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NONFULL-RANK FACTORIZATIONS OF ELEMENTARY 3-GROUPS

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Abstract. If a finite elementary 3-groups is a direct product of two of its subsets that contains the identity element such that one of the factors has three or nine elements then one of the factors does not span the whole group. This is an extension of two earlier results.

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1. Introduction

Let G be a finite abelian group. We will use multiplicative notation in connection with abelian groups. The identity element of G will be denoted by e. Let $A_1, \ldots, A_n$ be subsets of G. The product $A_1 \cdots A_n$ by definition is equal to

$$\{a_1 \cdots a_n \ : \ a_1 \in A_1, \ldots, a_n \in A_n\}.$$

If the elements on the list

$$a_1 \cdots a_n, \ a_1 \in A_1, \ldots, a_n \in A_n$$

are distinct, then we say that the product $A_1 \cdots A_n$ is direct. If the product $A_1 \cdots A_n$ is direct and is equal to G, then we say that the equation $G = A_1 \cdots A_n$ is a factorization of G.

If $e \in A$, then we say that the subset A is normalized. A factorization is called normalized if each of its factors is a normalized subset. Let $\langle A \rangle$ be the smallest subgroup of G that contains A. In other words let $\langle A \rangle$ be the span of A in G. A normalized subset A of G is called a full-rank subset if $\langle A \rangle = G$. A subset A of G is called periodic if there is an element $g \in G \setminus \{e\}$ such that $Ag = A$.

If the finite abelian group G is a direct product of its cyclic subgroups of orders $t_1, \ldots, t_n$, then we say that G is of type $(t_1, \ldots, t_n)$. Let p be a prime. If $t_1 = \cdots =$

$t_n = p$, then G is called an elementary p-group. The number n is called the rank of G.

Let G be an elementary 3-group of rank k and let $G = AB$ be a normalized factorization of G. It was shown that if $k = 4$ or $k = 5$, then one of the factors is not a full-rank subset of G in [6] and [1] respectively. In these cases one of the factors must have at most 9 elements. One might wonder if these results hold in general for each elementary 3-groups regardless of the rank of the group with the extra condition that $|A|$ is 3 or 9. The main result of this note gives an answer to this question in the affirmative. An example, exhibited in [5], shows that if $|A| = |B| = 27$ then, there is a normalized factorization $G = AB$ of an elementary 3-group of rank 6 such that A and B are full-rank subsets of G.

2. Lemmas

Let $G = AB$ be a normalized factorization of the finite abelian group G. Suppose that the factor B is periodic with period d. Set $H = \langle d \rangle$. There is a subset D of B such that the product DH is direct and is equal to B. From the factorization $G = AB = ADH$ by considering the factor group G/H we get a factorization

$$G/H = (AH)/H \cdot (DH)/H. \tag{1}$$

Lemma 1. *If $G = AB$ is a full rank factorization then so is (1).*

Proof. Assume that $G = AB$ is a full-rank factorization of G. First we will show that $(AH)/H$ spans G/H. We would like to verify that for each element gH of G/H there are elements $a_1H, \ldots, a_sH$ of $(AH)/H$ and integers $\alpha(1), \ldots, \alpha(s)$, such that $gH = (a_1H)^{\alpha(1)} \cdots (a_sH)^{\alpha(s)}$. Since $\langle A \rangle = G$, for each $g \in G$ there are elements $a_1, \ldots, a_s$ of A and integers $\alpha(1), \ldots, \alpha(s)$, such that $g = a_1^{\alpha(1)} \cdots a_s^{\alpha(s)}$. So

$$\begin{aligned} gH &= a_1^{\alpha(1)} \cdots a_s^{\alpha(s)} H \\ &= (a_1^{\alpha(1)} H) \cdots (a_s^{\alpha(s)} H) \\ &= (a_1H)^{\alpha(1)} \cdots (a_sH)^{\alpha(s)}. \end{aligned}$$

This means that $(AH)/H$ spans G/H.

Next we will show that $(DH)/H$ spans G/H. We would like to establish that for each element gH of G/H there are elements $d_1H, \ldots, d_sH$ of $(DH)/H$ and integers $\beta(1), \ldots, \beta(s)$ such that $gH = (d_1H)^{\beta(1)} \cdots (d_sH)^{\beta(s)}$. Since $B = DH$ spans G, for each $g \in G$, there are elements $d_1, \ldots, d_s \in D$, $h_1, \ldots, h_s \in H$ and integers

$\beta(1), \ldots, \beta(s)$ such that $g = (d_1h_1)^{\beta(1)} \cdots (d_sh_s)^{\beta(s)}$. Therefore

$$\begin{aligned} gH &= (d_1h_1)^{\beta(1)} \cdots (d_sh_s)^{\beta(s)} H \\ &= [(d_1h_1)^{\beta(1)} H] \cdots [(d_sh_s)^{\beta(s)} H] \\ &= (d_1^{\beta(1)} H) \cdots (d_s^{\beta(s)} H) \\ &= (d_1H)^{\beta(1)} \cdots (d_sH)^{\beta(s)}. \end{aligned}$$

This means that $(DH)/H$ spans G/H. □

Lemma 2. *Let G be a finite abelian 3-group. Let $G = AB$ be a normalized factorization of G, where $|A| = 3$. Then either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$.*

Proof. Let G be a finite abelian 3-group and let $G = AB$ be a normalized factorization of G with $|A| = 3$. In order to prove the lemma assume on the contrary that $\langle A \rangle = \langle B \rangle = G$. Among the counter-examples we choose one for which $|G|$ is minimal.

Since $G = AB$ is a factorization of G and $|A| = 3$, it follows that $|G| = |A||B| \geq |A| = 3$. If $|G| = 3$, then $|B| = 1$ and so $B = \{e\}$. Now $\langle B \rangle \neq G$. Thus in the remaining part of the proof we may assume that $|G| \geq 9$.

Let $A = \{e, a, b\}$. Suppose first that $|a| \geq 9$. By Lemma 3 of [4], in the factorization $G = AB$ the factor A can be replaced by $A' = \{e, a, a^2\}$ to get the factorization $G = A'B$. The factorization $G = A'B$ is equivalent to that the sets

$$eB,\ aB,\ a^2B \tag{2}$$

form a partition of G. Multiplying the factorization $G = A'B$ by a we get the factorization $G = (A'a)B$. This factorization is equivalent to that the sets

$$aB,\ a^2B,\ a^3B \tag{3}$$

form a partition of G. Comparing the partitions (2) and (3) gives that $B = a^3B$. This means that a^3 is a period of B. Set $H = \langle a^3 \rangle$. There is a subset D of B such that the product DH is direct and is equal to B. From the factorization $G = AB = ADH$ by considering the factor group G/H we get the factorization (1). The order of G/H is smaller than the order of G. By the minimality of the counter-example, either $(AH)/H$ or $(DH)/H$ does not span G/H. On the other hand by Lemma 1, $(AH)/H$ and $(DH)/H$ are full-rank subsets of G/H. This contradiction shows that $|a| = 3$. By symmetry we may assume that $|b| = 3$ also holds.

If $b = a^2$, then A is a subgroup of G. Namely, $A = \langle a \rangle$. Now $\langle A \rangle$ has three elements and so it cannot be equal to G which has at least nine elements. Thus we may assume that $b \neq a^2$. There is an element $d \in G \setminus \{e\}$ such that $b = a^2 d$. By Lemma 3 of [4], in the factorization $G = AB$ the factor A can be replaced by $A' = \{e, a, a^2\}$ to get the factorization $G = A'B$. The factorization $G = AB$ is equivalent to that the sets

$$eB,\ aB,\ a^2 dB \tag{4}$$

form a partition of G. The factorization $G = A'B$ is equivalent to that the sets

$$eB,\ aB,\ a^2 B \tag{5}$$

form a partition of G. Comparing the partitions (4) and (5) gives that $a^2 B = a^2 dB$ and so $B = Bd$. This means that d is a period of B. Let $H = \langle d \rangle$. There is a subset D of B such that the product DH is direct and is equal to B. From the factorization $G = AB = ADH$ by considering the factor group G/H we have the factorization (1). Using the minimality of the counter-example this leads to a contradiction.

This completes the proof. □

The anonymous referee of the paper has pointed out that Lemma 2 holds for each finite abelian groups not only for 3-groups as stated. By Lemma 1 of [3] if $|A| = 3$, then A or B must be periodic without the assumption that G is a 3-group and the rest of the proof follows roughly as given above.

3. Computations

Suppose we are given a finite abelian group G and a normalized subset A of G such that $|A|$ divides $|G|$. The problem is to find each normalized subset B of G for which $G = AB$ is a factorization of G. Here B is a factor complementing A and so we can call this problem the complementer factor problem. The fact that $G = AB$ is a factorization of G is equivalent to that the sets Ab, $b \in B$ form a partition of G. This suggests the following solution to the complementer factor problem. Let us form the family of the subsets Ag, $g \in G$. Then look for members of this family that form a partition of G. One can generate all possible partitions of G in a systematic manner. We will refer to this way of solution as the exact covering method. The method first was used in [2] to study factorings of elementary 2-groups.

Lemma 3. *Let G be an elementary 3-group of rank 6 and let $G = AB$ be a normalized factorization of G, where $|A| = 9$, $|B| = 81$. Then either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$.*

Proof. Let G be an elementary 3-group of rank 6 and let $G = AB$ be a normalized factorization of G such that $|A| = 9$, $|B| = 81$. In order to prove the claim of the lemma assume on the contrary that $\langle A\rangle = \langle B\rangle = G$. Since A spans G we can choose a basis $x_1, \dots, x_6$ of G such that

$$A = \{e, x_1, \dots, x_6, a, b\},$$

where $a, b \in G$. Set $d = x_1 \cdots x_6 ab$. We claim that $d = e$. In order to prove the claim assume on the contrary that $d \neq e$. By Lemma 1 of [1], d is an element of the Corrádi subgroup of A. If $d \neq e$, then by Lemma 2 of [1], d is a period of B. Set $H = \langle d\rangle$. There is a subset D of B such that the product DH is direct and is equal to B. From the factorization $G = AB = ADH$ by considering the factor group G/H we get the factorization (1). The rank of G/H is 5. By Theorem 1 of [1], either $(AH)/H$ or $(DH)/H$ does not span G/H. By Lemma 1, this is a contradiction. Thus $d = e$ and consequently $b = (x_1 \cdots x_6 a)^{-1}$. In other words one can compute b if one knows a.

Let

$$a = x_1^{\alpha(1)} \cdots x_6^{\alpha(6)},\ b = x_1^{\beta(1)} \cdots x_6^{\beta(6)},$$

$$w(a) = \alpha(1) + \cdots + \alpha(6),\ w(b) = \beta(1) + \cdots + \beta(6),$$

where $0 \leq \alpha(i), \beta(i) \leq 2$. We call $w(a)$, $w(b)$ the weight of the elements a and b respectively. We may assume that $w(a) \leq w(b)$ since this is only a matter of interchanging a and b. We may assume that $\alpha(1) \leq \cdots \leq \alpha(6)$, since this is only a matter of permuting the elements $x_1, \dots, x_6$. There are

$$\binom{3+6-1}{6} = \binom{8}{6} = \binom{8}{2} = 28$$

choices for $\alpha(1), \dots, \alpha(6)$. However some of these choices can be discarded.

In the $\alpha(1) = \cdots = \alpha(6) = 0$ case $a = e$ would appear in A twice. In the $\alpha(1) = \cdots = \alpha(5) = 0$, $\alpha(6) = 1$ case $a = x_6$ would appear in A twice. In the $\alpha(1) = \cdots = \alpha(6) = 2$ case $b = e$ would appear in A twice. In the $\alpha(1) = 1$, $\alpha(2) = \cdots = \alpha(6) = 2$ case $b = x_1$ would appear in A twice. We also have to discard the cases when $w(a) > w(b)$. We are left with 13 cases listed in Table 1. The exact cover algorithm provided 27 336 possible complementer factors B to A. An inspection revealed that none of these is a full-rank subset of G. $\square$

Lemma 4. *Let G be an elementary 3-group of rank 7 and let $G = AB$ be a normalized factorization of G, where $|A| = 9$, $|B| = 243$. Then either $\langle A\rangle \neq G$ or $\langle B\rangle \neq G$.*

TABLE 1. Choices for a and b and the number of solutions for B.

a	b	number of B's
011112	211110	2760
011111	211111	2760
001122	221100	864
001112	221110	864
001111	221111	72
000222	222000	72
000122	222100	72
000112	222110	2760
000111	222111	72
000022	222200	5760
000012	222210	2760
000011	222211	2760
000002	222220	5760

Proof. Let G be an elementary 3-group of rank 7 and let $G = AB$ be a normalized factorization of G such that $|A| = 9$, $|B| = 243$. In order to prove the claim of the lemma assume on the contrary that $\langle A \rangle = \langle B \rangle = G$. Since A spans G we can choose a basis $x_1, \ldots, x_7$ of G such that $A = \{e, x_1, \ldots, x_7, a\}$, where $a \in G$. Set $d = x_1 \cdots x_7 a$. In the way we have seen in the proof of Lemma 3 we can establish that $d = e$. Consequently $a = (x_1 \cdots x_7)^{-1}$. In other words a can be computed. The exact cover algorithm gives 158 760 possible solutions for B. An inspections shows that none of these is a full-rank subset of G. $\square$

4. The result

We are ready to prove the main result of the paper.

Theorem 1. *Let G be a finite elementary 3-group. Let $G = AB$ be a normalized factorization of G, where $3 \leq |A| \leq 9$. Then either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$.*

Proof. Let G be a finite elementary 3-group and let $G = AB$ be a normalized factorization of G, where $3 \leq |A| \leq 9$. In order to prove the theorem assume on the contrary that $\langle A \rangle = \langle B \rangle = G$. Among the counter-examples we choose one with the smallest possible $|G|$.

If $|A| = 3$, then by Lemma 2, either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$. Thus for the remaining part of the proof we may assume that $|A| = 9$.

Let k be the rank of G. Since $|A| = 9$, it follows that $k \geq 2$. If $k = 2$, then $|B| = 1$ and so $B = \{e\}$. Now $\langle B \rangle \neq G$. Hence we may assume that $k \geq 3$. If $k = 3$, then $|B| = 3$ and by Lemma 2, either $\langle A \rangle \neq G$ or $\langle B \rangle \neq G$. Thus we may assume that $k \geq 4$. The $k = 4$ case is settled in [6]. The $k = 5$ case is settled in [1]. The $k = 6$ case is settled in Lemma 3. The $k = 7$ case is settled in Lemma 4.

Suppose that $k = 8$. Now since $\langle A \rangle = G$ we can choose a basis $x_1, \dots, x_8$ of G such that $A = \{e, x_1, \dots, x_8\}$. Set $d = x_1 \cdots x_8$, $H = \langle d \rangle$. Clearly $d \neq e$. By Lemma 1 of [1], d is an element of the Corrádi subgroup of A and by Lemma 2 of [1], B is periodic with period d. There is a subset D of B such that the product DH is direct and is equal to B. From the factorization $G = AB = ADH$ we get the factorization (1). By Lemma 1 we get that $(AH)/H$ and $(DH)/H$ both span G/H. On the other hand the rank of G/H is 7 and so by Lemma 4, either $(AH)/H$ or $(DH)/H$ does not span G/H. This contradiction gives that $k \geq 9$. However in this case A does not have enough elements to span G. This completes the proof. $\square$

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