International Electronic Journal of Algebra
Volume 32 (2022) 157-175
DOI: 10.24330/ieja.1077596

# CATEGORY OF $n$-FCP-GR-PROJECTIVE MODULES WITH RESPECT TO SPECIAL COPRESENTED GRADED MODULES

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Received: 13 August 2021; Revised: 19 October 2021; Accepted: 8 November 2021
Communicated by Tuğçe Pekacar Çalcı

Abstract. Let $R$ be a ring graded by a group $G$ and $n \geq 1$ an integer. We introduce the notion of $n$-FCP-gr-projective $R$-modules and by using of special finitely copresented graded modules, we investigate that (1) there exist some equivalent characterizations of $n$-FCP-gr-projective modules and graded right modules of $n$-FCP-gr-projective dimension at most $k$ over $n$-gr-cocoherent rings, (2) $R$ is right $n$-gr-cocoherent if and only if for every short exact sequence $0 \to A \to B \to C \to 0$ of graded right $R$-modules, where $B$ and $C$ are $n$-FCP-gr-projective, it follows that $A$ is $n$-FCP-gr-projective if and only if ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory, where $gr$-$\mathcal{FCP}_n$ denotes the class of $n$-FCP-gr-projective right modules. Then we examine some of the conditions equivalent to that each right $R$-module is $n$-FCP-gr-projective.

**Mathematics Subject Classification (2020)**: 16D80, 16E05, 16E30, 16E65, 16P70
**Keywords**: $n$-gr-Cocoherent ring, special gr-copresented module, $n$-FCP-gr-projective module

## 1. Introduction

In 1969, Jans in [11] gave a definition of finitely cogenerated modules as a dual notion of finitely generated modules when he introduced co-Noetherian rings as a dual notion of Noetherian rings. A right $R$-module $M$ is said to be *finitely cogenerated* if for every family $\{M_i\}_{i\in I}$ of submodules of $M$ with $\cap_{i\in I} M_i = 0$, there is a finite subset $J \subset I$ such that $\cap_{i\in J} M_i = 0$. In 1994, Costa in [7] introduced the notion of $n$-coherent rings for a nonnegative integer $n$. A left $R$-module $M$ is said to be *$n$-presented* if it has a finite $n$-presentation, that is, there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ with each $F_i$ finitely generated free $R$-module, and a ring $R$ is called left *$n$-coherent* if every $n$-presented left $R$-module is $(n+1)$-presented, for more details see [6,8].

As we know, cocoherent rings as a dual notion of coherent rings have been characterized in various ways, and many nice properties were obtained for such rings in [12,19,22]. In 1999, Weimin Xue in [24] via finitely cogenerated modules introduced $n$-copresented modules and $n$-cocoherent rings as a dual notion of $n$-presented modules and $n$-coherent rings, respectively. A right $R$-module $M$ is said to be *$n$-copresented* if there is an exact sequence $0 \to M \to E^0 \to E^1 \to \cdots \to E^n$ of right $R$-modules, where each $E^i$ is finitely cogenerated injective. A ring $R$ is called right *$n$-cocoherent* if every $n$-copresented $R$-module is $(n+1)$-copresented. $n$-cocoherent rings have been studied by several authors (see, for example [1,2,5,21]).

The homological theory of graded rings and modules is a classical topic in algebra, because of its applications in algebraic geometry, see ([14,15,16]). Several authors have investegated the graded aspect of some notions in relative homological algerbra. For example, Asensio et al. in [4] introduced the notions of FP-gr-injective modules, then Yang and Liu in [18] investigated homological behavior of the FP-gr-injective modules on gr-coherent rings. Recently in 2018, Zhao, Gao and Huang [20] gave a definition of $n$-presented graded modules and $n$-gr-coherent rings and also, by using of $n$-presented graded modules, they introduced the concept of $n$-FP-gr-injective and $n$-gr-flat modules, and then examined the homological behavior of these modules over $n$-gr-coherent rings. In case $n = 1$, see [13,18].

The aim of this paper is to introduce and study $n$-copresented graded right modules, $n$-gr-cocoherent right rings and $n$-FCP-gr-projective right modules as a dual notion of $n$-presented graded left modules, $n$-gr-coherent left rings and $n$-FP-gr-injective left modules, respectively. Then, we study the relative homological theory of these modules and also, the properties of special finitely copresented graded modules, defined via finitely cogenerated gr-injective resolutions of $n$-copresented graded modules, play a crucial role.

This paper is organized in three sections as follows:

In Section 2, some fundamental concepts and some preliminary results are stated.

In Section 3, we first introduce the notions of $n$-copresented graded right modules, special gr-cogenerated and special gr-copresented modules with respect to any $n$-copresented graded right module, and also $n$-cocoherent graded right rings (or, right $n$-gr-cocoherent rings). Then via $n$-copresented graded modules, we give a concept of $n$-FCP-gr-projective right modules and investigate some characterizations of these modules. In this section, examples are given in order to show that $m$-FCP-gr-projectivity does not imply $n$-FCP-gr-projectivity for any $m > n$.

In Section 4, we prove that there exist some equivalent characterizations of graded right modules of $n$-FCP-gr-projective dimension at most $k$ on right $n$-gr-coherent rings. We obtain some equivalent characterizations of right $n$-gr-cocoherent rings in terms of $n$-FCP-gr-projective right modules on the short exact sequences. For example, $R$ is a right $n$-gr-cocoherent ring if and only if for every exact sequence

$0 \to A \to B \to C \to 0$ of graded right $R$-modules, where $B$ and $C$ are $n$-FCP-gr-projective, it follows that $A$ is $n$-FCP-gr-projective if and only if ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory, where $gr$-$\mathcal{FCP}_n$ denotes the class of $n$-FCP-gr-projective right modules. Moreover, if gr-$\mathcal{FI}_n$ is a class of $n$-FP-gr-injective left modules, then $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$. Hence, we prove that every graded right $R$-module is $n$-FCP-gr-projective if and only if ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is a perfect hereditary cotorsion theory and $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$ has an $n$-FCP-gr-projective cover with the unique mapping property if and only if $R$ is right $n$-gr-cocoherent and $N$ is $n$-FCP-gr-projective if and only if $N(\sigma)$ is gr-injective for any $\sigma \in G$.

## 2. Preliminaries

Throughout this paper, all rings considered are associative with identity element and the $R$-modules are unital. By Mod-$R$ and $R$-Mod we will denote the Grothendieck category of all right $R$-modules and left $R$-modules, respectively.

Let $G$ be a multiplicative group with neutral element $e$. A graded ring $R$ is a ring with identity 1 together with a direct decomposition $R = \bigoplus_{\sigma \in G} R_\sigma$ (as additive subgroups) such that $R_\sigma R_\tau \subseteq R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Thus, $Re$ is a subring of $R$, $1 \in Re$ and $R_\sigma$ is an $Re$-bimodule for every $\sigma \in G$. A *graded* right (resp. left) $R$-module is a right (resp. left) $R$-module $M$ endowed with an internal direct sum decomposition $M = \bigoplus_{\sigma \in G} M_\sigma$, where each $M_\sigma$ is a subgroup of the additive group of $M$ such that $M_\tau R_\sigma \subseteq M_{\tau\sigma}$ for all $\sigma, \tau \in G$. For any graded right $R$-modules $M$ and $N$, set $\mathrm{Hom}_{\mathrm{gr}-R}(M, N) := \{f : M \to N \mid f$ is $R$-linear and $f(M_\sigma) \subseteq N_\sigma$ for any $\sigma \in G\}$, which is the group of all morphisms from $M$ to $N$ in the category gr-$R$ of all graded right $R$-modules ($R$-gr will denote the category of all graded left $R$-modules). It is well known that gr-$R$ is a Grothendieck category. An $R$-linear map $f : M \to N$ is said to be a *graded morphism of degree* $\tau$ with $\tau \in G$ if $f(M_\sigma) \subseteq M_{\sigma\tau}$ for all $\sigma \in G$. Graded morphisms of degree $\sigma$ build an additive subgroup $\mathrm{HOM}_R(M, N)_\sigma$ of $\mathrm{Hom}_R(M, N)$. Then $\mathrm{HOM}_R(M, N) = \bigoplus_{\sigma \in G} \mathrm{HOM}_R(M, N)_\sigma$ is a graded abelian group of type $G$. We will denote by $\mathrm{Ext}^i_{\mathrm{gr}-R}$ and $\mathrm{EXT}^i_R$ the right derived functors of $\mathrm{Hom}_{\mathrm{gr}-R}$ and $\mathrm{HOM}_R$, respectively. Given a graded right $R$-module $M$, the *graded character module* of $M$ is defined as $M^* := \mathrm{HOM}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Q}$ is the rational numbers field and $\mathbb{Z}$ is the integers ring. It is easy to see that $M^* = \bigoplus_{\sigma \in G} \mathrm{Hom}_{\mathbb{Z}}(M_{\sigma^{-1}}, \mathbb{Q}/\mathbb{Z})$.

Let $M$ be a graded right $R$-module and $N$ a graded left $R$-module. The abelian group $M \bigotimes_R N$ may be graded by putting $(M \bigotimes_R N)_\sigma$ with $\sigma \in G$ to be the additive subgroup generated by elements $x \bigotimes y$ with $x \in M_\alpha$ and $y \in N_\beta$ such that $\alpha\beta = \sigma$. The object of $\mathbb{Z}$-gr thus defined will be called the *graded tensor product* of

$M$ and $N$. If $M$ is a graded right $R$-module and $\sigma \in G$, then $M(\sigma)$ is the graded right $R$-module obtained by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for any $\tau \in G$. The graded module $M(\sigma)$ is called the *$\sigma$-suspension* of $M$. We may regard the $\sigma$-suspension as an isomorphism of categories $T_\sigma : \text{gr-}R \to \text{gr-}R$, given on objects as $T_\sigma(M) = M(\sigma)$ for any $M \in \text{gr-}R$.

The forgetful functor $U : \text{gr-}R \to \text{Mod-}R$ associates to $M$ the underlying ungraded $R$-module. This functor has a right adjoint $F$ which associated to $M \in \text{Mod-}R$ the graded $R$-module $F(M) = \bigoplus_{\sigma \in G}({}^\sigma M)$, where each ${}^\sigma M$ is a copy of $M$ written $\{{}^\sigma x : x \in M\}$ with $R$-module structure defined by $r {*}^\tau x ={}^{\sigma\tau} (rx)$ for each $r \in R_\sigma$. If $f : M \to N$ is $R$-linear, then $F(f) : F(M) \to F(N)$ is a graded morphism given by $F(f)({}^\sigma x) ={}^\sigma f(x)$.

The injective (resp. projective) objects of gr-$R$ will be called *injective graded right* (resp. *projective graded right*) modules because $M$ is gr-injective (resp. gr-projective) in gr-$R$ if and only if it is a injective (resp. projective) graded right module. Similarly, it is defined for the graded left modules in $R$-gr. Let $n \geq 0$ be an integer. Then, a graded left $R$-module $F$ is called *$n$-presented* [20] if there exists an exact sequence $P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to F \to 0$ in $R$-gr with each $P_i$ is finitely generated free $R$-module. A graded ring $R$ is called left *$n$-gr-coherent* if each $n$-presented graded left $R$-module is $(n+1)$-presented. A graded left module $M$ is called *$n$-FP-gr-injective* if $\text{EXT}^n_R(F, M) = 0$ for any $n$-presented graded left $R$-module $F$. A graded left module $M$ is called *$n$-gr-flat* if $\text{Tor}^R_n(M, F) = 0$ for any $n$-presented graded left $R$-module $F$, see [20].

For a graded ring $R$, let $\mathcal{X}$ be a class of graded right $R$-modules and $M$ a graded left $R$-module. Following [3,20], we say that a graded morphism $f : X \to M$ is an $\mathcal{X}$-precover of $M$ if $X \in \mathcal{X}$ and $\text{HOM}_R(X', X) \to \text{HOM}_R(X', M) \to 0$ is exact for all $X' \in \mathcal{X}$. Moreover, if whenever a graded morphism $g : X \to X$ such that $fg = f$ is an automorphism of $X$, then $f : X \to M$ is called an $\mathcal{X}$-cover of $M$. The class $\mathcal{X}$ is called (pre)covering if each object in *gr-$R$* has an $\mathcal{X}$-(pre)cover. $\mathcal{X}$-envelope and $\mathcal{X}$-preenvelope are defined dually. Recall that a $\mathcal{X}$-cover $\phi : M \to N$ has the unique mapping property if for any homomorphism $f : A \to N$ with $A \in \mathcal{X}$, there exists a unique $g : A \to M$ such that $\phi g = f$.

## 3. $n$-FCP-gr-projective modules

In this section, we first introduce the special gr-copresented and special gr-cogenerated modules via $n$-copresented graded right modules. Then by using of these modules, some properties of $n$-FCP-gr-projective modules are discussed.

**Definition 3.1.** Let $n \geq 0$ be an integer. Then, a graded right $R$-module $U$ is called $n$-copresented if there exists an exact sequence

$$0 \longrightarrow U \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \cdots \longrightarrow E^{n-1} \longrightarrow E^n$$

in gr-$R$ with each $E^i$ is finitely cogenerated injective. Set $K^{n-1} = \mathrm{Coker}(E^{n-2} \to E^{n-1})$ and $K^n = \mathrm{Coker}(E^{n-1} \to E^n)$. Then, we shall say the sequence

$$\Delta^{'} : 0 \longrightarrow K^{n-1} \longrightarrow E^n \longrightarrow K^n \longrightarrow 0$$

in gr-$R$ is a special short exact sequence. Moreover, we call the objects $K^n$ and $K^{n-1}$ special finitely cogenerated and special finitely copresented graded (special gr-cogenerated and special gr-copresented for short) right $R$-modules, respectively. Then, it follows that $\mathrm{EXT}^1_R(M, K^{n-1}) \cong \mathrm{EXT}^n_R(M, U)$ for any graded right $R$-module $M$. Also, a short exact sequence $0 \to A \to B \to C \to 0$ in gr-$R$ is called special gr-copure, if for every special gr-copresented $K^{n-1}$, there exists the following exact sequence:

$$0 \longrightarrow \mathrm{HOM}_R(C, K^{n-1}) \longrightarrow \mathrm{HOM}_R(B, K^{n-1}) \longrightarrow \mathrm{HOM}_R(A, K^{n-1}) \longrightarrow 0,$$

where $A$ is said to be special gr-copure in $B$. A graded ring $R$ is called right $n$-gr-cocoherent if each $n$-copresented graded right $R$-module is $(n+1)$-copresented.

The following lemma is the graded version of [23, Theorem 3].

**Lemma 3.2.** *Let $0 \to A \to B \to C \to 0$ in gr-$R$ be a short exact sequence. Then the following statements hold for any $n \geq 0$.*

(1) *If $A$ and $C$ are $n$-copresented, then so is $B$.*
(2) *If $C$ is $n$-copresented and $B$ is $(n+1)$-copresented, then $A$ is $(n+1)$-copresented.*
(3) *If $A$ and $B$ are $(n+1)$-copresented, then $C$ is $n$-copresented.*

**Definition 3.3.** Let $n \geq 1$ be an integer. A graded right $R$-module $M$ is called $n$-FCP-gr- projective if $\mathrm{EXT}^n_R(M, U) = 0$ for any $n$-copresented graded right $R$-module $U$.

Notice that for ungraded, a right $R$-module $M$ is called $n$-FCP-projective if $\mathrm{Ext}^n_R(M, U) = 0$ for any $n$-copresented right $R$-module $U$, and in case $n = 1$, $M$ is called FCP-projective, see [21].

**Remark 3.4.** (1) Every $m$-copresented graded right $R$-module is $n$-copresented for any $m \geq n$.
(2) If $n = 1$, then every 1-FCP-gr- projective is FCP-gr- projective.

(3) If $M$ is $n$-FCP-gr- projective and $U$ is $(n+1)$-copresented graded right $R$-module, then there exists an exact sequence $0 \to U \to E^0 \to C \to 0$ in gr-$R$, where $E^0$ is finitely cogenerated injective and $C$ is $n$-copresented by Lemma 3.2. So we deduce that the sequence $0 \to \mathrm{EXT}_R^n(M, C) \to \mathrm{EXT}_R^{n+1}(M, U) \to 0$ is exact. But, $\mathrm{EXT}_R^n(M, C) = 0$ since $M$ is $n$-FCP-gr- projective and $C$ is $n$-copresented. Hence, $\mathrm{EXT}_R^{n+1}(M, U) = 0$ and consequently $M$ is $(n+1)$-FCP-gr- projective. Therefore,

$$gr\text{-}\mathcal{FCP}_1 \subseteq gr\text{-}\mathcal{FCP}_2 \subseteq \cdots \subseteq gr\text{-}\mathcal{FCP}_n \subseteq gr\text{-}\mathcal{FCP}_{n+1} \subseteq \cdots .$$

In general, $m$-FCP-gr-projective right $R$-modules need not be $n$-FCP-gr-projective whenever $m > n$, see Example 3.6(1).

(4) Every gr-projective right $R$-module is $n$-FCP-gr-projective.

**Definition 3.5.** (1) The $n$-FCP-gr-projective dimension of a graded right module $M$ is defined by
$n$.FCP-gr-pd$M = inf\{\mathrm{k} : \mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$ for every special gr-copresented $K^{n-1}\}$.
(2) The $n$-FCP-gr-projective global dimension of a graded ring $R$ is defined by
$r.n$.FCP-gr-dim$R = sup\{$n.FCP-gr-pd$M \mid M$ is a graded right $R$-module$\}$.

A graded ring $R$ is called a right gr-$V$-ring if every simple graded right $R$-module is gr-injective. A graded ring $R$ is called right gr-hereditary if every submodule of a projective graded right $R$-module is projective. A graded ring $R$ is called right gr-cosemihereditary if every submodule of a projective graded right $R$-module is FCP-gr-projective, see the graded version of [21, Definition 3.6]. It is clear that gr-hereditary rings are gr-cosemihereditary.

**Example 3.6.** Let $R$ be a gr-cosemihereditary ring but not gr-$V$-ring, for example gr-hereditary ring $R = k[X]$ where $k$ is field. Then by graded vesion of [21, Theorem 3.9], there exists a graded $R$-module which is not 1-FCP-gr-projective. Also by graded vesion of [21, Theorem 3.7], $r$.FCP-gr-$dimR \leq 1$. So, FCP-gr-$pdM \leq 1$ for any graded right $R$-module $M$. Hence by Lemma 4.2, there is an exact sequence $0 \to P_1 \to P_0 \to M \to 0$ in gr-$R$, where $P_0$ and $P_1$ are 1-FCP-gr-projective. If $U$ is a 2-copresented graded right module, then $\mathrm{EXT}_R^1(P_1, U) = 0$, since every 2-copresented graded right module is 1-copresented. Therefore from $\mathrm{EXT}_R^1(P_1, U) \cong \mathrm{EXT}_R^2(M, U)$ we get that every graded right $R$-module $M$ is 2-FCP-gr-projective.

We have the following lemma before the next proposition.

**Lemma 3.7.** *Let $R$ be a ring graded by a group $G$, $M$ a graded right $R$-module and $k$ non-negative integer. Then* $\mathrm{EXT}_R^k(M, -)_\sigma \cong \mathrm{Ext}_{\mathrm{gr}-R}^k(M(\sigma^{-1}), -)$ *for any* $\sigma \in G$.

**Proof.** It is clear that for $k = 0$, $\mathrm{HOM}_R(M,-)_\sigma \cong \mathrm{Hom}_{\mathrm{gr}-R}(M(\sigma^{-1}),-)$, see [10]. Let $N$ be a graded right $R$-module. Then, there exists an exact sequence $0 \to N \to E \to L \to 0$, where $E$ is gr-injective. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{EXT}_R^{k-1}(M,L)_\sigma & \longrightarrow & \mathrm{EXT}_R^{k}(M,N)_\sigma & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \\ 0 & \longrightarrow & \mathrm{Ext}_{\mathrm{gr}-R}^{k-1}(M(\sigma^{-1}),L) & \longrightarrow & \mathrm{Ext}_{\mathrm{gr}-R}^{k}(M(\sigma^{-1}),N) & \longrightarrow & 0 \end{array}$$

By induction hypothesis, $\alpha$ is an isomorphism and then so is $\beta$. □

In the following proposition, we give some equivalent characterizations of $n$-FCP-gr-projective modules with respect to special gr-copure short exact sequences.

**Proposition 3.8.** *Let $R$ be a ring graded by a group $G$ and $n \geq 1$ an integer. Then the following statements are equivalent for a graded right $R$-module $M$.*

(1) *$M$ is n-FCP-gr-projective;*
(2) *$M$ is gr-projective with respect to all special short exact sequences in gr-R;*
(3) *The exact sequence $0 \to A \to B \to M \to 0$ in gr-R is special gr-copure;*
(4) *$M(\sigma^{-1})$ is n-FCP-gr-projective for any $\sigma \in G$;*
(5) *There exists a special gr-copure short exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where $P$ is gr-projective;*
(6) *There exists a special gr-copure short exact sequence $0 \to K \to P \to M \to 0$ in gr-R, where $P$ is n-FCP-gr-projective;*
(7) *$M(\sigma^{-1})$ is gr-projective with respect to all special short exact sequences in gr-R for any $\sigma \in G$.*

**Proof.** (1) $\Longrightarrow$ (2) Let $0 \to K^{n-1} \to E^n \to K^n \to 0$ be a special short exact sequence with respect to any $n$-copresented graded right $R$-module $U$. Then, $\mathrm{EXT}_R^1(M, K^{n-1}) \cong \mathrm{EXT}_R^n(M,U) = 0$.

(2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (1), (5) $\Longrightarrow$ (6) and (6) $\Longrightarrow$ (1) are clear.

(4) $\Longrightarrow$ (1) It is clear, since by Lemma 3.7, $\mathrm{EXT}_R^n(M,U)_\sigma \cong \mathrm{Ext}_{\mathrm{gr}-R}^n(M(\sigma^{-1}),U)$ for any $n$-copresented graded right $R$-module $U$ and any $\sigma \in G$.

(1) $\Longrightarrow$ (5) Let $M$ be a graded right $R$-module. Then, there is exact sequence $0 \to K \to P \to M \to 0$ in gr-$R$ with $P$ is gr-projective. By (1), $\mathrm{EXT}_R^1(M, K^{n-1}) \cong \mathrm{EXT}_R^n(M,U) = 0$ for any special gr-copresented $K^{n-1}$ and any $n$-copresented graded right module $U$. So, (5) follows.

(2) $\Longleftrightarrow$ (7) It is trivial, since $\mathrm{Hom}_{\mathrm{gr}-R}(M(\sigma^{-1}),-) \cong \mathrm{HOM}_R(M,-)_\sigma$.

(2) $\Longrightarrow$ (4) Assume that $U$ is an $n$-copresented graded right $R$-module and $0 \to K^{n-1} \to E^n \to K^n \to 0$ is a special short exact sequence in gr-$R$, where $E^n$ is finitely cogenerated injective. Then, we have the following exact sequence for any $\tau \in G$:

$$0 \longrightarrow \mathrm{HOM}_R(M, K^{n-1})_\tau \longrightarrow \mathrm{HOM}_R(M, E^n)_\tau \longrightarrow \mathrm{HOM}_R(M, K^n)_\tau \longrightarrow 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{HOM}_R(M, K^{n-1})_{\tau\sigma} & \longrightarrow & \mathrm{HOM}_R(M, E^n)_{\tau\sigma} & \longrightarrow & \mathrm{HOM}_R(M, K^n)_{\tau\sigma} & \longrightarrow 0 \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, K^{n-1}) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, E^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(M(\tau\sigma)^{-1}, K^n) & \\
 & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1})_\tau & \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), E^n)_\tau & \longrightarrow & \mathrm{HOM}_R(M(\sigma^{-1}), K^n)_\tau &
\end{array}$$

with the upper row exact for every $\tau \in G$. So, we deduce that

$$0 \to \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1})_\tau \to \mathrm{HOM}_R(M(\sigma^{-1}), E^n)_\tau \to \mathrm{HOM}_R(M(\sigma^{-1}), K^n)_\tau \to 0$$

is exact, which gives rise to the exactness of

$$0 \to \mathrm{HOM}_R(M(\sigma^{-1}), K^{n-1}) \to \mathrm{HOM}_R(M(\sigma^{-1}), E^n) \to \mathrm{HOM}_R(M(\sigma^{-1}), K^n) \to 0.$$

Hence, $0 = \mathrm{EXT}^1_R(M(\sigma^{-1}), K^{n-1}) \cong \mathrm{EXT}^n_R(M(\sigma^{-1}), U)$ and consequently, $M(\sigma^{-1})$ is $n$-FCP-gr-projective. $\square$

Transfer result of $n$-FCP-gr-projective modules with respect to the functor $F$ is given in the following result.

**Proposition 3.9.** *Let $R$ be a ring graded by a group $G$. If $M$ is an $n$-FCP-projective right $R$-module, then $F(M)$ is $n$-FCP-gr-projective.*

**Proof.** Let $0 \longrightarrow K^{n-1} \longrightarrow E^n \xrightarrow{g} K^n \longrightarrow 0$ be a special short exact sequence in gr-$R$ and $f : F(M) \to K^n$ a graded morphism. Since $F$ is a right adjoint functor of the forgetful functor, we have the commutative diagram:

$$\begin{array}{ccccccccc}
 & & & & M & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \longrightarrow & K^n & \longrightarrow & 0
\end{array}$$

Now, again by the adjoint situation between the forgetful functor and $F$ we have a graded morphism $F(M) \to E^n$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc} & & & & F(M) & & & & \\ & & & & \downarrow & \searrow & & & \\ 0 & \longrightarrow & K^{n-1} & \longrightarrow & E^n & \xrightarrow{g} & K^n & \longrightarrow & 0 \end{array}$$

which shows that $F(M)$ is gr-projective with respect to all the special short exact sequences in gr-$R$. Let $f : F(M)(\sigma^{-1}) \to K^n$ be a graded morphism for any $\sigma \in G$. Since, the exact sequence

$$0 \longrightarrow K^{n-1}(\sigma) \longrightarrow E^n(\sigma) \xrightarrow{T_\sigma(g)} K^n(\sigma) \longrightarrow 0$$

exists and $K^{n-1}(\sigma)$ is special gr-copresented, there exists a graded morphism $h : F(M) \to E^n(\sigma)$ such that $T_\sigma(g)h = T_\sigma(f)$, and so $gT_{\sigma^{-1}}(h) = f$ for $T_{\sigma^{-1}}(h) : F(M)(\sigma^{-1}) \to E^n$. Therefore for any $\sigma \in G$, $F(M)(\sigma^{-1})$ is gr-projective with respect to all the special short exact sequences and consequently by Proposition 3.8, $F(M)$ is $n$-FCP-gr-projective. □

Also, as for the classical projective notion, the class *gr-*$\mathcal{FCP}_n$ is closed under direct limits.

**Proposition 3.10.** *Let $R$ be a graded ring by a group $G$, $K^{n-1}$ a special gr-copresented and $\{M_i\}_{i\in I}$ a direct system of graded right $R$-modules with $I$ directed. Then:*

(1) $\mathrm{HOM}_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{HOM}_R(M_i, K^{n-1})$.
(2) $\mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1})$.

**Proof.** (1) For any $\sigma \in G$, we have

$$\mathrm{HOM}_R(\varinjlim M_i, K^{n-1}) = \bigoplus_{\sigma\in G} \mathrm{HOM}_R(\varinjlim M_i, K^{n-1})_\sigma \cong \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}).$$

By [17, Proposition 5.26],

$$\bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(\varinjlim M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(M_i(\sigma^{-1}), K^{n-1}).$$

So,

$$\varprojlim \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(M_i(\sigma^{-1}), K^{n-1}) \cong \varprojlim \bigoplus_{\sigma\in G} \mathrm{HOM}_R(M_i, K^{n-1})_\sigma = \varprojlim \mathrm{HOM}_R(M_i, K^{n-1}).$$

(2) Let $0 \to K^{n-1} \to E^n \to K^n \to 0$ be a special short exact sequence in gr-$R$. Then by (1), the following commutative diagram exists:

$$\begin{array}{ccccccc}
\mathrm{HOM}_R(\varinjlim M_i, E^n) & \longrightarrow & \mathrm{HOM}_R(\varinjlim M_i, K^n) & \longrightarrow & \mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \\
\varprojlim \mathrm{HOM}_R(M_i, E^n) & \longrightarrow & \varprojlim \mathrm{HOM}_R(M_i, K^n) & \longrightarrow & \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1}) & \longrightarrow & 0
\end{array}$$

Therefore, $\mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1})$. □

**Corollary 3.11.** *Let $R$ be a graded ring. Then, the class gr-$\mathcal{FCP}_n$ is closed under direct limits.*

**Proof.** Let $U$ be an $n$-copresented graded right module and let $\{M_i\}_{i\in I}$ be a family of $n$-FCP-gr-projective right modules . Then by Proposition 3.10,

$$\mathrm{EXT}^n_R(\varinjlim M_i, U) \cong \mathrm{EXT}^1_R(\varinjlim M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^1_R(M_i, K^{n-1}) \cong \varprojlim \mathrm{EXT}^n_R(M_i, U),$$

where $K^{n-1}$ is special gr-copresented. □

## 4. $n$-gr-cocoherent rings

In this section, some characterizations of $n$-FCP-gr-projective right modules on right $n$-gr-cocoherent rings are given.

We state the following lemmas that are derived from [21, Theorem 2.12].

**Lemma 4.1.** *Let $R$ be a right $n$-gr-cocoherent ring and $M$ a graded right $R$-module. Then the following statements are equivalent:*

(1) *$n$.FCP-gr-pd$M \leq k$;*
(2) $\mathrm{EXT}^{k+1}_R(M, K^{n-1}) = 0$ *for any special gr-copresented $K^{n-1}$.*

**Proof.** (2) $\Longrightarrow$ (1) is trivial by Definition 3.5.

(1) $\Longrightarrow$ (2) Use induction on $k$. Clear if $n$.FCP-gr-pd$M = k$. Let $n$.FCP-gr-pd$M \leq k-1$. If $0 \to K^{n-1} \to E^n \to K^n \to 0$ is a special short exact sequence in gr-$R$ with respect to any $n$-copresented graded right $R$-module $U$, then from the $n$-gr-cocoherence of $R$ we deduce that $K^n$ is special gr-copresented. Also, we have $\mathrm{EXT}^k_R(M, K^n) \cong \mathrm{EXT}^{k+1}_R(M, K^{n-1})$. So by induction hypothesis, $\mathrm{EXT}^k_R(M, K^n) = 0$ and consequently $\mathrm{EXT}^{k+1}_R(M, K^{n-1}) = 0$ which completes the proof. □

**Lemma 4.2.** *Let $R$ be a right $n$-gr-cocoherent ring, $M$ a graded right $R$-module and $k$ a non-negative integer. Then the following statements are equivalent:*

(1) *$n$.FCP-gr-pd$M \leq k$;*

(2) $\mathrm{EXT}_R^{k+l}(M, K^{n-1}) = 0$ *for any special gr-copresented* $K^{n-1}$ *and all positive integers* $l$*;*

(3) $\mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$ *for any special gr-copresented* $K^{n-1}$*;*

(4) *There exists an exact sequence*

$$0 \longrightarrow P_k \xrightarrow{f_k} P_{k-1} \xrightarrow{f_{k-1}} \cdots P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

*in gr-*$R$ *with* $P_0, P_1, \cdots, P_k$ *are* $n$*-FCP-gr-projective;*

(5) $n$*.FCP-gr-pd*$M(\sigma^{-1}) \leq k$ *for any* $\sigma \in G$*.*

**Proof.** (1) $\Longrightarrow$ (2) If $n$.FCP-gr-pd$M \leq k$, then $n$.FCP-gr-pd$M \leq k + l - 1$. So by Lemma 4.1, $\mathrm{EXT}_R^{k+l}(M, K^{n-1}) = 0$.

(4) $\Longrightarrow$ (1) Since $R$ is right $n$-gr-cocoherent, by Lemma 4.1, $\mathrm{EXT}_R^{j}(P_i, K^{n-1}) = 0$ for any special gr-copresented $K^{n-1}$, all positive integers $j$ and any $0 \leq i \leq k$. So by (4), we have:

$$\mathrm{EXT}_R^{k+1}(M, K^{n-1}) \cong \mathrm{EXT}_R^{k}(ker(f_0), K^{n-1}) \cong \cdots \cong \mathrm{EXT}_R^{1}(P_k, K^{n-1}).$$

Hence by Lemma 4.1, $n$.FCP-gr-pd$M \leq k$.

(2) $\Longrightarrow$ (3) It is obvious.

(3) $\Longrightarrow$ (4) For every graded right $R$-module $M$, there exists an exact sequence

$$0 \longrightarrow P_k \longrightarrow P_{k-1} \longrightarrow \cdots P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

in gr-$R$ with $P_0, P_1, \cdots, P_{k-1}$ are gr-projective. Therefore for any positive integers $l$, we have $\mathrm{EXT}_R^{l}(P_i, K^{n-1}) = 0$ for all special gr-copresented modules $K^{n-1}$ and any $0 \leq i \leq k-1$. Let $K_i = \ker(P_i \to P_{i-1})$. Then,

$$\mathrm{EXT}_R^{k+1}(M, K^{n-1}) \cong \mathrm{EXT}_R^{k}(K_0, K^{n-1}) \cong \mathrm{EXT}_R^{k-1}(K_1, K^{n-1}) \cong \mathrm{EXT}_R^{1}(P_k, K^{n-1}).$$

By (3), $\mathrm{EXT}_R^{k+1}(M, K^{n-1}) = 0$, and so $\mathrm{EXT}_R^{1}(P_k, K^{n-1}) = 0$, which means that $P_k$ is $n$-FCP-gr-projective.

(1) $\Longleftrightarrow$ (5) Use induction on $k$. If $k = 0$, then by Proposition 3.8, $M$ is $n$-FCP-gr-projective if and only if $M(\sigma^{-1})$ is $n$-FCP-gr-projective for any $\sigma \in G$. Assume that $k > 0$. There is an exact sequence $0 \to L \xrightarrow{f} P \to M \to 0$ in gr-$R$, where $P$ is gr-projective. For any $\sigma \in G$, it follows that the exact sequence

$$0 \longrightarrow L(\sigma^{-1}) \xrightarrow{T_{\sigma^{-1}}(f)} P(\sigma^{-1}) \longrightarrow M(\sigma^{-1}) \longrightarrow 0 \ ,$$

where $P(\sigma^{-1})$ is gr-projective exists. So by Lemma 3.7, for every special *gr*-copresented $K^{n-1}$ and any $\tau \in G$, we have:

$$\mathrm{EXT}_R^{k+1}(M(\sigma^{-1}), K^{n-1})_\tau \cong \mathrm{EXT}_R^{k}(L(\sigma^{-1}), K^{n-1})_\tau \cong \mathrm{Ext}_{\mathrm{gr}-R}^{k}(L(\tau\sigma)^{-1}, K^{n-1}) \cong$$

$\mathrm{EXT}_R^k(L, K^{n-1})_{\tau\sigma} \cong \mathrm{EXT}_R^{k+1}(M, K^{n-1})_{\tau\sigma}$. By induction hypothesis, $n$.FCP-gr-pd$L(\sigma^{-1}) \leq k-1$ if and only if $n$.FCP-gr-pd$L \leq k-1$. So, we deduce that $n$.FCP-gr-pd$M(\sigma^{-1}) \leq k$ if and only if $n$.FCP-gr-pd$M \leq k$. □

**Corollary 4.3.** *Let $R$ be a right $n$-gr-cocoherent ring of type $G$. Then the following statements are equivalent:*

1. *If $\phi : N \to M$ is an $n$-FCP-gr-projective preenvelope, then $N$ has an epic $n$-FCP-gr-projective preenvelope;*
2. *The cokernel of any $n$-FCP-gr-projective preenvelope of a graded right $R$-module is $n$-FCP-gr-projective;*
   *Moreover, if every submodule of an $n$-FCP-gr-projective graded right $R$-module has an $n$-FCP-gr-projective preenvelope, the above are equivalent to:*
3. *r.n.FCP-gr-dim$R \leq 1$.*

**Proof.** (1) $\Longrightarrow$ (2) Let $\phi : N \to M$ be an $n$-FCP-gr-projective preenvelope. Then $f : \mathrm{Im}(\phi) \to M$ is an $n$-FCP-gr-projective preenvelope. By (1), there is an epic $n$-FCP-gr-projective preenvelope $g : \mathrm{Im}(\phi) \to C$. Consider the following commutative diagram, where $D$ is a pushout of two maps $f$ and $g$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Im}(\phi) & \xrightarrow{f} & M & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle id} & & \\
 & & C & \xrightarrow{\alpha} & D & \longrightarrow & \mathrm{Coker}(\phi) & \longrightarrow & 0
\end{array}$$

By [17, Exercise 5.10], $\alpha$ is injective and $\beta$ is surjective. On the other hand, $D = \alpha(C) + \beta(M)$. Since $\beta$ is surjective, $D = \alpha(C) + D$ and so, $\alpha(C) \subseteq D$. Also, by using of preenvelopes $f$ and $g$, there is a graded morphism $h : D \to C$ such that $h\alpha = 1_C$. Hence, $D \cong C \oplus \mathrm{Coker}(\phi)$. Similarly $D \cong M$. Therefore from $n$-FCP-gr-projectivity $M$, we deduce that $\mathrm{Coker}(\phi)$ is $n$-FCP-gr-projective.

(2) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an $n$-FCP-gr-projective preenvelope. It is enough to show that $\mathrm{Im}(\phi)$ is $n$-FCP-gr-projective. Consider the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$. By hypothesis, $\mathrm{Coker}(\phi)$ is $n$-FCP-gr-projective, and so for every special gr-copresented $K^{n-1}$, we have:

$$0 = \mathrm{EXT}_R^1(M, K^{n-1}) \longrightarrow \mathrm{EXT}_R^1(\mathrm{Im}(\phi), K^{n-1}) \longrightarrow \mathrm{EXT}_R^2(\mathrm{Coker}(\phi), K^{n-1}).$$

Hence by Lemm 4.2 and (2), $\mathrm{EXT}_R^2(\mathrm{Coker}(\phi), K^{n-1}) = 0$. Hence,

$$0 = \mathrm{EXT}_R^1(\mathrm{Im}(\phi), K^{n-1}) \cong \mathrm{EXT}_R^n(\mathrm{Im}(\phi), U)$$

for any $n$-copresented graded right module $U$, and then it follows that $\mathrm{Im}(\phi)$ is $n$-FCP-gr-projective.

(2) $\Longrightarrow$ (3) Let $M$ be a graded right $R$-module. Then, there exists an exact sequence $0 \to K \to P \to M \to 0$ in gr-$R$, where $P$ is gr-projective. If $K$ is $n$-FCP-gr-projective, then $r.n.\mathrm{FCP}$-gr-$\dim(R) \leq 1$. So, we show that $K$ is $n$-FCP-gr-projective. If $\phi : K \to N$ is an $n$-FCP-gr-projective preenvelope, then $\phi$ is injective. So similar to proof of (2) $\Longrightarrow$ (1), we get that $K$ is $n$-FCP-gr-projective.

(3) $\Longrightarrow$ (1) Let $\phi : N \to M$ be an $n$-FCP-gr-projective preenvelope. Then by Lemma 4.2, the exact sequence $0 \to \mathrm{Im}(\phi) \to M \to \mathrm{Coker}(\phi) \to 0$ implies that $\mathrm{Im}(\phi)$ is $n$-FCP-gr-projective, and so $N \to \mathrm{Im}(\phi)$ is an epic $n$-FCP-gr-projective preenvelope of $N$. □

In the following, we present one of the main results in this paper.

**Theorem 4.4.** *Let $R$ be a graded ring. Then the following statements are equivalent:*

(1) *$R$ is right $n$-gr-cocoherent;*
(2) *For every exact sequence $0 \to A \to B \to C \to 0$ in gr-$R$, $A$ is $n$-FCP-gr-projective if $B$ and $C$ are $n$-FCP-gr-projective.*

**Proof.** (1) $\Longrightarrow$ (2) Let $U$ be an $n$-copresented graded right $R$-module. Then by Lemm 4.2(2), we have: $0 = \mathrm{EXT}^n_R(B, U) \to \mathrm{EXT}^n_R(A, U) \to \mathrm{EXT}^{n+1}_R(C, U) = 0$, since $B$ and $C$ are $n$-FCP-gr-projective. So $\mathrm{EXT}^n_R(A, U) = 0$, and hence $A$ is $n$-FCP-gr-projective.

(2) $\Longrightarrow$ (1) Let $U$ be an $n$-copresented graded right $R$-module and $0 \to K^{n-1} \to E^n \to K^n \to 0$ a special short exact sequence in gr-$R$ with respect to $U$. We show that $U$ is $(n+1)$-copresented. For this, enough to say that $K^n$ is special gr-copresented. Let $M$ be a 1-FCP-projective right $R$-module and $0 \to K \to P \to M \to 0$ an exact sequence in Mod-$R$ with $P$ is projective. Then $0 \to F(K) \to F(P) \to F(M) \to 0$ is exact in gr-$R$, where by Proposition 3.9, $F(P)$ and $F(M)$ are 1-FCP-gr-projective. Hence, Remark 3.4 imply that $F(P)$ and $F(M)$ are $n$-FCP-gr-projective. So by hypothesis, it follows that $F(K)$ is $n$-FCP-gr-projective. Since $E^n$ is *gr*-injective, we have:

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), E^n) \longrightarrow \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \longrightarrow \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \longrightarrow 0.$$

So, $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1})$. Also, we have:

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(P), K^{n-1}) \to \mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \to \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \to 0.$$

Consequently $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1})$ and so by Lemma 3.7, $n$-FCP-gr-projectivity $F(K)$ imply that $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(K)(\sigma^{-1}), K^{n-1}) = 0$ for any $\sigma \in G$. Thus,

$$0 = \mathrm{Ext}^1_{\mathrm{gr}-R}(F(K), K^{n-1}) \cong \mathrm{Ext}^2_{\mathrm{gr}-R}(F(M), K^{n-1}) \cong \mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n).$$

Now, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(M), K^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(P), K^n) & \longrightarrow & \mathrm{Hom}_{\mathrm{gr}-R}(F(K), K^n) & \longrightarrow & 0 \\
 & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathrm{Hom}_R(M, K^n) & \longrightarrow & \mathrm{Hom}_R(P, K^n) & \longrightarrow & \mathrm{Hom}_R(K, K^n) & &
\end{array}$$

with the upper row exact. Therefore, $\mathrm{Ext}^1_{\mathrm{gr}-R}(F(M), K^n) \cong \mathrm{Ext}^1_R(M, K^n) = 0$ for any 1-FCP-projective $R$-module $M$, and it follows that $K^n$ is 1-copresented. Hence, $U$ is $(n+1)$-copresented. □

**Corollary 4.5.** *Let $R$ be a right $n$-gr-cocoherent ring. Then, graded right $R$-module $M$ is $n$-FCP-gr-projective if and only if every copure epimorphic image and copure submodule of $M$ is $n$-FCP-gr-projective.*

**Proof.** ($\Longrightarrow$) Let $N$ be a copure submodule of $n$-FCP-gr- projective right $R$-module $M$. Then, the exact sequence $0 \to N \to M \to \frac{M}{N} \to 0$ is special gr-copure. So by Proposition 3.8, $\frac{M}{N}$ is $n$-FCP-gr-projective and hence by Theorem 4.4, $N$ is $n$-FCP-gr-projective.

($\Longleftarrow$) It is clear. □

Before the next results, we first introduce the following symbols and definitions given in [9,20].

For every class $\mathcal{Y}$ of graded right $R$-modules, denote the classes

$$\mathcal{Y}^{\perp} = \{X \in gr\text{-}R \ : \mathrm{Ext}^1_{\mathrm{gr}-R}(Y, X) = 0 \ for\ all\ Y \in \mathcal{Y}\}$$

and

$$^{\perp}\mathcal{Y} = \{X \in gr\text{-}R \ : \mathrm{Ext}_{\mathrm{gr}-R}(X, Y) = 0 \ for\ all\ Y \in \mathcal{Y}\}.$$

Given two classes of graded right $R$-modules $\mathcal{F}$ and $\mathcal{C}$, then we say that $(\mathcal{F}, \mathcal{C})$ is a cotorsion theory in gr-$R$ if $\mathcal{F}^{\perp} = \mathcal{C}$ and $\mathcal{F} = {}^{\perp}\mathcal{C}$. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is called hereditary if whenever $0 \to F' \to F \to F'' \to 0$ is exact in gr-$R$ with $F, F'' \in \mathcal{F}$ then $F'$ is also in $\mathcal{F}$.

A duality pair over a graded ring $R$ is a pair $(\mathcal{F}, \mathcal{C})$, where $\mathcal{F}$ is a class of graded right (resp. left) $R$-modules and $\mathcal{C}$ is a class of graded left (resp. right) $R$-modules, subject to the following conditions: (1) For any graded module $F$, one has $F \in \mathcal{F}$ if and only if $F^* \in \mathcal{C}$. (2) $\mathcal{C}$ is closed under direct summands and finite direct sums.

A duality pair $(\mathcal{F}, \mathcal{C})$ is called (co)product-closed if the class of $\mathcal{F}$ is closed under graded direct (co)products, and a duality pair $(\mathcal{F}, \mathcal{C})$ is called perfect if it is coproduct-closed, $\mathcal{F}$ is closed under extensions and $R$ belongs to $\mathcal{F}$.

**Theorem 4.6.** *The pair ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory if and only if $R$ is a right $n$-gr-cocoherent ring.*

**Proof.** ($\Longrightarrow$) Let $M$ be an $n$-FCP-gr-projective right $R$-module. Then, there is an exact sequence $0 \to K \to P \to M \to 0$ in gr-$R$, where $P$ is gr-projective. Thus by Remark 3.4, $P$ is $n$-FCP-gr-projective, too. Since ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory, we deduce that $K$ is $n$-FCP-gr-projective, and then by Theorem 4.4, it follows that $R$ is right $n$-gr-cocoherent.

($\Longleftarrow$) Note that we have to show that ${}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp}) = gr\text{-}\mathcal{FCP}_n$. Let $K^{n-1}$ be a special gr-copresented with respect to any $n$-copresented graded right $R$-module $U$ and $M \in {}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp})$. Then, $K^{n-1} \in gr\text{-}\mathcal{FCP}_n^{\perp}$ and $M(\sigma^{-1}) \in {}^{\perp}(gr\text{-}\mathcal{FCP}_n^{\perp})$ for all $\sigma \in G$ by analogy with the proof of Proposition 3.9. Therefore by Lemma 3.7, $\mathrm{EXT}_R^1(M, K^{n-1})_\sigma \cong \mathrm{Ext}^1_{\mathrm{gr}-R}(M(\sigma^{-1}), K^{n-1}) = 0$ and consequently by Lemma 4.2, $M$ is $n$-FCP-gr-projective, and hence $M \in gr\text{-}\mathcal{FCP}_n$. Let $0 \to F^{'} \to F \to F^{''} \to 0$ be a exact sequence of modules in gr-$R$, where $F$ and $F^{''}$ are $n$-FCP-gr-projective. Then by Theorem 4.4, $F^{'}$ is $n$-FCP-gr-projective, since $R$ is $n$-gr-cocoherent. So, it follows that ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is a hereditary cotorsion theory. □

We denote $(gr\text{-}\mathcal{FCP}_n)^* = \{M^* \mid\ M \in gr\text{-}\mathcal{FCP}_n\}$. The following lemma shows the connection between $n$-FCP-gr-projective and $n$-FP-gr-injective modules.

**Lemma 4.7.** *Let $R$ be a graded ring of type $G$.*

(1) *If $U$ is an $n$-presented graded left $R$-module, then $U^*$ is an $n$-copresented graded right $R$-module.*
(2) $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$.

**Proof.** (1) Let $U$ be an $n$-presented graded left $R$-module. Then, there exists an exact sequence

$$F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow U \longrightarrow 0$$

in $R$-gr with each $F_i$ is finitely generated free. So by graded version of [17, Lemma 3.53], there is an exact sequence

$$0 \longrightarrow U^* \longrightarrow F_0^* \longrightarrow \cdots \longrightarrow F_{n-1}^* \longrightarrow F_n^*$$

with $R$-modules in gr-$R$. It suffices to show that every $F_i^*$ is finitely cogenerated injective. It is clear that any $F_i^*$ is finitely cogenerated. So, we prove that every $F_i^*$ is injective, too. Consider the short exact sequence $0 \to A \to B \to C \to 0$ in gr-$R$. Then, there exists the following commutative diagram with the upper row exact for any $\sigma \in G$:

$$\begin{array}{ccccccc}
0 \longrightarrow & \mathrm{HOM}_R(C, F_i^*) & \longrightarrow & \mathrm{HOM}_R(B, F_i^*) & \longrightarrow & \mathrm{HOM}_R(A, F_i^*) & \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(C(\sigma^{-1}), F_i^*) & \longrightarrow & \bigoplus_{\sigma\in G} \mathrm{Hom}_{\mathrm{gr}-R}(B(\sigma^{-1}), F_i^*) & \longrightarrow & \bigoplus \mathrm{Hom}_{\mathrm{gr}-R}(A(\sigma^{-1}), F_i^*) & \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \bigoplus_{\sigma\in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, C^*(\sigma)) & \longrightarrow & \bigoplus_{\sigma\in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, B^*(\sigma)) & \longrightarrow & \bigoplus_{\sigma\in G} \mathrm{Hom}_{R-\mathrm{gr}}(F_i, A^*(\sigma)) & \\
& \downarrow \cong & & \downarrow \cong & & \downarrow \cong & \\
0 \longrightarrow & \mathrm{HOM}_R(F_i, C^*) & \longrightarrow & \mathrm{HOM}_R(F_i, B^*) & \longrightarrow & \mathrm{HOM}_R(F_i, A^*) & \longrightarrow 0
\end{array}$$

So, $\mathrm{EXT}_R^1(C, F_i^*) = 0$ and hence any $F_i^*$ is injective.

(2) By [20, Definition 3.1], let $0 \to K_{n-1} \to P_{n-1} \to K_n \to 0$ be a special short exact sequence in $R$-gr with respect to any $n$-presented graded left $R$-module $U$. Then by (1), $K_{n-1}^*$ is special gr-copresented in gr-$R$. So if $M$ is $n$-FCP-gr-projective right $R$-module, then similar to the proof (1), $0 = \mathrm{EXT}_R^1(M, K_{n-1}^*) \cong \mathrm{EXT}_R^1(K_{n-1}, M^*)$. On the other hand, $\mathrm{EXT}_R^n(U, M^*) \cong \mathrm{EXT}_R^1(K_{n-1}, M^*)$ by [20, Definition 3.1]. Therefore, $M^*$ is an $n$-FP-gr-injective left $R$-module, and then we conclude that $(gr\text{-}\mathcal{FCP}_n)^* \subseteq gr\text{-}\mathcal{FI}_n$. □

In the following theorem, by using the previous results, we present some equivalent characterizations to that each graded right $R$-module is $n$-FCP-gr-projective.

**Theorem 4.8.** *Let $R$ be a graded ring of type $G$. Then, the following statements are equivalent:*

(1) *Every graded right module is $n$-FCP-gr-projective;*
(2) *gr-id$(U) \leq n-1$ for any $n$-copresented graded right $R$-module $U$;*
(3) *Every special gr-copresented right module is gr-injective;*
(4) *$(gr\text{-}\mathcal{FCP}_n,\ gr\text{-}\mathcal{FCP}_n^{\perp})$ is perfect hereditary cotorsion theory and $N$ has an $n$-FCP-gr-projective cover with the unique mapping property for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$;*
(5) *$N$ is gr-injective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$;*
(6) *$N(\sigma)$ is gr-injective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$ and any $\sigma \in G$;*
(7) *Every graded right module has an $n$-FCP-gr-projective cover with the unique mapping property;*
(8) *$R$ is right $n$-gr-cocoherent and $N$ is $n$-FCP-gr-projective for any $N \in gr\text{-}\mathcal{FCP}_n^{\perp}$.*

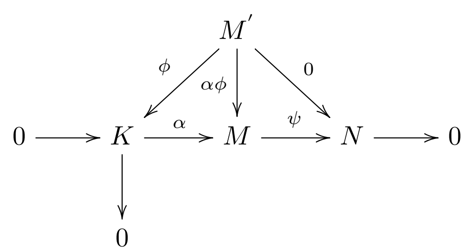
**Proof.** (1) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1) are clear by Proposition 3.8.

(1) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (3) are obvious.

(5) $\Longleftrightarrow$ (6) it follows from $\mathrm{HOM}_R(-,N)_\sigma \cong \mathrm{Hom}_{\mathrm{gr}-R}(-,N(\sigma))$.

(1) $\Longrightarrow$ (7) First, we show that the class $gr$-$\mathcal{FCP}_n$ is covering. If $M \in gr$-$\mathcal{FCP}_n$, then by Lemma 4.7, $M^* \in gr$-$\mathcal{FP}_n$. Contrary, if $M^* \in gr$-$\mathcal{FP}_n$, then [20, Proposition 3.8] implies that $M$ is $n$-gr-flat, and hence by (1), $M \in gr$-$\mathcal{FCP}_n$. On the other hand, the class $gr$-$\mathcal{FP}_n$ is closed under direct summands and direct sums by [20, Propositions 3.7 and 3.16]. So, we obtain that ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FI}_n$) is a duality pair. Also By (1), it follows that the class $gr$-$\mathcal{FCP}_n$ is closed under gr-pure submodules, gr-pure quotients and gr-pure extensions. Therefore by Corollary 3.11 and [20, Theorem 4.2], the class $gr$-$\mathcal{FCP}_n$ is covering and hence by hypothesis, (7) follows.

(7) $\Longrightarrow$ (1) Let $N$ be a graded right $R$-module. Then there is a commutative diagram with exact rows:




where,$\psi$ and $\phi$ are $n$-FCP-gr-projective cover with the unique mapping property. Since $\psi\alpha\phi = 0 = \psi$, we have $\alpha\phi = 0$ by (7). Therefore, $K = \mathrm{im}(\phi) \subseteq \ker(\alpha) = 0$ and so $K = 0$. Thus $N \cong M$ and hence every graded right module $N$ is $n$-FCP-gr-projective.

(1) $\Longrightarrow$ (8) Let $M$ be an $n$-FCP-gr-projective right $R$-module. Then, there is an exact sequence $0 \to K \to P \to M \to 0$ in gr-$R$, where $P$ and $K$ are $n$-FCP-gr-projective by (1). So by Theorem 4.4, it follows that $R$ is right $n$-gr-cocoherent. Also by hypothesis, $N$ is $n$-FCP-gr-projective for any $N \in gr$-$\mathcal{FCP}_n^{\perp}$.

(8) $\Longrightarrow$ (3) Cosider the special short exact sequence $\Delta^{'} : 0 \to K^{n-1} \to E^{n-1} \to K^n \to 0$ in gr-$R$ with respect to any $n$-copresented graded right module $U$, where $K^{n-1}$ is gr-copresented and $K^n$ is $n$-gr-cogenerated. But $K_n$ is gr-copresented, too, since $R$ is $n$-gr-cocoherent. Thus $K^n \in gr$-$\mathcal{FCP}_n^{\perp}$ and consequently $0 = \mathrm{EXT}_R^n(K^n, U) \cong \mathrm{EXT}_R^1(K^n, K^{n-1})$, since $K_n$ is $n$-FCP-gr-projective by (8). Therefore $\Delta^{'}$ is split and we deduce that $K^{n-1}$ is gr-injective.

(4) $\Longrightarrow$ (8) By Theorem 4.6, $R$ is right $n$-gr-cocoherent. Let $N \in gr$-$\mathcal{FCP}_n^{\perp}$. If $\phi : M \to N$ is an $n$-FCP-gr-projective cover with the unique mapping property, then $\ker\phi \in gr$-$\mathcal{FCP}_n^{\perp}$. Thus, similar to the proof of (7) $\Longrightarrow$ (1), we get that $N$ is $n$-FCP-gr-projective.

(8) $\Longrightarrow$ (4) By Theorem 4.6, ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is hereditary cotorsion theory. Also $R \in gr$-$\mathcal{FCP}_n$ and by Corollary 3.11 and (8) $\Longrightarrow$ (3) $\Longrightarrow$ (1), $gr$-$\mathcal{FCP}_n$ is closed under direct sum and extensions. Therefore, we deduce that ($gr$-$\mathcal{FCP}_n$, $gr$-$\mathcal{FCP}_n^{\perp}$) is a perfect hereditary cotorsion theory. If $N$ is $n$-FCP-gr-projective for any $N \in gr$-$\mathcal{FCP}_n^{\perp}$, then it is clear that $N$ has an $n$-FCP-gr-projective cover with the unique mapping property. □

**Acknowledgment.** The authors would like to thank the referee for the helpful suggestions and valuable comments.

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