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ON A GENERALIZATION OF z-IDEALS IN MODULES OVER COMMUTATIVE RINGS

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Abstract. In this article, we introduce and study the concept of z-submodules as a generalization of z-ideals. Let M be a module over a commutative ring with identity R. A proper submodule N of M is called a z-submodule if for any $x \in M$ and $y \in N$ such that every maximal submodule of M containing y also contains x, then $x \in N$ as well. We investigate the properties of z-submodules, particularly considering their stability with respect to various module constructions. Let $\mathcal{Z}({}_RM)$ denote the lattice of z-submodules of M ordered by inclusion. We are concerned with certain mappings between the lattices $\mathcal{Z}({}_RR)$ and $\mathcal{Z}({}_RM)$. The mappings in question are $\phi : \mathcal{Z}({}_RR) \to \mathcal{Z}({}_RM)$ defined by setting for each z-ideal I of R, $\phi(I)$ to be the intersection of all z-submodules of M containing IM and $\psi : \mathcal{Z}({}_RM) \to \mathcal{Z}({}_RR)$ defined by $\psi(N)$ is the colon ideal $(N : M)$. It is shown that ϕ is a lattice homomorphism, and if M is a finitely generated multiplication module, then ψ is also a lattice homomorphism. In particular, $\mathcal{Z}({}_RM)$ is a homomorphic image of $\mathcal{R}({}_RM)$, the lattice of radical submodules of M. Finally, we show that if Y is a finite subset of a compact Hausdorff P-space X, then every submodule of the $C(X)$-module $\mathbb{R}^Y$ is a z-submodule of $\mathbb{R}^Y$.

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1. Introduction

We assume all rings are commutative with identity and all modules are unitary. In 1957, Kohls [11] was the first to use the concept of z-ideals in the study of the ring of real-valued continuous functions $C(X)$ on a completely regular Hausdorff space X. Nearly two decades later, Mason [13] extended the concept of z-ideals to any commutative ring with identity. In recent years, the theory of z-ideals has been developed in several directions (see, for example, [1,2,3,5,6,10,14]). In this article, we introduce the concept of z-submodules generalizing z-ideals. This article consists of four sections. In section 2, we study the basic properties of z-submodules and investigate their behavior under some standard operations in commutative algebra. Let R be a ring and M an R-module. Also, let $\mathrm{Max}(M)$ denote the set of maximal submodules of M. For each $x \in M$, we set

$$\mathcal{M}(x) := \{K \in \mathrm{Max}(M) \mid \ x \in K\}.$$

A proper submodule N of M is called a *z-submodule* if for any $x \in M$ and $y \in N$, $\mathcal{M}(x) \supseteq \mathcal{M}(y)$ implies that $x \in N$. If $\mathcal{M}(y) = \emptyset$ for some $y \in N$, then N is a z-submodule of M if and only if $N = M$. Evidently, z-submodules of the R-module R coincide with the z-ideals of R. Maximal submodules of any R-module M are z-submodules of M. For any two submodules N and L of M, we take $(N : L) := \{r \in R \mid rL \subseteq N\}$ which is the colon ideal of L into N. It is shown that if N is a z-submodule of M, then $(N : M)$ is a z-ideal of R (Lemma 2.2). For any submodule N of M, the *z-taking* of N, denoted N_z, is the intersection of all z-submodules of M containing N. It is clear that N is a z-submodule of M if and only if $N_z = N$.

Let M be an R-module. A proper submodule P of M is called a *prime submodule* if for $\mathrm{p} = (P : M)$, whenever $rm \in P$ for $r \in R$ and $m \in M$, we have $r \in \mathrm{p}$ or $m \in P$. The *radical* of a submodule N of M, denoted $\mathrm{rad}\, N$, is the intersection of all prime submodules of M containing N or, in case there are no such prime submodules, $\mathrm{rad}\, N$ is M. For an ideal I of a ring R, we assume that $\sqrt{I}$ denotes the radical of I. A submodule N of M is called a *radical submodule* if $\mathrm{rad}\, N = N$ (For more information on prime and radical submodules, the reader may consult [12] for example). It is shown that every z-submodule of a multiplication module is a radical submodule (Proposition 2.4). It is seen that the z-taking of submodules enjoy analogs of many properties of radical submodules. For instance, it is shown that for any ideal I of R, $(IM)_z = (I_z M)_z$ (Theorem 2.6). For any subset S of an R-module M, let $\mathcal{M}(S)$ denote the set of maximal submodules of M containing S. As a generalization of z-submodules, any submodule N of M is called a *strongly z-submodule* of M or briefly *sz-submodule* if for any two finite subsets S and T of M such that $S \subseteq N$ and $\mathcal{M}(S) \subseteq \mathcal{M}(T)$, we have $T \subseteq N$. Also, an I of R is called a *sz-ideal* if it is a z-submodule of the R-module R. It is shown that, if M is a finitely generated faithful multiplication R-module and I is a sz-ideal of R, then IM is a z-submodule of M (Theorem 2.7). Note that if $R = C(X)$, then by [1, p. 255] the concept of z-ideal coincides with the sz-ideal. Using this fact, it is proved that if $R = C(X)$, then every sz-submodule of a finitely generated faithful multiplication R-module is an intersection of prime z-submodules (Corollary 2.9). It is shown that if F is a free R-module, then for any z-ideal I of R, IF is a z-submodule of F (Corollary 2.16) and in particular, $(IF)_z = I_z F$ (Corollary 2.17).

Let M be an R-module. The collection $\mathcal{Z}({}_RM)$ consisting of all z-submodules of M forms a lattice with the operations $N \vee L = (N + L)_z$ and $N \wedge L = N \cap L$, for all z-submodules N and L of M. Recently, various properties of certain mappings between different types of module lattices have been examined by the second author and others (see [9,15,16,17,20]) whose motivation sterns back to P. F. Smith's works (see [23,24,25]). In section 3, we will deal with the mappings $\phi : \mathcal{Z}({}_RR) \to \mathcal{Z}({}_RM)$ defined by $\phi(I) = (IM)_z$ and $\psi : \mathcal{Z}({}_RM) \to \mathcal{Z}({}_RR)$ defined by $\psi(N) = (N : M)$. It is shown that ϕ is a lattice homomorphism (Lemma 3.1), but ψ is not in general (Example 3.3). In particular, if M is a finitely generated multiplication R-module, then $\mathcal{Z}({}_RM)$ is a homomorphic image of the lattice $\mathcal{R}({}_RM)$ consisting of all radical submodules of M (Corollary 3.2). It is also shown that if $R = C(X)$ and M is a finitely generated multiplication R-module, then ψ is a lattice homomorphism (Theorem 3.4). In particular, if M is a finitely generated faithful multiplication R-module, then ϕ is a lattice isomorphism, and ψ is its inverse (Corollary 3.11).

Finally, in Section 4, we present a non-trivial example of a finitely generated faithful multiplication module over the ring of continuous functions $C(X)$, where X is a compact Hausdorff P-space, all of whose submodules are z-submodules. Indeed, if Y is a finite subset of a compact Hausdorff space X, then $\mathbb{R}^Y$ consisting of all real-valued functions with domain Y is a multiplication $C(X)$-module (Theorem 4.1), and if in addition X is a P-space, then $\mathbb{R}^Y$ is a flat $C(X)$-module (Theorem 4.2). In particular, $\mathbb{R}^Y$ is a finitely generated faithful multiplication $C(X)$-module (Corollary 4.3), and therefore every submodule of it is a z-submodule of $\mathbb{R}^Y$ (Corollary 4.4).

2. z-Submodules

Let M be an R-module and N be a submodule of M. Recall that $\mathcal{M}(x)$ denotes the set of all maximal submodules of M containing x. To begin, let's consider the following lemma.

Lemma 2.1. *Let R be a ring and M an R-module. If for any $r, s \in R$, $\mathcal{M}(r) \subseteq \mathcal{M}(s)$, then $\mathcal{M}(rm) \subseteq \mathcal{M}(sm)$ for all $m \in M$.*

Proof. Let $m \in M$ and $K \in \mathcal{M}(rm)$. If $m \in K$, then $sm \in K$ and so $K \in \mathcal{M}(sm)$, otherwise $(K : Rm)$ is a maximal ideal of R and in particular, $(K : Rm) \in \mathcal{M}(r)$

(note that if K is a maximal submodule of M, then M/K is a non-zero simple R-module, and hence $(K : M) = \mathrm{Ann}(M/K)$ is a maximal ideal of R. In particular, since $(K : M) \subseteq (K : Rm)$ for all $m \in M$, it follows that $(K : Rm)$ is a maximal ideal of R). So by the assumption $(K : Rm) \in \mathcal{M}(s)$. Hence we have $sm \in K$ which implies that $K \in \mathcal{M}(sm)$. □

The next result relates the z-submodules of an R-module M to the z-ideals of R.

Lemma 2.2. *Let M be an R-module. If N is a z-submodule of M, then $(N : M)$ is a z-ideal of R.*

Proof. Assume that $\mathcal{M}(r) \subseteq \mathcal{M}(s)$ for $r \in (N : M)$ and $s \in R$. By Lemma 2.1, we have $\mathcal{M}(rm) \subseteq \mathcal{M}(sm)$ for all $m \in M$. Now, since N is a z-submodule of M, we conclude that $sm \in N$ for all $m \in M$, and so $s \in (N : M)$. □

The following lemma collects some frequently used facts on z-taking of submodules.

Lemma 2.3. *Let N and L be submodules of an R-module M and $\{N_i\}_{i\in I}$ be a collection of submodules of M. Then:*

(1) $N \subseteq N_z$;
(2) *If* $N \subseteq L$, *then* $N_z \subseteq L_z$;
(3) $N_z = (N_z)_z$;
(4) $(\cap_{i\in I} N_i)_z \subseteq \cap_{i\in I}(N_i)_z$;
(5) $(\sum_{i\in I} N_i)_z = (\sum_{i\in I}(N_i)_z)_z$;
(6) $(N : M)_z \subseteq (N_z : M)$;
(7) $\sqrt{(N : M)} \subseteq (N_z : M)$.

Proof. (1)-(5) are straightforward.
(6) It is clear that for any submodule N of M, $(N : M) \subseteq (N_z : M)$. Thus by Lemma 2.2, $(N : M)_z \subseteq (N_z : M)_z = (N_z : M)$.
(7) Since every z-ideal is radical, we conclude by Lemma 2.2 that $\sqrt{(N : M)} \subseteq \sqrt{(N_z : M)} = (N_z : M)$. □

An R-module M is called a *multiplication* R-module, if for every submodule N of M, there exists an ideal I of R such that $N = IM$. It is easy to see that M is a multiplication R-module if and only if for each submodule N of M, $N = (N : M)M$. Cyclic modules, ideals of Dedekind domains, and ideals of regular rings are well-known examples of multiplication modules. It is noted that by Lemma 2.2 and [5, Corollary 1], every z-submodule of a multiplication R-module M is of the form nM for some square-free integer n.

As shown in [13, p. 281], every z-ideal of a ring R is a radical ideal of R. Using this fact, we give a similar result for z-submodules of multiplication modules.

Proposition 2.4. *Every z-submodule of any multiplication R-module M is a radical submodule of M.*

Proof. Let N be a z-submodule of M. Then by [7, Theorem 2.12] and Lemma 2.2, we have $\operatorname{rad} N = \sqrt{(N : M)}M = (N : M)M = N$. □

As stated in [12, Proposition 3.1], for each radical ideal I of a ring R and any finitely generated R-module M, we have $(IM : M) = I$ if and only if $I \supseteq \operatorname{Ann}(M)$. This fact is used in the following proposition.

Proposition 2.5. *Let M be a finitely generated R-module and let I be an ideal of R. Then $(IM : M)_z = (I + \operatorname{Ann(M)})_z$.*

Proof. Let J be a z-ideal of R containing $(IM : M)$. Then $\operatorname{Ann(M)} \subseteq J$ and $I \subseteq (IM : M) \subseteq J$ which implies $(I + \operatorname{Ann(M)}) \subseteq J$. Therefore $(I + \operatorname{Ann(M)})_z \subseteq (IM : M)_z$. For the revers inclusion, let J be a z-ideal of R containing $(I+\operatorname{Ann(M)})$. Then since J is a radical ideal of R, $(IM : M) \subseteq (JM : M) = J$. Hence we have $(IM : M)_z \subseteq (I + \operatorname{Ann(M)})_z$. □

Theorem 2.6. *Let M be an R-module. For any ideal I of R, $(IM)_z = (I_zM)_z$. In particular, if M is a multiplication R-module, then for each submodule N of M, $N_z = ((N : M)_zM)_z$.*

Proof. Assume that K is a z-submodule of M containing IM. Since $(K : M)$ is a z-ideal of R, $I_z \subseteq (K : M)$ and hence $I_zM \subseteq (K : M)M \subseteq K$. It follows that $(I_zM)_z \subseteq (IM)_z$. The reverse inclusion is obvious. The "in particular" part follows by taking $I = (N : M)$. □

Let M be an R-module. For any subset S of M, we recall that $\mathcal{M}(S)$ is the set of maximal submodules of M containing S. Let $\mathcal{M}_S$ denote the intersection of all elements of $\mathcal{M}(S)$. Evidently, N is a sz-submodule of M iff for any finite subset S of N, $\mathcal{M}_S \subseteq N$ (see for example [1,2] for more details about sz-ideals).

Theorem 2.7. *Let R be a ring and M be a finitely generated R-module. Then:*

(1) *If M is a faithful multiplication R-module and I is a sz-ideal of R, then IM is a sz-submodule (and therefore a z-submodule) of M;*

(2) *If M is a faithful R-module and IM is a z-submodule of M, then I is a z-ideal of R.*

Proof. (1) Let $M = Rx_1 + Rx_2 + \cdots + Rx_n$. Moreover, let $S = \{y_1, \cdots, y_s\}$ and $T = \{z_1, \cdots, z_t\}$ be two subsets of M such that $S \subseteq IM$ and $\mathcal{M}(S) \subseteq \mathcal{M}(T)$. Since $S \subseteq IM$, there exist $r_{ij} \in I$ such that for any $1 \leq i \leq s$, $y_i = \sum_{j=1}^{n} r_{ij}x_j$. Also, since $T \subseteq (RT : M)M$, there exist $s_{ij} \in (RT : M)$ such that for any $1 \leq i \leq t$, $z_i = \sum_{j=1}^{n} s_{ij}x_j$. We set $U = \{r_{i_j} \mid 1 \leq i \leq s,\ 1 \leq j \leq n\}$ and $V = \{s_{i_j} \mid 1 \leq i \leq t,\ 1 \leq j \leq n\}$, and show that $\mathcal{M}(U) \subseteq \mathcal{M}(V)(*)$. For this, we assume that $\mathrm{m} \in \mathcal{M}(U)$. It follows that $S \subseteq UM \subseteq \mathrm{m}M$. Now, since by [7, Theorem 2.5] $\mathrm{m}M$ is a maximal submodule of M, we have $\mathrm{m}M \in \mathcal{M}(S)$ and so $\mathrm{m}M \in \mathcal{M}(T)$. Therefore $V \subseteq (RT : M) \subseteq (\mathrm{m}M : M) = \mathrm{m}$, which yields that $\mathrm{m} \in \mathcal{M}(V)$. Thus $(*)$ holds and since I is a *sz*-ideal, we have $V \subseteq I$. Then $T \subseteq IM$, as desired.
(2) Since I is a radical ideal of R, we have $(IM : M) = I$ by [12, Proposition 3.1]. Thus, the result follows from Lemma 2.2. □

Let M be an R-module. For any submodule N of M, we let N_{sz} denote the intersection of all *sz*-submodules of M containing N. Note that, since any *sz*-submodule is a z-submodule, we have $N_z \subseteq N_{sz}$.

Corollary 2.8. *Let R be a ring and M be a finitely generated faithful multiplication R-module and N a submodule of M. Then $(N : M)_z \subseteq (N_z : M) \subseteq (N : M)_{sz}$. In particular, if $R = C(X)$, then $(N : M)_z = (N_z : M) = (N : M)_{sz}$.*

Proof. By Lemma 2.3(6), $(N : M)_z \subseteq (N_z : M)$. To establish the reverse inclusion, we assume that I is a *sz*-ideal of R containing $(N : M)$. Then $N \subseteq IM$, and hence by Theorem 2.7(1), we have $N_z \subseteq IM$, and so $(N_z : M) \subseteq I$. Therefore $(N_z : M) \subseteq (N : M)_{sz}$, as required. The "in particular part" follows from the previous part and a fact given in [1, p. 225] which follows that the concept of z-ideal coincides with the *sz*-ideal in $C(X)$. □

Corollary 2.9. *Let $R = C(X)$ and M be a finitely generated faithful multiplication R-module. Then every sz-submodule of M is an intersection of prime z-submodules of M.*

Proof. Let N be a *sz*-submodule of M. Then N is a z-submodule of M and so $(N : M)$ is a radical ideal of R. Thus $(N : M) = \cap_{\mathrm{p} \in \mathrm{Min}(N:M)} \mathrm{p}$. Since $(N : M)$ is a z-ideal of R, it is also a *sz*-ideal of R, and hence by [1, Theorem 3.13], every $\mathrm{p} \in \mathrm{Min}(N : M)$ is a *sz*-ideal of R. Thus by [7, Lemma 2.10 and Corollary 2.11] $\mathrm{p}M \in \mathrm{Min}(N)$

for all $\mathrm{p} \in \mathrm{Min}(N : M)$, and by Theorem 2.7(1), these pM's are z-submodules of M. Now, since $N = (N : M)M = (\bigcap_{\mathrm{p}\in\mathrm{Min}(N:M)} \mathrm{p})M = \bigcap_{\mathrm{p}\in\mathrm{Min}(N:M)} \mathrm{p}M$ by [7, Theorem 1.6], we conclude that N is an intersection of prime z-submodules of M. □

Theorem 2.10. *If I and J are two ideals in R, then*

$$(IJM)_z = ((I \cap J)M)_z = (IM)_z \cap (JM)_z.$$

In particular, for any positive integer n, $(I^n M)_z = (IM)_z$.

Proof. To establish the given equality, it suffices to show that $(IM)_z \cap (JM)_z$ is the smallest z-submodule containing IJM. For this, let K be a z-submodule of M containing IJM. Then $(K : M)$ is a z-ideal of R containing IJ, and so $(K : M) = \bigcap_{\mathrm{p}\in\mathrm{Min}(K:M)} \mathrm{p}$. Consequently, for every $\mathrm{p} \in \mathrm{Min}(K : M)$, we have $I \subseteq \mathrm{p}$ or $J \subseteq \mathrm{p}$. In any case, $I_z M \subseteq \mathrm{p}M$ or $J_z M \subseteq \mathrm{p}M$. Thus for any $\mathrm{p} \in \mathrm{Min}(K : M)$, we have $(I_z M)_z \subseteq (\mathrm{p}M)_z$ or $(J_z M)_z \subseteq (\mathrm{p}M)_z$ which implies that $(IM)_z \subseteq K$ or $(JM)_z \subseteq K$. Therefore $(IM)_z \cap (JM)_z \subseteq K$, as required. The "in particular" part is obtained easily by induction on n. □

Theorem 2.11. *Let M and M' be R-modules. Let $f : M \longrightarrow M'$ be a surjective R-module homomorphism, and* $\mathrm{Ker}\, f$ *is contained in each maximal submodule of M. Then:*

1. *If M is a finitely generated R-module and N' is a z-submodule of M', then $f^{-1}(N')$ is a z-submodule of M;*
2. *If M' is a finitely generated R-module and N is a submodule of M such that $N + \mathrm{Ker}\, f$ is a z-submodule of M, then $f(N)$ is a z-submodule of M'.*

Proof. (1) Suppose that N' is a z-submodule of M', and $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ for $a \in f^{-1}(N')$ and $b \in M$. We show that $\mathcal{M}(f(a)) \subseteq \mathcal{M}(f(b))$. For this, we let $K' \in \mathrm{Max}(M)$ and $f(a) \in K'$. Since M is finitely generated and $f^{-1}(K') \neq M$, there exists a maximal submodule K of M containing $f^{-1}(K')$. Note that if $f(K) = M'$, we get $M = K + \mathrm{Ker}\, f = K$, which is a contradiction. Hence, we have $f(K) = K'$. Then, by hypothesis, $f^{-1}(K') = K$. Since $a \in f^{-1}(K')$, we have $f^{-1}(K') \in \mathcal{M}(a)$. So, $b \in f^{-1}(K')$, and $f(b) \in K'$.

(2) Suppose that $N + \mathrm{Ker}\, f$ is a z-submodule of M, $\mathcal{M}(f(a)) \subseteq \mathcal{M}(f(b))$ for $f(a) \in f(N)$ and $b \in M$. We show that $\mathcal{M}(a) \subseteq \mathcal{M}(b)$. For this, we assume that $K \in \mathrm{Max}(M)$ and $a \in K$. It is noted that if $f(K) = M'$, since f is surjective, we have $M = K + \mathrm{Ker}\, f = K$, a contradiction. Thus since M' is finitely generated and $f(K) \neq M'$, there exists $L' \in \mathrm{Max}(M')$ such that $f(K) \subseteq L'$. Letting $L' = f(L)$, we conclude that $K \subseteq L + \mathrm{Ker}\, f \subseteq M$. Consequently, $K = L + \mathrm{Ker}\, f$ (note that

if $L + \operatorname{Ker} f = M$, then we get $L' = f(L) = f(M) = M'$ which is a contradiction). Hence we have $f(K) \in \operatorname{Max}(M')$ and $f(K) \in \mathcal{M}(f(a))$. It follows that $f(b) \in f(K)$ and so $b \in K + \operatorname{Ker} f = K$, we are done. Now, since $\mathcal{M}(a) \subseteq \mathcal{M}(b)$ and $a \in N + \operatorname{Ker} f$, we have $b \in N + \operatorname{Ker} f$. Thus $f(b) \in f(N)$, as required. □

The following example illustrates Theorem 2.11.

Example 2.12. Let $\mathbb{Z}$ be the ring of integers and $M_n = \mathbb{Z}/p^n\mathbb{Z}$ be the $\mathbb{Z}$-module of integers modulo $p^n\mathbb{Z}$. Since M_n is cyclic, it is clear that every proper submodule of M_n is of the form $(\overline{p^k})$ for some $1 \leq k < n$. In particular, $(\overline{p})$ is the only maximal submodule of M_n, and so $\mathcal{M}(\overline{p^k}) \subseteq \mathcal{M}(\overline{p})$. It follows that if $k > 1$, then $(\overline{p^k})$ is not a z-submodule of M_n. Now, for any two positive integers m, n with $m > n$, we consider the mapping $f : M_m \longrightarrow M_n$ defined by $f(x+p^m\mathbb{Z}) = x+p^n\mathbb{Z}$. Evidently, f is a surjective non-isomorphism whose kernel is contained in $(\overline{p})$, and Theorem 2.11 holds by considering $N = (\overline{p})$ modulo $p^n\mathbb{Z}$ and $N' = (\overline{p})$ modulo $p^m\mathbb{Z}$.

Corollary 2.13. *Let M be a finitely generated R-module and L be a submodule of M contained in each maximal submodule of M. If N is a z-submodule of M containing L, then N/L is a z-submodule of M/L.*

Proof. Consider the natural projection $\pi : M \to M/L$ and apply Theorem 2.11(2). □

As usual, $\operatorname{Spec}(M)$ denotes the set of prime submodules of M.

Proposition 2.14. *Let R be a ring, M a multiplication R-module and $S = R \setminus \cup_{P\in \operatorname{Spec(M)}}(P : M)$. If N is a z-submodule of M, then $S^{-1}N$ is a z-submodule of $S^{-1}M$.*

Proof. Suppose that N is a z-submodule of M, $\mathcal{M}(\frac{x}{s}) \subseteq \mathcal{M}(\frac{y}{t})$ and $\frac{x}{s} \in S^{-1}N$. Then $\frac{x}{s} = \frac{n}{s'}$ for some $n \in N$ and $s' \in S$. It follows that $us'x = usn \in N$ for some $u \in S$. We first show that $\mathcal{M}(us'x) \subseteq \mathcal{M}(y)$. For this, we let $P \in \operatorname{Max}(M)$ and $us'x \in P$. Now since $us' \notin (P : M)$, then we get $x \in P$. This implies that $\frac{x}{s} \in S^{-1}P$. Since M is a multiplication R-module, $S^{-1}M$ is clearly a multiplication $S^{-1}R$-module, and thus by [7, Theorem 2.5], $S^{-1}P \subseteq S^{-1}Q$ for some maximal submodule $S^{-1}Q$ of $S^{-1}M$. In particular, by [18, Theorem 3.1], Q is a prime submodule of M and $(Q : M) \cap S = \emptyset$. Therefore $P \subseteq Q$ and so by maximality of P, $P = Q$. It follows that $S^{-1}P = S^{-1}Q$, and so $S^{-1}P \in \mathcal{M}(\frac{x}{s})$. Hence we have $\frac{y}{t} \in S^{-1}P$ which implies that $y \in P$, and therefore $P \in \mathcal{M}(y)$. Now, since N is a z-submodule of M we have $y \in N$, and so $\frac{y}{t} \in S^{-1}N$, as required. □

Theorem 2.15. *Let $\{M_i\}_{i\in I}$ be a non-empty collection of R-modules and $M = \oplus_{i\in I} M_i$. If N_i is a z-submodule of M_i for each $i \in I$, then $N = \oplus_{i\in I} N_i$ is a z-submodule of M.*

Proof. Let $\{x_i\} \in N$, $\{y_i\} \in M$, and assume that $\mathcal{M}(\{x_i\}) \subseteq \mathcal{M}(\{y_i\})$. We first show that $\mathcal{M}(x_i) \subseteq \mathcal{M}(y_i)$ for all $i \in I$. For this, we let $K \in \mathcal{M}(x_j)$ for fixed $j \in I$. Thus $\{x_i\} \in K \oplus (\oplus_{i\neq j} M_i)$. Now since $K \oplus (\oplus_{i\neq j} M_i) \in \mathcal{M}(\{x_i\})$, we have $\{y_i\} \in K \oplus (\oplus_{i\neq j} M_i)$. Consequently, we can conclude that $y_j \in K$, which means that $\mathcal{M}(x_j) \subseteq \mathcal{M}(y_j)$. Now, since N_i's are z-submodules and $x_i \in N_i$, we have $y_i \in N_i$. Therefore $\{y_i\} \in N$, as desired. □

Corollary 2.16. *Let F be a free R-module and I be a z-ideal of R. Then IF is a z-submodule of F.*

Proof. It is clear that for any ideal I, the R-module IF is isomorphic to a direct sum of I's. Now the result follows from Theorem 2.15. □

Corollary 2.17. *Let F be a free R-module and I be an ideal of R. Then $(IF)_z = I_z F$.*

Proof. First note that for any ideal I, we have $I_z = (IF : F)_z \subseteq ((IF)_z : F)$ which shows $I_z F \subseteq (IF)_z$. For the reverse inclusion, let J be a z-ideal of R containing I. By Corollary 2.16, JF is a z-submodule of F containing $(IF)_z$ and so $(IF)_z \subseteq \cap\{JF \mid J \text{ is a } z\text{-ideal of } R\}$. Thus, by [21, p. 51], $(IF)_z \subseteq (\cap J)F$ where J runs through the set of z-ideals containing I, namely $(IF)_z \subseteq I_z F$, as required. □

3. Mappings between lattices of z-submodules

Let R be a ring and M be an R-module. We recall that the collection of z-submodules of M forms a lattice with respect to inclusion order for which $N \vee L = (N + L)_z$ and $N \wedge L = N \cap L$ are respectively the supremum and infimum of any two element set $\{N, L\}$ of z-submodules of M. We shall denote the lattice of z-submodules by $\mathcal{Z}({}_R M)$. It should be noted that by [3, Example 2.3], the finite sum of z-ideals of a ring R is not necessarily a z-ideal, and so $\mathcal{Z}({}_R M)$ is not in general a sublattice of the usual lattice $\mathcal{L}({}_R M)$ consisting of all submodules of M. (Of course, if $R = C(X)$ is the ring of continuous functions on a completely regular Hausdorff space X, then by [8, p. 198], any finite sum of z-ideals is a z-ideal.)

For lattices L and L', a map $f : L \to L'$ is a homomorphism of lattices, if $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$. Note the following result.

Lemma 3.1. *Let R be a ring and M an R-module. Then*

(1) *The mapping $\phi : \mathcal{Z}({}_RR) \to \mathcal{Z}({}_RM)$ defined by $\phi(I) = (IM)_z$ is a lattice homomorphism;*

(2) *The mapping $\psi : \mathcal{Z}({}_RM) \to \mathcal{Z}({}_RR)$ defined by $\psi(N) = (N : M)$ is a lattice homomorphism if and only if $((N + L)_z : M) = ((N : M) + (L : M))_z$ for all z-submodules N and L of M.*

Proof. (1) First, we verify that ϕ preserves the operation $\vee$. For this, let $I, J \in \mathcal{Z}({}_RR)$. Using Lemma 2.3(5) and Theorem 2.6, we have

$$\begin{aligned} \phi(I \vee J) &= \phi((I + J)_z) = ((I + J)_z M)_z = ((I + J)M)_z \\ &= (IM + JM)_z = ((IM)_z + (JM)_z)_z \\ &= (IM)_z \vee (JM)_z = \phi(I) \vee \phi(J). \end{aligned}$$

Moreover, by Theorem 2.10, we have

$$\phi(I \wedge J) = \phi(I \cap J) = ((I \cap J)M)_z = (IM)_z \cap (JM)_z = \phi(I) \wedge \phi(J).$$

(2) Clearly for any $N, L \in \mathcal{Z}({}_RM)$ we have

$$\psi(N \wedge L) = (N \cap L : M) = (N : M) \cap (L : M) = \psi(N) \wedge \psi(L).$$

Thus ψ is a lattice homomorphism if and only if $\psi(N \vee L) = \psi(N) \vee \psi(L)$ if and only if $((N + L)_z : M) = ((N : M) + (L : M))_z$. $\square$

Let M be an R-module. It is easy to see that the set $\mathcal{R}({}_RM)$ consisting of radical submodules of M is a lattice with the operations $N \vee L = \mathrm{rad}(N + L)$ and $N \wedge L = N \cap L$ for all radical submodules N and L of M. As shown in [15, Theorem 2.11], if M is a finitely generated multiplication R-module, then $\sigma : \mathcal{R}({}_RR) \to \mathcal{R}({}_RM)$ given by $\sigma(N) = (N : M)$ is a lattice homomorphism. Also, as stated in [10, page 5], $\kappa : \mathcal{R}({}_RR) \to \mathcal{Z}({}_RR)$ defined by $\kappa(I) = I_z$ is a lattice homomorphism. Considering these lattice homomorphisms, we have the following result:

Corollary 3.2. *Let M be an R-module. If M is a finitely generated multiplication R-module. Then the assignment $N \mapsto N_z$ is a lattice epimorphism from $\mathcal{R}({}_RM)$ to $\mathcal{Z}({}_RM)$.*

Proof. Considering the composition $\mathcal{R}({}_RM) \xrightarrow{\sigma} \mathcal{R}({}_RR) \xrightarrow{\kappa} \mathcal{Z}({}_RR) \xrightarrow{\phi} \mathcal{Z}({}_RM)$ of lattice homomorphisms ϕ, σ and κ, and by using Theorem 2.6, we get that

$$(\phi\kappa\sigma)(N) = \phi\kappa((N : M)) = \phi((N : M)_z) = ((N : M)_z M)_z = ((N : M)M)_z = N_z,$$

which indicates the rule of $\phi\kappa\sigma$. Moreover, by Proposition 2.4, the lattice homomorphism $\phi\kappa\sigma$ is surjective. □

Note that ψ is not generally a lattice homomorphism, as the following example shows.

Example 3.3. Let V be a vector space with a dimension greater than one over a field F, and N and L be two proper subspaces of V such that $V = N \oplus L$. Then $((N+L)_z : V) = (V : V) = F$, while $((N : M) + (L : M))_z = ((0))_z = (0)$. Thus by Lemma 3.1, $\psi : \mathcal{Z}({}_RM) \to \mathcal{Z}({}_RR)$ is not a lattice homomorphism.

It will be convenient for us to call an R-module M a ψ-module if the mapping ψ, given in Lemma 3.1, is a homomorphism.

Theorem 3.4. *Let $R = C(X)$ and M a finitely generated multiplication R-module. Then M is a ψ-module. In particular, every cyclic module is a ψ-module.*

Proof. Let N and L be submodules of M. Now by Proposition 2.5 and Corollary 2.8, we have

$$\begin{aligned}
((N : M) + (L : M))_z &= ((N : M) + (0 : M/L))_z \\
&= ((N : M)(M/L) : M/L)_z \\
&= (((N : M)M + L)/L : M/L)_z \\
&= ((N : M)M + L : M)_z \\
&= (N + L : M)_z \\
&= ((N + L)_z : M).
\end{aligned}$$

Thus by Lemma 3.1, M is a ψ-module. The first part obtains the "in particular" part. □

Corollary 3.5. *Let $R = C(X)$ and M be an R-module. If every finitely generated submodule of M is a ψ-module, then $R = (Rx : Ry) + (Ry : Rx)$ for all elements $x, y \in M$. If, in addition, every submodule of M is multiplication, then the converse holds.*

Proof. For the first part, let $x, y \in M$. Since $Rx + Ry$ is a ψ-module, we have

$$\begin{aligned}
R &= ((Rx + Ry)_z : Rx + Ry) \\
&= ((Rx : Rx + Ry) + (Ry : Rx + Ry))_z \\
&= ((Rx : Rx) \cap (Rx : Ry) + (Ry : Rx) \cap (Ry : Ry))_z \\
&= ((Rx : Ry) + (Ry : Rx))_z.
\end{aligned}$$

Thus $R = (Rx : Ry) + (Ry : Rx)$. For the converse, M is a ψ-module by Theorem 3.4 and [23, Corollary 3.9]. □

Theorem 3.6. *Let ϕ and ψ be a before. Then, the following hold.*

(1) $\psi\phi\psi = \psi$.
(2) $\phi\psi\phi = \phi$.

Proof. (1) Let N be a z-submodule of M. Then

$$\psi\phi\psi(N) = \psi\phi((N : M)) = \psi(((N : M)M)_z) = (((N : M)M)_z : M).$$

Now since N is a z-submodule of M, we have $((N : M)M)_z \subseteq N$, and so $(((N : M)M)_z : M) \subseteq (N : M)$. Moreover, $(N : M) \subseteq ((N : M)M : M) \subseteq (((N : M)M)_z : M)$. Therefore $(N : M) = ((N : M)M)_z : M) = \psi(N)$ which shows that $\psi\phi\psi(N) = \psi(N)$.
(2) Let I be a z-ideal of R. Then

$$\phi\psi\phi(I) = \phi\psi((IM)_z) = \phi(((IM)_z : M)) = (((IM)_z : M)M)_z.$$

Now, $((IM)_z : M)M \subseteq (IM)_z$, implies that $(((IM)_z : M)M)_z \subseteq ((IM)_z)_z = (IM)_z$. Also, $IM \subseteq (IM)_z$ implies that $I \subseteq ((IM)_z : M)$ which gives $(IM)_z \subseteq (((IM)_z : M)M)_z$. Thus $(((IM)_z : M)M)_z = (IM)_z = \phi(I)$, and hence $\phi\psi\phi = \phi$. □

The next two results are obtained immediately.

Corollary 3.7. *Let M be an R-module. Then the following statements are equivalent:*

(1) *ϕ is a surjection.*
(2) $\phi\psi = 1$.
(3) $N = ((N : M)M)_z$ *for every z-submodule N of M.*
(4) *ψ is an injection.*

Corollary 3.8. *Let M be an R-module. Then the following statements are equivalent:*

(1) *ϕ is an injection.*
(2) $\psi\phi = 1$.
(3) $I = ((IM)_z : M)$ *for every z-ideal I of R.*
(4) *ψ is a surjection.*

Corollary 3.9. *If ϕ is an injection, then $((0) : M)_z = ((0)_z : M)$.*

Proof. By Corollary 3.8(3) and Theorem 2.6, we have

$$((0) : M)_z = ((((0) : M)_z M)_z : M) = ((((0) : M)M)_z : M) = ((0)_z : M). \quad □$$

Corollary 3.10. *Let M be an R-module. Then the mapping ϕ is a bijection if and only if ψ is a bijection. In particular, if ϕ is a bijection, then ϕ is a lattice isomorphism and ψ is its inverse.*

Proof. The first part follows from Corollary 3.7 and Corollary 3.8. These and Lemma 3.1 conclude the "in particular" part. □

Corollary 3.11. *Let $R = C(X)$ and M be a finitely generated faithful multiplication R-module. Then, ϕ is a lattice isomorphism.*

Proof. Firstly by Corollary 2.8 and Proposition 2.5, we have $((IM)_z : M) = (IM : M)_z = I_z = I$ for all z-ideals I of R which implies that ϕ is an injection by Corollary 3.8. On the other hand, since M is multiplication, we have $((N : M)M)_z = N_z = N$ for every z-submodule N of M which shows that ϕ is a surjection by Corollary 3.7. Thus, the assertion holds by Corollary 3.10. □

4. A finitely generated multiplication module over C(X)

Let m be a maximal ideal of R. An R-module M is called m-*cyclic* provided there exist $x \in M$ and $a \in \mathrm{m}$ such that $(1-a)M \subseteq Rx$. By [7, Theorem 1.2], every m-cyclic module is a multiplication module. Assume that Y is a subset of a topological space X. Then $\mathbb{R}^Y$ consisting of all functions from Y to $\mathbb{R}$ is a $C(X)$-module with the usual multiplication of functions as the scalar multiplication. If Y is a finite subset of a compact Hausdorff space X and $\mathrm{m}_x := \{f \in C(X) \mid f(x) = 0\}$ for each fixed point $x \in X$, we show that the $C(X)$-module $\mathbb{R}^Y$ (consisting of all functions from Y to $\mathbb{R}$) is m_x-cyclic (see [4, Exercise 26, p. 14] for that m_x is a maximal ideal of C(X)). In particular, we have the following result:

Theorem 4.1. *If Y is a finite subset of a compact Hausdorff space X, then $\mathbb{R}^Y$ is a multiplication $C(X)$-module.*

Proof. Since X is Hausdorff, the finite subset Y is closed in X, and the subspace topology of Y is discrete. Therefore $C(Y) = \mathbb{R}^Y$. Now if $f \in \mathrm{m}_x$ and $g \in \mathbb{R}^Y$, then $(1-f)g = (1-f)|_Y \tilde{g}$, where $(1-f)|_Y$ denotes the restriction of $(1-f)$ to Y and $\tilde{g}$ is the Tietze extension of g [19, Theorem 3.2]. It implies that $(1-f)\mathbb{R}^Y \subseteq C(X)(1-f)|_Y$, as required. Thus $\mathbb{R}^Y$ is an m_x-cyclic $C(X)$-module, and so by [7, Theorem 1.2], $\mathbb{R}^Y$ is a multiplication $C(X)$-module. □

Recall that any completely regular space X is said to be a *P-space* if every prime ideal of $C(X)$ is a maximal ideal. If X is a compact Hausdorff P-space, then by [8, 4J] and [13, Theorem 1.2], $C(X)$ is a regular ring. This fact is used in the following result.

Theorem 4.2. *If Y is a finite subset of a compact Hausdorff P-space X, then $\mathbb{R}^Y$ is a flat $C(X)$-module.*

Proof. First, we consider the mapping $\phi : \mathbb{R}^Y \to \prod_{x\in Y} C(X)/\mathrm{m}_x$ defined by $\phi(g) = (C_{g(x)} + \mathrm{m}_x)_{x\in Y}$, where $C_{g(x)}$ is the constant function which maps the whole of X to $g(x)$. Clearly, ϕ is a $C(X)$-module homomorphism and its inverse is the mapping $\psi : \prod_{x\in Y} C(X)/\mathrm{m}_x \to \mathbb{R}^Y$ defined by $\psi((f_x + \mathrm{m}_x)_{x\in Y})(y) = f_y(y)$, i.e., ϕ is a $C(X)$-module isomorphism. Now, since $C(X)$ is regular and $C(X)/\mathrm{m}_x$ is a simple $C(X)$-module, we conclude that $C(X)/\mathrm{m}_x$ is an injective $C(X)$-module by [26, Theorem 2]. But by [22, Proposition 1.4], the injectivity of $C(X)/\mathrm{m}_x$ is equivalent to its flatness. Consequently, $\prod_{x\in Y} C(X)/\mathrm{m}_x$ is a flat $C(X)$-module and so is $\mathbb{R}^Y$. □

It is clear that for any non-empty finite subset Y of a compact Hausdorff P-space X and for any $x \in X$, the submodule $\mathrm{m}_x\mathbb{R}^Y$ of the $C(X)$- module $\mathbb{R}^Y$ dose not contain the non-zero constant functions from Y to $\mathbb{R}$, and so $(\mathrm{m}_x\mathbb{R}^Y : \mathbb{R}^Y) = \mathrm{m}_x$ for all $x \in X$. Now, by Theorem 4.1, $\mathbb{R}^Y$ is a multiplication $C(X)$-module, and so the flatness of the $C(X)$-module $\mathbb{R}^Y$ (Theorem 4.2) implies that $\mathbb{R}^Y$ is a finitely generated $C(X)$-module by [12, Propositions 2.4 and 3.8]. Thus, we have the following without further proof.

Corollary 4.3. *If Y is a finite subset of a compact Hausdorff P-space X, then $\mathbb{R}^Y$ is a finitely generated faithful multiplication $C(X)$-module.*

Corollary 4.4. *Let X be a compact Hausdorff P-space and Y be a finite subset of X. Then every submodule of the $C(X)$-module $\mathbb{R}^Y$ is a z-submodule of $\mathbb{R}^Y$.*

Proof. Let N be a submodule of $\mathbb{R}^Y$. By Theorem 4.1, $N = IM$ for some ideal I of $C(X)$. But, since X is a P-space, I is a z-ideal of $C(X)$ by [8, 4J], and so by [3, page 1], I is a *sz*-ideal of $C(X)$. Hence N is a z-submodule by Theorem 2.7(1) and Corollary 4.3. □

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References

[1] A. R. Aliabad, M. Badie and S. Nazari, *An extension of z-ideals and z^0-ideals*, Hacet. J. Math. Stat., 49(1) (2020), 254-272.

[2] A. R. Aliabad and R. Mohamadian, *On z-ideals and z^0-ideals of power series rings*, J. Math. Ext., 7(2) (2013), 93-108.

[3] A. R. Aliabad and R. Mohamadian, *Prime z-ideal rings (pz-Rings)*, Bull. Iranian Math. Soc., 48(3) (2022), 1177-1192.

[4] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Co., 1969.

[5] A. Benhissi and A. Maatallah, *A question about higher order z-ideals in commutative rings*, Quaest. Math., 43(8) (2020), 1155-1157.

[6] T. Dube and O. Ighedo, *On lattices of z-ideals of function rings*, Math. Slovaca, 68(2) (2018), 271-284.

[7] Z. A. El-Bast and F. P. Smith, *Multiplication Modules*, Comm. Algebra, 16(4) (1988), 755-779.

[8] L. Gillman and M. Jerison, Rings of Continuous Functions, Univ. Ser. Higher Math., D. Van Nostrand Co., New York, 1960.

[9] J. B. Harrehdashti and H. F. Moghimi, *Complete homomorphisms between the lattices of radical submodules*, Math. Rep., 20(70)(2) (2018), 187-200.

[10] O. Ighedo and W. Wm. McGovern, *On the lattice of z-ideals of a commutative ring*, Topology Appl., 273 (2020), 106969 (16 pp).

[11] C. W. Kohls, *Ideals in rings of continuous functions*, Fund. Math., 45 (1957), 28-50.

[12] C. P. Lu, *A module whose prime spectrum has the surjective natural map*, Houston J. Math., 33(1) (2007), 125-143.

[13] G. Mason, *z-ideals and prime ideals*, J. Algebra, 26 (1973), 280-297.

[14] G. Mason, *Prime z-ideals of $C(X)$ and related rings*, Canad, Math. Bull., 23(4) (1980), 437-443.

[15] H. F. Moghimi and J. B. Harehdashti, *Mappings between lattices of radical submodules*, Int. Electron. J. Algebra, 19 (2016), 35-48.

[16] H. F. Moghimi and M. Noferesti, *Mappings between the lattices of varieties of submodules*, J. Algebra Relat. Topics, 10(1) (2022), 35-50.

[17] H. F. Moghimi and M. Noferesti, *On the distributivity of the lattice of radical submodules*, J. Mahani Math. Res., 13(1) (2024), 347-355.

[18] M. E. Moore and S. J. Smith, *Prime and radical submodules of modules over commutative rings*, Comm. Algebra, 30(10) (2002), 5037-5064.

[19] J. R. Munkres, Topology: A First Course, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[20] M. Noferesti, H. F. Moghimi and M. H. Hosseini, *Mappings between the lattices of saturated submodules with respect to a prime ideal*, Hacet. J. Math. Stat., 50(1) (2021), 243-254.

[21] J. Ohm and D. E. Rush, *Content modules and algebras*, Math. Scand., 31 (1972), 49-68.

[22] V. S. Ramamurthi, *On the injectivity and flatness of certain cyclic modules*, Proc. Amer. Math. Soc., 48 (1975), 21-25.

[23] P. F. Smith, *Mapping between module lattices*, Int. Electron. J. Algebra, 15 (2014), 173-195.

[24] P. F. Smith, *Complete homomorphisms between module lattices*, Int. Electron. J. Algebra, 16 (2014), 16-31.

[25] P. F. Smith, *Anti-homomorphisms between module lattices*, J. Commut. Algebra, 7(4) (2015), 567-592.

[26] E. M. Vechtomov, *Modules of all functions over the ring of continuous functions*, Mathematical notes of the Academy of Sciences of the USSR, 28 (1980), 701-705.

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