{U_{a,b}} \otimes ((\theta_X^{ab})^{-1} \theta_X^a \theta_X^b), \tag{14}$$

$$\gamma_{a,b,c}(\theta^a_{U_{b,c}} \otimes t_{bc,a})\gamma_{b,c,a} = (t_{b,c} \otimes \mathrm{id}_{U_{ba,c}})\gamma_{b,a,c}(t_{c,a} \otimes \mathrm{id}_{U_{b,ac}}), \tag{15}$$

$$\gamma^{-1}_{c,a,b}(\tau^b_{c,a} \otimes t_{b,ca})\gamma^{-1}_{b,c,a} = (t_{b,a} \otimes \mathrm{id}_{U_{c,ba}})\gamma^{-1}_{c,b,a}(t_{b,c} \otimes \mathrm{id}_{U_{bc,a}}). \tag{16}$$

By Lemma 2.1, each monoidal natural isomorphism of the identity functor comes from a left H-comodule algebra isomorphism, then $\theta_X^a := (\varepsilon v^a \otimes \mathrm{id})\rho_X$ and $\tau_X^a := (\varepsilon w^a \otimes \mathrm{id}_X)\rho_X$ for all $X \in \mathrm{Comod}(H)$. Since $U_{g(a,b)} = \Bbbk_{g(a,b)}$, we can take $t_{a,b} \in \Bbbk^*$.

Each $t_{a,b}$ is a left H-comodule isomorphism if and only if $g(a,b) \otimes t_{a,b}\,\mathrm{id}_\Bbbk = g(b,a) \otimes t_{a,b}\,\mathrm{id}_\Bbbk$ which gives $g(a,b) = g(b,a)$ for all $a, b \in \Gamma$. Moreover, each $t_{a,b}$ is a braided morphism if and only if $\sigma_X^{a,b} t_{a,b} = \sigma_X^{b,a} t_{a,b}$ for $a, b \in \Gamma$ and $X \in \mathrm{Comod}(H)$ if and only if $\sigma^{a,b} = \sigma^{b,a}$. Then $t_{a,b}$ is an isomorphism in $\mathcal{Z}(\mathrm{Comod}(H))$ if and only if Condition (7) holds.

Condition (12) is equivalent to $v^1 = w^1 = \mathrm{id}_H$ and $t_{a,1} = t_{1,a} = 1$, since $\theta^a_\Bbbk = \mathrm{id}_\Bbbk = \tau^a_\Bbbk$ is always true. Condition (13) is equivalent to

$$F^{a,b}(x_{-2})r(x_{-1} \otimes g(a,b))x_0 \otimes k = W^b(x_{-3})W^a(x_{-2})(W^{ab})^{-1}(x_{-1})x_0 \otimes k,$$

for $x \otimes k \in X \otimes \Bbbk_{g(a,b)}$, which is equivalent to Condition (8). In the same way, Condition (14) is equivalent to Condition (9). Condition (15) is equivalent to $\gamma_{a,b,c}V^a(g(b,c))t_{bc,a}\gamma_{b,c,a} = t_{b,c}\gamma_{b,a,c}t_{c,a}$ but if we take $c = 1$ then

$$1 = t_{b,a}, \text{ for } a, b \in \Gamma,$$

so, this Condition is equivalent to Condition (10), and Condition (16) is equivalent to Condition (11).

By [3, Theorem 5.4], this pair produces a braiding over $\mathcal{C}$ given by

$$\mathbf{c}_{[V,a],[W,b]} = [c_{V,W}(\theta_V^a \otimes \tau_W^a), ab], \quad \text{for all } V, W \in \mathrm{Comod}(H), a, b \in \Gamma, \tag{17}$$

and all braidings come from such a pair. □

Now, we focus our attention into the case $\Gamma = C_2$. By Lemma 2.8, a datum $\Upsilon' = (g, f, \gamma)$ with $g \in G(H)$ a group-like element, $f : H^g \to H$ a bicomodule algebra isomorphism and $\gamma \in \Bbbk^\times$, $\gamma^2 = 1$; generates a tensor category $\mathcal{C} = \mathrm{Comod}(H)(\Upsilon')$.

The following theorem gives us the third and last condition to decide if our categories are braidable.

Theorem 3.3. *The category* $\mathrm{Comod}(H)(\Upsilon')$ *is a braided* C_2*-extension if and only if, there exists a pair of isomorphisms of left* H*-comodule algebras* $v, w : H \to H$ *such that for all* $X \in \mathrm{Comod}\, H$ *and* $x \in X$

a. $\varepsilon(w(x_{-2})w^{-1}(x_{-1}))x_0 = x,$

b. $\varepsilon(w(x_{-2})w(x_{-1}))x_0 = \varepsilon f(x_{-2})r(x_{-1} \otimes g)x_0$,
c. $\varepsilon(v(x_{-2})v^{-1}(x_{-1}))x_0 = x$,
d. $\varepsilon(v(x_{-2})v(x_{-1}))x_0 = r(x_{-2} \otimes g)\varepsilon f(x_{-1})x_0$,
e. $\varepsilon(v(g)) = \gamma^{-1}$,
f. $\varepsilon(w(g)) = \gamma$.

Proof. Condition (7) is always true. Condition (8) is equivalent to $r(x_{-1} \otimes 1)x_0 = x$, and items a,b. Condition (9) is equivalent to $r(x_{-1} \otimes 1)x_0 = x$, and items c,d. Condition (10) is equivalent to item e. Condition (11) is equivalent to item f.

Regarding condition $r(x_{-1} \otimes 1)x_0 = x$, it is always true over a CoQuasi-triangular Hopf algebra. □

If $H = \wedge V \# \Bbbk C_2$, as Example 2.4, by [8, Proposition 4.10], the isomorphisms v and w are identities. Then if the extension is braided the only possible braiding is the trivial, see Equation (17), since the category $\operatorname{Comod} H$ has a braiding giving by the r-form. With this information, Conditions a-f are equivalent to

a'. $\varepsilon(x_{-2}x_{-1})x_0 = x$,
b'. $\varepsilon(x_{-2}x_{-1})x_0 = \varepsilon f(x_{-2})r(x_{-1} \otimes g)x_0$,
c'. $\varepsilon(x_{-2}x_{-1})x_0 = r(x_{-2} \otimes g)\varepsilon f(x_{-1})x_0$,
d'. $\varepsilon(g) = \gamma^{-1}$,
e'. $\varepsilon(g) = \gamma$.

Since g is a group-like element, d' and e' imply that $\gamma = 1$. Thus, the only categories that could be braided are $\mathcal{D}(1, \mathrm{id}, 1)$ and $\mathcal{D}(u, \iota, 1)$.

Corollary 3.4. *A C_2-extension over* $\operatorname{Comod}(\wedge V \# \Bbbk C_2)$ *is braided if and only if, for all comodule X, $r(f(x_{-1}) \otimes g)x_0 = x$, for all $x \in X$.*

Proof. Condition a' is always true over comodules. Since $x_1 y_1 r(x_2 \otimes y_2) = r(x_1 \otimes y_1)y_2 x_2$ for $x, y \in H$ we have

$$(x_{-1}g)r(x_{-2} \otimes g) \otimes x_0 = r(x_{-1} \otimes g)gx_{-2} \otimes x_0.$$

Applying $\varepsilon f \otimes \mathrm{id}_X$, we obtain $r(x_{-2} \otimes g)\varepsilon f(x_{-1})x_0 = \varepsilon f(x_{-2})r(x_{-1} \otimes g)x_0$. This implies that Conditions b' and c' are equivalent. Since

$$r(f(x) \otimes g) = r(f(x)_1 \otimes g)\varepsilon(g(f(x)_2)) = r(x_1 \otimes g)\varepsilon(f(x_2))$$

we have $r(f(x_{-1}) \otimes g)x_0 = \varepsilon f(x_{-2})r(x_{-1} \otimes g)x_0$, then Condition b' is equivalent to

$$r(f(x_{-1}) \otimes g)x_0 = x.$$

□

We are ready for our main result.

Theorem 3.5. *The categories $\mathcal{D}(1, \mathrm{id}, 1)$ and $\mathcal{D}(u, \iota, 1)$ are braided tensor categories. The remaining 6 categories found in* [8] *are non-braidable.*

Proof. By [3, Theorem 5.4], the only possible option for v and w is for there to be the identity. Then the categories $\mathcal{D}(1, \mathrm{id}, 1)$ and $\mathcal{D}(u, \iota, 1)$ have associated at most a single pair $(\mathrm{id}, \mathrm{id})$, which would give it a braided structure For the remaining six categories, we already know that they are non-braidable.

Since $\mathcal{D}(1, \mathrm{id}, 1)$ has trivial associativity and $\mathrm{Comod}(H)$ is braided then the braiding for $\mathcal{D}(1, \mathrm{id}, 1)$ is

$$\mathbf{c}_{[V,a],[W,b]} = [c_{V,W}, ab], \quad \text{for all } V, W \in \mathrm{Comod}(H), a, b \in C_2. \tag{18}$$

Over $\mathcal{D}(u, \iota, 1)$ it is enough to check Equations (1) and (2) where the associativity is not trivial. Since $(f \otimes \mathrm{id})(\mathrm{id} \otimes g) = (\mathrm{id} \otimes g)(f \otimes \mathrm{id})$ for any f, g morphisms in the category, also the braiding given in (18) also satisfies the desired Equations. □

Corollary 3.6. *For $X \in \mathrm{Comod}\,(\wedge V \# \Bbbk C_2)$, $r(\iota(x_{-1}) \otimes g)x_0 = x$, for all $x \in X$.*

Remark 3.7. Since $\mathrm{Comod}(H)$ is not symmetric, then these two categories are not symmetric either.

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