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UNITIFICATION OF WEAKLY RICKART AND WEAKLY P.Q.-BAER $*$-RINGS

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Abstract. S. K. Berberian raised the problem "Can every weakly Rickart $*$-ring be embedded in a Rickart $*$-ring with preservation of right projections?". Berberian has given a partial solution to this problem. Khairnar and Waphare raised a similar problem for p.q.-Baer $*$-rings and gave a partial solution. In this paper, we give more general partial solutions to both the problems.

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1. Introduction

Kaplansky [6] introduced Baer rings and Baer $*$-rings to abstract various properties of AW^*-algebras (i.e., a C^*-algebra which is also a Baer $*$-ring), von Neumann algebras and complete $*$-regular rings. The concept of a Baer $*$-ring is naturally motivated from the study of functional analysis. For example, every von Neumann algebra is a Baer $*$-algebra. One can refer [4,7,9,10,13,14] for recent work on rings with involution.

Throughout this paper, R denotes an associative ring. An *ideal* of a ring R, we mean a two sided ideal. A ring R is said to be *reduced* if it does not have a nonzero nilpotent element. A ring R is said to be *abelian* if its every idempotent element is central. Let S be a nonempty subset of R. We write $r_R(S) = \{a \in R \mid sa = 0, \ \forall \ s \in S\}$, and is called the *right annihilator* of S in R, and $l_R(S) = \{a \in R \mid as = 0, \ \forall \ s \in S\}$, is the *left annihilator* of S in R. A $*$-*ring* R is a ring equipped with an involution $x \to x^*$, that is, an additive anti-automorphism of the period at most two. An element e of a $*$-ring R is called a *projection* if it is self-adjoint (i.e., $e = e^*$) and idempotent (i.e., $e^2 = e$). A $*$-ring R is said to be a *Rickart $*$-ring* if for each $x \in R$, $r_R(\{x\}) = eR$, where e is a projection in R. For each element a in a Rickart $*$-ring, there is unique projection e such that

$ae = a$ and $ax = 0$ if and only if $ex = 0$, called the *right projection* of a, denoted by $RP(a)$. Similarly, the left projection $LP(a)$ is defined for each element a in a Rickart $*$-ring. A $*$-ring R is said to be a *weakly Rickart $*$-ring* if for any $x \in R$, there exists a projection e such that (1) $xe = x$, and (2) if $xy = 0$, then $ey = 0$. Recall the following propositions and an open problem from [1].

Proposition 1.1. [1, Proposition 2, page 13] *If R is a Rickart $*$-ring, then R has a unity element and the involution of R is proper.*

Proposition 1.2. [1, Proposition 2, page 28] *The following conditions on a $*$-ring R are equivalent:*

(a) *R is a Rickart $*$-ring.*
(b) *R is a weakly Rickart $*$-ring with unity.*

Proposition 1.1 says that the unity element exists in any Rickart $*$-ring and the Proposition 1.2 naturally creates the following problem.
Problem 1: Can every weakly Rickart $*$-ring be embedded in a Rickart $*$-ring with preservation of RP's?
In [1], Berberian has given a partial solution to this problem.

According to Birkenmeier et al. [2], a $*$-ring R is said to be a *quasi-Baer $*$-ring* if the right annihilator of every ideal of R is generated, as a right ideal, by a projection in R. In [3], Birkenmeier et al. introduced principally quasi-Baer (p.q.-Baer) $*$-rings as a generalization of quasi-Baer $*$-rings. A $*$-ring R is said to be a *p.q.-Baer $*$-ring* if for every principal right ideal aR of R, $r_R(aR) = eR$, where e is a projection in R. It follows that $l_R(Ra) = Rf$, for a suitable projection f. Note that an abelian Rickart $*$-ring is a p.q.-Baer $*$-ring, and a reduced p.q.-Baer $*$-ring is a Rickart $*$-ring. We say that an element x of a $*$-ring R possesses a *central cover* if there exists a smallest central projection $h \in R$ such that $hx = x$. If such a projection h exists, then it is unique, it is called the central cover of x, denoted by $h = C(x)$. In [8], Khairnar and Waphare proved that the central cover exists for every element in any p.q.-Baer $*$-ring.

Theorem 1.3. [8, Theorem 2.5] *Let R be a p.q.-Baer $*$-ring and $x \in R$. Then x has a central cover $e \in R$. Further, $xRy = 0$ if and only if $yRx = 0$ if and only if $ey = 0$, that is, $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1 - e)R = R(1 - e)$.*

In [8], Khairnar and Waphare introduced the concept of a weakly p.q.-Baer $*$-ring. A $*$-ring R is said to be a *weakly p.q.-Baer $*$-ring* if every $x \in R$ has a central cover $e \in R$ such that $xRy = 0$ if and only if $ey = 0$. According to [3], the involution

$*$ of a $*$-ring R is *semi-proper* if for any $a \in R$, $aRa^* = 0$ implies $a = 0$. Recall the following results and an open problem from [8].

Proposition 1.4. [8, Proposition 2.4] *If R is a p.q.-Baer $*$-ring, then R has the unity element and the involution of R is semi-proper.*

Theorem 1.5. [8, Theorem 3.9] *The following conditions on a $*$-ring R are equivalent:*

(a) *R is a p.q.-Baer $*$-ring.*
(b) *R is a weakly p.q.-Baer $*$-ring with unity.*

In view of the above theorem, the following problem is raised in [8].

Problem 2: Can every weakly p.q.-Baer $*$-ring be embedded in a p.q.-Baer $*$-ring with preservation of central covers?

In [8], Khairnar and Waphare provided a partial solution to Problem 2.

In the second section of this paper, we give a more general partial solution of Problem 1 and in Section 3, we give a more general partial solution of Problem 2.

2. Unitification of weakly Rickart $*$-rings

Recall the definition of unitification of a $*$-ring given by Berberian [1]. Let R be a $*$-ring. We say that R_1 is a *unitification* of R if there exists a ring K such that

(1) K is an integral domain with involution (necessarily proper), that is, K is a commutative $*$-ring with unity and without divisors of zero (the identity involution is permitted),
(2) R is a $*$-algebra over K (i.e., R is a left K-module such that identically $1a = a$, $\lambda(ab) = (\lambda a)b = a(\lambda b)$, *and* $(\lambda a)^* = \lambda^* a^*$, for $\lambda \in K$ and $a, b \in R$),
(3) R is a torsion-free K-module (that is, $\lambda a = 0$ implies $\lambda = 0$ or $a = 0$).

Define $R_1 = R \oplus K$ (the additive group direct sum), thus $(a, \lambda) = (b, \mu)$ means, by the definition that $a = b$ and $\lambda = \mu$, and addition in R_1, is defined by the formula $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$. Define $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$, $(a, \lambda)^* = (a^*, \lambda^*)$. Evidently, R_1 is also a $*$-algebra over K, has unity element $(0, 1)$ and R is a $*$-ideal in R_1. The following lemmas are elementary facts about unitification R_1 of a $*$-ring R.

Lemma 2.1. [1, Lemma 1, page 30] *With notations as in the definition of unitification, if an involution on R is proper, then so is the involution of R_1.*

Lemma 2.2. [1, Lemma 3, page 30] *With notations as in the definition of unitification, let $x \in R$ and let e be a projection in R. Then $RP(x) = e$ in R if and only if $RP((x, 0)) = (e, 0)$ in R_1.*

Berberian has given a partial solution to Problem 1 as follows.

Theorem 2.3. [1, Theorem 1, page 31] *Let R be a weakly Rickart $*$-ring. If there exists an involutory integral domain K such that R is a $*$-algebra over K and it is a torsion-free K-module, then R can be embedded in a Rickart $*$-ring with preservation of RP's.*

After 1972, there was not much headway towards the solution of Problem 1. In 1996, Thakare and Waphare supplied partial solutions wherein the condition on the underlying weakly Rickart $*$-rings was weakened in two distinct ways. For the solution of this open problem, Berberian used the condition that R is a torsion-free left K-module, and K is an integral domain. Thakare and Waphare gave another solution in which the condition of torsion-free is replaced by other condition. They establish the following.

Theorem 2.4. [11, Theorem 2] *A weakly Rickart $*$-ring R can be embedded into a Rickart $*$-ring provided there exists a ring K such that*

(1) *K is an integral domain with involution,*
(2) *R is a $*$-algebra over K,*
(3) *For any $\lambda \in K - \{0\}$, there exists a projection e_λ that is an upper bound for the set of left projections of the right annihilators of λ, that is, if $x \in R$ and $\lambda x = 0$, then $LP(x) \leq e_\lambda$.*

Theorem 2.5. [11, Theorem 7] *A weakly Rickart $*$-ring R can be embedded into Rickart $*$-ring provided the characteristic of R is nonzero.*

The $*$-ring $C_\infty(T) \bigoplus M_2(\mathbb{Z}_3)$ has no embedding in the sense of Theorem 2.3 as the characteristic of R is zero though it has a unitification in the sense of Theorem 2.4. This example shows that Theorem 2.4 is an improvement over Theorem 2.3 of Berberian.

For projections e and f in a $*$-ring R, we say that $e \leq f$ if $e = ef$. Note that $\leq$ is a partial order on the set of all projections in a $*$-ring. Now we prove the existence of largest projection corresponding to the self-adjoint element by using the condition (3) of the above theorem.

Lemma 2.6. *Let R be a weakly Rickart $*$-ring with the condition* (3) *in Theorem 2.4. Then for any self-adjoint element a and $\lambda \neq 0$, there exists a largest projection g such that $ag = \lambda g$.*

Proof. Let $RP(a) = e'$ and e_λ be the projection as given by the condition (3) of Theorem 2.4. Let $e = e' \vee e_\lambda$, then $e' \leq e$ and $e' = e'e = ee'$. Since $ae' = a$, we have

$ae'e = ae$. Hence $a = ae' = ae$. Also, $a^* = a$ implies that $a = ea = eae \in eRe$. Thus $a - \lambda e \in eRe$. Let $h = RP(a - \lambda e)$ and $g = e - h$. This gives $(a - \lambda e)g = 0$, hence $ag = \lambda g$. Let k be any projection in R such that $ak = \lambda k$. Consider $\lambda(ek - k) = e\lambda k - \lambda k = eak - \lambda k = ak - \lambda k = 0$. Let $LP(ek - k) = f$. Therefore $f \leq e_\lambda \leq e$. That is, $ek - k = f(ek - k) = fe(ek - k) = f(ek - ek) = 0$. Consider $(a - \lambda e)k = ak - \lambda ek = ak - \lambda k = 0$. Therefore $RP(a - \lambda e)k = hk = 0$. Hence $kg = k(e - h) = ke - kh = k - 0 = k$. That is, $k \leq g$. Therefore g is the largest projection such that $ag = \lambda g$. □

Recall the following lemma from [1].

Lemma 2.7. [1, Lemma 5, page 31] *Let B be a $*$-ring with proper involution, $x \in B$ and e be a projection in B. Then e is the right projection of x if and only if e is the right projection of x^*x.*

We give a solution of Problem 1 in which the condition "K is an integral domain" is replaced by "K is a commutative ring with unity".

Let R be a $*$-ring, K be a commutative $*$-ring with unity and R be an algebra over K. Write $\mathcal{R} = \mathcal{R}(R, +)$ for the endomorphism ring of the additive group of R. Each $a \in R$ determines an element L_a of $\mathcal{R}$ via $L_a x = ax$ and each $\lambda \in K$ an element λI of $\mathcal{R}$ via $(\lambda I)x = \lambda x$. Let $R_1 = R \oplus K$ with the $*$-algebra operations as follows $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$, $\mu(a, \lambda) = (\mu a, \mu\lambda)$, $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\, \mu)$, $(a, \lambda)^* = (a^*, \lambda^*)$. Each $(a, \lambda) \in R_1$ determines an element $L_a + \lambda I$ of $\mathcal{R}$ and the mapping $(a, \lambda) \to L_a + \lambda I$ is a ring homomorphism of R_1 onto a subring S of R, namely the subring of $\mathcal{R}$ generated by L_a and λI. Define $\mu(L_a + \lambda I)$ to be the ring product $(\mu I)(L_a + \lambda I)$, then S becomes an algebra over K and $(a, \lambda) \to L_a + \lambda I$ is an algebra homomorphism of R_1 onto S. Let N be the kernel of this mapping and write $\hat{R_1} = R_1/N$ for the quotient algebra. Denote the coset $(a, \lambda) + N$ by $[a, \lambda]$. Hence $[a, \lambda]$ is an equivalence class of (a, λ) under equivalence relation $(a, \lambda) \equiv (b, \mu)$ if and only if $ax + \lambda x = bx + \mu x, \ \forall\, x \in R$.

The following result leads to the partial solution of Problem 1.

Theorem 2.8. *With above notations, we have the following.*

(1) *The mapping $a \to \ \bar{a} = [a, 0]$ is an algebra homomorphism of R into $\hat{R_1}$.*
(2) *If $L(R) = \{x \in R \mid xy = 0, \ \forall\, y \in R\} = \{0\}$ (that is, if the involution of R is proper), then the mapping $a \to \bar{a}$ is injective.*
(3) *If the involution of R is proper, then $[a, \lambda] = 0$ if and only if $[a^*, \lambda^*] = 0$ and the formula $[a, \lambda]^* = [a^*, \lambda^*]$ defines unambiguously proper involution in $\hat{R_1}$.*

(4) *If R is a weakly Rickart $*$-ring, $a \in R$ and e is the right projection of a in R, then $\bar{e}$ is the right projection of $\bar{a}$ in $\hat{R_1}$.*

Proof. (1) and (2) are easy verification.
(3) Observe that $[a, \lambda] = 0$ if and only if $(a, \lambda) + N = N$ if and only if $(a, \lambda) \in N$ if and only if $(L_a + \lambda I)x = 0,\ \forall\ x \in R$ if and only if $ax + \lambda x = 0,\ \forall\ x \in R$. Therefore in order to show $[a^*, \lambda^*] = 0$ whenever $[a, \lambda] = 0$, it is enough to show $a^*x + \lambda^*x = 0,\ \forall x \in R$. Consider $(a^*x + \lambda^*x)^*(a^*x + \lambda^*x) = (x^*a + \lambda x^*)(a^*x + \lambda^*x) = x^*aa^*x + x^*a\lambda^*x + \lambda x^*a^*x + \lambda x^*\lambda^*x = x^*\{a(a^*x) + \lambda(a^*x)\} + x^*\{a(\lambda^*x) + \lambda(\lambda^*x)\}$ $= x^*0 + x^*0 = 0,\ \forall x \in R$. Therefore $a^*x + \lambda^*x = 0,\ \forall x \in R$. Hence $[a, \lambda]^* = [a^*, \lambda^*]$ defines an involution in $\hat{R_1}$. Also, $[a, \lambda]^*[a, \lambda] = 0$ implies that $[a^*, \lambda^*][a, \lambda] = 0$. That is, $[a^*a + \lambda a^* + \lambda^*a, \lambda^*\lambda] = 0$. This gives $(a^*a + \lambda a^* + \lambda^*a)x + \lambda^*\lambda x = 0,\ \forall x \in R$. Therefore $a^*ax + \lambda a^*x + \lambda^*ax + \lambda^*\lambda x = 0,\ \forall x \in R$. Also, $(ax + \lambda x)^*(ax + \lambda x) = (x^*a^* + \lambda^*x^*)(ax + \lambda x) = x^*a^*ax + x^*a^*\lambda x + \lambda^*x^*ax + \lambda^*x^*\lambda x = x^*[a^*ax + a^*\lambda x + \lambda^*ax + \lambda^*\lambda x] = x^*[a^*ax + \lambda a^*x + \lambda^*ax + \lambda^*\lambda x] = x^*0 = 0,\ \forall x \in R$. That is, $ax + \lambda x = 0,\ \forall x \in R$. This gives $[a, \lambda] = 0$. Hence the involution $*$ is proper.
(4) Let R be a weakly Rickart $*$-ring, $a \in R$ and $e = RP(a)$. Then $ae = a$ and $ay = 0$ implies that $ey = 0$, for $y \in R$. We prove that $\bar{e} = RP(\bar{a})$. Consider $\bar{a}\bar{e} = [a, 0][e, 0] = [ae, 0] = [a, 0] = \bar{a}$. Let $\bar{y} = [b, \mu]$ and $\bar{a}\bar{y} = 0$. Then $[a, 0][b, \mu] = [ab + \mu a, 0] = 0$. This gives $(ab + \mu a)x = 0,\ \forall x \in R$. That is, $a(bx + \mu x) = 0,\ \forall x \in R$. This implies that $(eb + \mu e)x = 0,\ \forall x \in R$. Therefore $[eb + \mu e, 0] = 0$. That is, $[e, 0][b, \mu] = 0$. This gives $\bar{e}\bar{y} = 0$. Therefore $\bar{e} = RP(\bar{a})$. □

The following theorem gives a more general partial solution to Problem 1, we give the solution in which we replace integral domain K by any commutative ring.

Theorem 2.9. *Let R be a weakly Rickart $*$-ring and K be a commutative $*$-ring with unity such that R is a $*$-algebra over K satisfying the condition* (3) *of Theorem 2.4. Then R can be embedded in a Rickart $*$-ring with preservation of right projections.*

Proof. Let $\hat{R_1} = R_1/N = \{[a, \lambda] \mid (a, \lambda) \in R_1\}$ and $\hat{R_1}$ have unity element $u = [0, 1]$. By Lemma 2.7, it is enough to show that every self-adjoint element of $\hat{R_1}$ has the right projection. Let $[a, \lambda] \in \hat{R_1}$ be a self-adjoint element. If $\lambda = 0$, then $e = RP(a)$ and by Theorem 2.8, $\bar{e} = RP(\bar{a})$. Suppose $\lambda \neq 0$. Then by Lemma 2.6, there exists a largest projection g such that $ag = -\lambda g$. Now we show that $RP([a, \lambda]) = [-g, 1]$. Note that $[-g, 1]$ is a projection. Also, $[a, \lambda][-g, 1] = [-ag - \lambda g + a, \lambda] = [a, \lambda]$. Moreover, $[a, \lambda][b, \mu] = 0$ if and only if $[ab + \mu a + \lambda b, \lambda\mu] = 0$ if and only if $abx + \mu ax + \lambda bx + \lambda\mu x = 0,\ \forall x \in R$ if and only if $a(bx + \mu x) + \lambda(bx + \mu x) =$

$0,\ \forall x \in R$ if and only if $(a+\lambda e_x)(bx+\mu x)=0$, where $e_x = LP(bx+\mu x)$ if and only if $(a+\lambda e_x)e_x = 0,\ \forall x \in R$ if and only if $ae_x = -\lambda e_x,\ \forall x \in R$. Since g is the largest projection such that $ag = -\lambda g$. Therefore $e_x \le g$. This gives $e_x g = g e_x = e_x$. Therefore $[a,\lambda][b,\mu] = 0$ if and only if $(e_x - g)e_x = 0,\ \forall x \in R$ if and only if $(e_x - g)(bx+\mu x) = 0,\ \forall x \in R$ if and only if $-g(bx+\mu x)+e_x(bx+\mu x) = 0,\ \forall x \in R$ if and only if $-gbx-\mu gx+bx+\mu x = 0,\ \forall x \in R$ if and only if $[-gb-\mu g+b,\mu] = 0$ if and only if $[-g,1][b,\mu] = 0$. Hence $\hat{R_1}$ is a Rickart $*$-ring. □

3. Unitification of weakly p.q.-Baer $*$-rings

We recall the following examples of p.q.-Baer $*$-rings. This also shows how the class of p.q.-Baer $*$-rings is different than the class of Rickart $*$-rings.

Example 3.1. [3, Exercise 10.2.24.4] Let A be a domain, $A_n = A$ for all $n = 1,2,\ldots,$ and $B = \{(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n \ :\ a_n \text{ is eventually constant}\}$, which is a subring of $\prod_{n=1}^{\infty} A_n$. Take $R = M_n(B)$, where n is an integer such that $n > 1$ with transpose of matrix as an involution. Then R is a p.q.-Baer $*$-ring which is not quasi-Baer (hence not a quasi-Baer $*$-ring). Also, if A is commutative which is not Prüfer, then R is not a Rickart $*$-ring.

Example 3.2. [3, Exercise 10.2.24.5] Let R be a $*$-ring. If R is a right (or left) p.q.-Baer ring and $*$ is semiproper, then R is a p.q.-Baer $*$-ring. Hence, if R is biregular and $*$ is semiproper, then R is a p.q.-Baer $*$-ring.

Example 3.3. [8, Example 1.7] Let

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d,\ b \equiv 0,\ \text{and } c \equiv 0\ (mod\ 2) \right\}.$$

Consider involution $*$ on R as the transpose of the matrix. In [5, Example 2(1)], it is shown that R is neither right p.p. nor left p.p. (hence not a Rickart $*$-ring) but $r_R(uR) = \{0\} = 0R$, for any nonzero element $u \in R$. Therefore R is a p.q.-Baer $*$-ring.

Recall the following result which gives the condition on m and n, so that the matrix ring $M_n(\mathbb{Z}_m)$ is a Baer $*$-ring and hence a Rickart $*$-ring.

Corollary 3.4. [12, Corollary 7]

(i) *$M_n(\mathbb{Z}_m)$ is a Baer $*$-ring for $n \ge 2$ if and only if $n = 2$ and m is a square free integer whose every prime factor is of the form $4k+3$.*

(ii) *$\mathbb{Z}_m$ is a Baer $*$-ring if and only if m is a square free integer.*

The following example shows that the right projections in a $*$-ring need not be central covers.

Example 3.5. [8, Example 2.8] Let $A = M_2(\mathbb{Z}_3)$, which is a Baer $*$-ring (hence a p.q.-Baer $*$-ring and a Rickart $*$-ring) with transpose as an involution. There is an element $x \in A$ such that $RP(x)$ is not equal to $C(y)$, for any $y \in A$.

The following is a partial solution of the Problem 2 given in [8].

Theorem 3.6. [8, Theorem 4.6] *A weakly p.q.-Baer $*$-ring R can be embedded in a p.q.-Baer $*$-ring provided that there exists a ring K such that*

(1) *K is an integral domain with involution,*
(2) *R is a $*$-algebra over K,*
(3) *For any $\lambda \in K - \{0\}$, there exists a projection $e_\lambda \in R$ that is an upper bound for the central covers of the right annihilators of λ, that is, for $t \in R$, if $\lambda\ t = 0$, then $C(t) \leq e_\lambda$.*

Let $\tilde{R}$ denote the set of all projections in a $*$-ring R. In a weakly p.q.-Baer $*$-ring, the following is called the *condition (β)*: For any $0 \neq \lambda \in K, \exists\ e_\lambda \in \tilde{R}$ such that $\lambda x = 0$ implies that $C(x) \leq e_\lambda$, where K is a commutative $*$-ring with unity.

Lemma 3.7. *Let R be a weakly p.q.-Baer $*$-ring which is a $*$-algebra over a commutative $*$-ring K with unity satisfying the condition (β). Then for any $a \in R$ and $0 \neq \lambda \in K$, there exists a largest central projection g such that $ag = \lambda g$.*

Proof. On the similar lines as in the proof of Lemma 2.6. □

The following result leads to the solution of Problem 2.

Theorem 3.8. *With the notation as defined earlier, we have the following.*

(1) *The mapping $a \to \bar{a} = [a, 0]$ is an algebra homomorphism of R into $\hat{R}_1$.*
(2) *If $L(R) = \{x \in R : xy = 0,\ \forall y \in R\} = \{0\}$, then the mapping $a \to \bar{a}$ is injective and we may regard R as embedded in $\hat{R}_1$.*
(3) *If the involution $*$ is semiproper, then $[a, \lambda] = 0$ if and only if $[a^*, \lambda^*] = 0$. Hence $[a, \lambda]^* = [a^*, \lambda^*]$ defines involution in $\hat{R}_1$.*
(4) *If R is a weakly p.q.-Baer $*$-ring, $a \in R, C(a) = e$, then $C(\bar{a}) = \bar{e}$ in $\hat{R}_1$.*

Proof. (1) Obvious.
(2) To prove $\phi : R \to \hat{R}_1$ given by $\phi(a) = \bar{a}$ is injective, let $\phi(a) = \phi(b)$. Then $\bar{a} = \bar{b}$, that is, $[a, 0] = [b, 0]$. This gives $ax = bx, \forall x \in R$. Therefore $(a - b)x = 0,\ \forall x \in R$. This gives $a - b = 0$. Hence $a = b$.

(3) Suppose that R has a semiproper involution, therefore for $a \in R$, $a^*Ra = 0$ implies that $a = 0$. Now, $[a, \lambda] = 0$ if and only if $ax + \lambda x = 0$, $\forall x \in R$. Also, for any $r \in R$, $(x^*a + \lambda x^*)r(a^*x + \lambda^*x) = x^*ara^*x + x^*ar\lambda^*x + \lambda x^*ra^*x + \lambda x^*r\lambda^*x$ $= x^*\{a(ra^*x) + \lambda(ra^*x)\} + x^*\{a(r\lambda^*x) + \lambda(r\lambda^*x)\} = x^*0 + x^*0 = 0$. Therefore $[a, \lambda] = 0$ if and only if $(x^*a + \lambda x^*)R(a^*x + \lambda^*x) = 0$ if and only if $(a^*x + \lambda^*x) = 0$ if and only if $[a^*, \lambda^*] = 0$. Hence $[a, \lambda]^* = [a^*, \lambda^*]$ defines an involution in $\hat{R}_1$.
(4) Let R be a weakly p.q.-Baer $*$-ring, $a \in R$ and $C(a) = e$. Consider $\bar{a}\bar{e} = [a, e][e, 0] = [ae, 0] = [a, 0] = \bar{a}$. Also, $\bar{a}\hat{R}_1[b, \mu] = 0$ if and only if $\bar{a}\bar{e}\hat{R}_1[b, \mu] = 0$ if and only if $\bar{a}\hat{R}_1\bar{e}[b, \mu] = 0$ if and only if $[a, 0]\hat{R}_1[eb + \mu e, 0] = 0$ if and only if $[a, 0][x, \lambda][eb + \mu e, 0] = 0$ if and only if $[a(x + \lambda e)(eb + \mu e), 0] = 0$ if and only if $a(x + \lambda e)(eb + \mu e) = 0$ if and only if $aR(eb + \mu e) = 0$ if and only if $e(eb + \mu e) = 0$ if and only if $eb + \mu e = 0$ if and only if $(eb + \mu e)x = 0$, $\forall x \in R$ if and only if $[eb + \mu e, 0] = 0$ if and only if $[e, 0][b, \mu] = 0$. Therefore $C(\bar{a}) = \bar{e}$. □

Now we give the more general partial solution to the Problem 2, in which we replace integral domain K by any commutative ring with unity.

Theorem 3.9. *Let R be a weakly p.q.-Baer $*$-ring and K be a commutative $*$-ring with unity such that R is a $*$-algebra over K satisfying the condition (β). Then R can be embedded in a p.q.-Baer $*$-ring with preservation of central covers.*

Proof. Let $\hat{R}_1 = R_1/N = \{[a, \lambda] \mid (a, \lambda) \in R_1\}$. Note that $u = [0, 1]$ is a unity element of $\hat{R}_1$. We show that $\hat{R}_1$ is a p.q.-Baer $*$-ring. It is enough to show that for every element $x \in \hat{R}_1$, there exists a central projection $e \in \hat{R}_1$ such that: (1) $xe = x$, (2) $x\hat{R}_1y = 0$ if and only if $ey = 0$. Let $x = [a, \lambda] \in \hat{R}_1$. If $\lambda = 0$, let $C(a) = e$. By Theorem 3.8, $C(\bar{a}) = \bar{e}$. Suppose $\lambda \neq 0$, then by Lemma 3.7, there exists the largest central projection g such that $ag = -\lambda g$. Clearly, $[-g, 1]$ is a central projection. Also, $[a, \lambda][-g, 1] = [-ag + a - \lambda g, \lambda] = [a, \lambda]$, that is, $xe = x$ with $e = [-g, 1], x = [a, \lambda]$. Suppose $[a, \lambda]\hat{R}_1[b, \mu] = 0$. Therefore $[a, \lambda][r, 0][b, \mu] = 0$ for all $r \in R$. This gives $[arb + \lambda rb + \mu ar + \lambda\mu r, 0] = 0$ for all $r \in R$. This implies $arbx + \lambda rbx + \mu arx + \lambda\mu rx = 0$ for all $r, x \in R$. That is, $ar(bx + \mu x) + \lambda r(bx + \mu x) = 0$ for all $r, x \in R$. Therefore $(ar + \lambda re_x)(bx + \mu x) = 0$, where $e_x = C(bx + \mu x)$. This gives $(a + \lambda e_x)r(bx + \mu x) = 0$ for all $r \in R$. That is, $(a + \lambda e_x)R(bx + \mu x) = 0$. Therefore $(a + \lambda e_x)e_x = 0$. Hence $ae_x = -\lambda e_x$. Since g is a largest central projection such that $ag = (-\lambda)g$, therefore $e_x \leq g$. Therefore $(1 - g)e_x = 0$. This gives $(1 - g)e_x(bx + \mu x) = 0$. Thus $(1 - g)(bx + \mu x) = 0$ for all $x \in R$. Hence $bx + \mu x - gbx - \mu gx = 0$ for all $x \in R$. Therefore $[-gb - \mu g + b, \mu] = 0$, that is, $[-g, 1][b, \mu] = 0$. Hence $\hat{R}_1$ is a p.q.-Baer $*$-ring. □

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