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CONCERNING EQUALLY COVERED GROUPS

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ABSTRACT. A finite group is equally covered if it has a covering by proper subgroups of equal orders. Among other results, it is shown that finite simple groups have no equal coverings, and for any finite group G the n^{th} Cartesian power of G has an equal covering for some n. Some related topics are also discussed.

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1. Introduction, definitions and notation

In this paper, our focus is solely on groups of finite order. We define a group G to be *covered by proper subgroups* $H_1, H_2, \ldots, H_n$ if

$$G = H_1 \cup H_2 \cup \cdots \cup H_n. \tag{1}$$

Each subgroup H_i in this coverage is referred to as a *component*. It is evident that a group is covered by proper subgroups if and only if it is not cyclic. The cover (1) is called *irredundant* if every proper subset of the set $\{H_1, \ldots, H_n\}$ does not cover G. In this paper, we look at irredundant covering of G by proper subgroups of *equal orders*, i.e.,

$$G = H_1 \cup H_2 \cup \cdots \cup H_n, \quad |H_1| = |H_2| = \cdots = |H_n|. \tag{2}$$

Such a covering is called an *equal covering*.

In certain instances, the components H_i in the equal covering (2) may possess specific properties. Some of these properties are outlined below, with the variable p representing a prime number.

(a) Equal covering (2) is termed an *equal partition* of G if $H_i \cap H_j = 1$ for every $i \neq j$. Isaacs [15] established that the noncyclic p-groups of exponent p are the only groups having an equal partition.

(b) Let S be a subgroup of G. Equal covering (2) is termed an *equal strict S-partition* (*ES-partition*) of G if $H_i \cap H_j = S$ for every $i \neq j$. Atanasov, Foguel and Penland [1] investigated the ES-partitions of finite p-groups, particularly when H_is are maximal in G.

(c) In equal covering (2), if H_is are proper isomorphic abelian subgroups of G, then G is called a *CIA-group*. Foguel and Ragland [10] conducted a study on CIA-groups.

When we are dealing with an equal covering (2), its components will face some restrictions. It is easy to see from (2) that

$$|G| \leq n|H_i| - (n-1) < n|H_i|,$$

and hence

$$1 < |G : H_i| < n, \quad \text{for } i = 1, 2, \ldots, n, \tag{3}$$

(see also [3, Remark 1.1] and [8, Theorem 1]). A special case is when $n = 3$, which forces $|G : H_i| = 2$ for $i = 1, 2, 3$. In particular, it is shown in [23] (see also [6,14]) that $G/(H_1 \cap H_2 \cap H_3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Furthermore, our calculations with `GAP` [12] show that the similar situation with $|G : H_i| = n-1$ occurs for $n \in \{4, 5, 6, 8, 9, 10, 12, 14\}$ among the groups of order at most 200.

The paper is organized as follows. In Section 2, we derive some auxiliary results. In Section 3, we investigate the conditions under which the n^{th} Cartesian power G^n of a group G has an equal covering. In Section 4, we prove that simple groups have no equal covering (Theorem 4.2).

Before continuing, we need some additional notation. Given a natural number n, we denote by $\pi(n)$ the set of prime divisors of n and for a group G we put $\pi(G) = \pi(|G|)$. Throughout, we use $\mathbb{A}_n$ and $\mathbb{S}_n$, to denote the alternating and the symmetric group of degree n, respectively, and write $\mathbb{Z}_n$ for a cyclic group of order n. Recall that the *commutator subgroup* of a group G is the subgroup G' generated by all commutators $[x, y] = x^{-1}y^{-1}xy$, with $x, y \in G$. For $X \subseteq G$, we write $X^{\#} = X \setminus \{1\}$. The *order* of an element g of a group G is denoted by $|g|$, and by the *exponent* of G, denoted $\exp(G)$, we mean the least common multiple of the orders of the elements of G, in other words, the exponent of G is the least positive integer n, such that $g^n = 1$ for all $g \in G$. For definitions of standard group theoretic terminology and notation not defined here, the reader is referred to [9,16].

2. Auxiliary results

We begin by establishing some preliminary facts about the groups having an equal covering.

Lemma 2.1. [24] *If G has an equal covering $\Pi = \{H_1, \ldots, H_n\}$, then $\exp(G)$ divides $|H_i|$ for all $H_i \in \Pi$. In particular, we have $\pi(H_i) = \pi(G)$ for all $H_i \in \Pi$.*

Proof. Suppose G is a group with equal covering $\Pi = \{H_1, \dots, H_n\}$. If $g \in G$, then $g \in H_i$ for some $H_i \in \Pi$ and so $|g|$ divides $|H_i|$. Since Π is an equal covering of G, $|g|$ divides the order of all members of Π. On the other hand, since g can be any element of G and $\exp(G)$ is divisible by $|g|$ for all $g \in G$, $\exp(G)$ divides $|H_i|$ for all $H_i \in \Pi$. The second statement follows from the first because $\pi(\exp(G)) = \pi(G)$. □

Lemma 2.2. *Let G be a group having an equal covering. Then G has at least two nonconjugate maximal subgroups M_1 and M_2 with $\pi(M_1) = \pi(M_2) = \pi(G)$.*

Proof. Let $\Pi = \{H_1, \dots, H_n\}$ be an equal covering of G. Let M_i be a maximal subgroup of G containing H_i, for each i. Then $\Psi = \{M_1, \dots, M_n\}$ is a covering of G. It follows from Lemma 2.1 that $\pi(M_i) \supseteq \pi(H_i) = \pi(G)$ which implies that $\pi(M_i) = \pi(G)$, for each i. Finally, since Ψ is a covering of G, the M_is cannot all be conjugate, and the proof is complete. □

The *spectrum* $\omega(G)$ of a finite group G is the set of orders of elements in G. Two groups are said to be *isospectral* if their spectra coincide. The set $\omega(G)$ determines the *Grüenberg-Kegel graph* (or *prime graph*) $\mathrm{GK}(G)$ of G whose vertex set is $\pi(G)$, and two vertices p and q are *adjacent* if and only if $pq \in \omega(G)$.

Lemma 2.3. *Let G be a group having a covering $\Pi = \{H_1, \dots, H_n\}$ by isomorphic subgroups. Then for each $H_i \in \Pi$, H_i is isospectral to G, that is $\omega(H_i) = \omega(G)$. In particular, $\exp(H_i) = \exp(G)$ and $\mathrm{GK}(H_i) = \mathrm{GK}(G)$.*

Proof. Clearly, $\omega(H_i) \subseteq \omega(G)$. We shall argue that $\omega(G) \subseteq \omega(H_i)$. If $g \in G$, then there exists $H_j \in \Pi$ such that $g \in H_j$, and so $|g| \in \omega(H_j)$. Since the subgroups $H_1, \dots, H_n$ are isomorphic, they are isospectral groups and so each of them contains an element of order $|g|$. Hence, $|g| \in \omega(H_i)$, and thus $\omega(G) \subseteq \omega(H_i)$, as required. The last assertions are now obvious from the definition. □

Corollary 2.4. *Let G be a group having an equal covering $\{H_1, \dots, H_n\}$ for which $H_i \leq M_i$, for each i, where the M_is are isomorphic proper subgroups of G. Then, for each i, we have $\omega(M_i) = \omega(G)$. In particular, $\exp(M_i) = \exp(G)$ and $\mathrm{GK}(M_i) = \mathrm{GK}(G)$.*

Proof. Note that the collection $\{M_1, \dots, M_n\}$ forms a covering of G by isomorphic subgroups and the result is now immediate from Lemma 2.3. □

Lemma 2.5. [24] *If $H \trianglelefteq G$ and G/H has an equal covering, then G has an equal covering.*

Proof. This is immediate, since if G/H has an equal covering, say

$$\Pi = \{N_1/H, N_2/H, \dots, N_k/H\},$$

then it follows by correspondence theorem that $\Psi = \{N_1, N_2, \ldots, N_k\}$ is an equal covering of G. □

The following is a useful special case of Lemma 2.5.

Corollary 2.6. [24] *If $G = H \rtimes K$ (in particular, if $G = H \times K$) and K has an equal covering, then G has an equal covering.*

We also have the following related result:

Corollary 2.7. *Let $P \in \mathrm{Syl}_p(G)$ where G is a supersolvable group and p is the smallest prime divisor of $|G|$ and assume that P is noncyclic. Then, G has an equal covering.*

Proof. In view of [22, 5.4.8], it follows that the Sylow p-subgroup P of G has a normal p'-complement in G, say Q, and so $G = Q \rtimes P$. The result is now immediate from Corollary 2.6. □

Note that the noncyclic p-groups of exponent p possess an equal partition (we can simply take the subgroups of order p), and thus possess an equal covering. It is proved in [2, Remark 3.5] (see also [3, Lemma 2.1] and [24, Theorem 15]) that every noncyclic p-group has an equal covering by maximal subgroups. The following lemma considers a more general case, when we deal with a noncyclic nilpotent group.

Lemma 2.8. [24] *If G is a noncyclic nilpotent group, then G has an equal covering. In particular, if G is a noncyclic abelian group, then G has an equal covering.*

Proof. Let G be a noncyclic nilpotent group. Since G is the direct product of its Sylow p-subgroups, we see that at least one of these Sylow p-subgroups is noncyclic. The result now follows from the fact that a noncyclic p-group always has an equal covering together with Corollary 2.6. □

Using the fact that G/G' is always abelian and Lemma 2.5, the following is then an immediate consequence of Lemma 2.8.

Corollary 2.9. *If G/G' is a noncyclic group, then G has an equal covering.*

We define a group to possess a *maximal equal covering* if it has an equal covering composed of maximal subgroups. For example, as mentioned earlier, every noncyclic p-group exhibits a maximal equal covering. It is worth noting that while a group G may have an equal covering, it is not necessarily guaranteed to have a maximal equal covering. For instance, utilizing `GAP` [12], we observe that the group $G = \mathbb{Z}_3 \times \mathbb{S}_4 = \mathrm{SmallGroup}(72, 42)$ possesses an equal covering but lacks a maximal equal covering (being the smallest group with this property). Conversely, we find that for noncyclic nilpotent groups and noncyclic groups with orders less than 31, the existence of a maximal equal covering holds true.

Lemma 2.10. *If G is a group with $\exp(G) = |G|/p$ where p is the smallest prime dividing $|G|$, then every equal covering of G is a maximal equal covering of G which contains proper normal subgroups of index p. Moreover, in this situation, G has a maximal equal covering consists of $p+1$ proper normal subgroups of index p.*

Proof. Suppose that $\Pi = \{H_1, H_2, \dots, H_n\}$ is an equal covering of G. Then, since $\exp(G) = |G|/p$ divides $|H_i|$ for all $H_i \in \Pi$ (by Lemma 2.1), and that p is the smallest prime dividing $|G|$, we conclude that $|G : H_i| = p$ for each $1 \leq i \leq n$. Hence, according to [17] the subgroups H_i are normal in G, and so Π is a maximal equal covering of G by proper normal subgroups of index p. Furthermore, for any i, j with $1 \leq i, j \leq n$, $G/H_i \cong G/H_j \cong \mathbb{Z}_p$, which implies that $G/(H_i \cap H_j) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Now we may conclude from Theorem 4 in [4] that G has a maximal equal covering consists of $p+1$ proper normal subgroups of index p. □

Additionally, it is noteworthy that certain groups G of even order with $\exp(G) = |G|/2$ may not possess an equal covering. For example, when G is either the alternating group $\mathbb{A}_5$ or the symmetric group $\mathbb{S}_5$, both having $\exp(G) = |G|/2$, these groups do not exhibit an equal covering.

3. The Cartesian powers of a group

In this section, our focus turns to the Cartesian product of groups, with particular emphasis on the n^{th} Cartesian power G^n of a group G. We start with the following Definition and Theorem.[1]

Definition 3.1. Let G be a group and $G^n = G \times G \times \cdots \times G$ the direct product of n-copies of G (the n^{th} Cartesian power of G). An element $x = (x_1, x_2, \dots, x_n)$ in G^n is called an *ij-pair diagonal* for $i \neq j$ if $x_i = x_j$.

Theorem 3.2. *Let G be a group and let $n \geq |G| + 1$ be an integer. Then the n^{th} Cartesian power G^n has an equal covering.*

Proof. Let $n \geq |G| + 1$. We observe, first of all, that every element of the n^{th} Cartesian power G^n is an ij-pair diagonal for some i and j. For $1 \leq i < j \leq n$, we now define $H_{i,j}$ to be the set of all ij-pair diagonals in G^n. It is routine to check that $H_{i,j}$ is a proper subgroup of G^n, and the collection $\Pi = \{H_{i,j} : 1 \leq i < j \leq n\}$ is an equal covering of G^n and the proof is complete. □

[1] Stephen M. Gagola Jr. of Kent State University brought attention to this definition and proposed Theorem 3.2 during the Zassenhaus meeting in 2022.

Definition 3.3. Given a group G, we denote by $\xi(G)$ the *smallest* integer $n \geq 1$ for which the n^{th} Cartesian power G^n has an equal covering.

Recall that a group G is called *perfect* if it equals its commutator subgroup G'; otherwise it is called a *nonperfect* group.

Theorem 3.4. *If G is a nonperfect group, then $\xi(G) \leq 2$.*

Proof. Suppose G is a nonperfect group and $\xi(G) \neq 1$. It follows by Lemma 2.5 that $\xi(G/G') \neq 1$, and so G/G' is a nonidentity cyclic group. The isomorphism

$$\frac{G \times G}{G' \times G'} \cong \frac{G}{G'} \times \frac{G}{G'},$$

shows that $(G \times G)/(G' \times G')$ is a noncyclic abelian group and by Lemma 2.8 has an equal covering. But now Lemma 2.5 implies that the 2^{nd} Cartesian power G^2 has an equal covering, and so $\xi(G) = 2$. □

Remark 3.5. Using `GAP` [12], we determined that

$$\xi(\mathbb{A}_5) = \xi(\mathrm{PSL}_2(4)) = \xi(\mathrm{PSL}_2(5)) = 2,$$

$$\xi(\mathbb{A}_6) = \xi(\mathrm{PSL}_2(7)) = \xi(\mathrm{PSL}_2(8)) = 3$$

and

$$\xi(\mathrm{PSL}_2(11)) = \xi(\mathrm{PSL}_2(13)) = 4,$$

while if S is one of the following simple groups:

$$\mathbb{A}_7,\ \mathrm{PSL}_2(q),\ q = 16, 17, 19 \text{ or } 23,\ \mathrm{PSL}_3(3),\ \mathrm{PSU}_3(3),$$

then $\xi(S) > 3$.

A question which arises naturally is to find or estimate an upper bound on $\xi(G)$.

Question 3.6. *Is there a natural number n such that for all groups G, $\xi(G) \leq n$?*

In the sequel, we will enhance Theorem 3.2 (see Theorem 3.8). However, it is crucial to note that the extent of improvement remains contingent upon the specific characteristics of the group G.

Definition 3.7. Let G be a group, $\mathrm{Aut}(G)$, the automorphism group of G, and $\mathrm{End}(G)$, the set of all endomorphisms of G. Let $x = (x_1, x_2, \ldots, x_n)$ be an element in the n^{th} Cartesian power G^n.

(a) The element x is called an *ij-automorphic pair diagonal* for $i \neq j$, if $x_j = \sigma(x_i)$ for some $\sigma \in \mathrm{Aut}(G)$.

(b) The element x is called an *ij-endomorphic pair diagonal* for $i \neq j$, if $x_j = \sigma(x_i)$ for some $\sigma \in \mathrm{End}(G)$.[2]

[2]For simple groups, the only endomorphism which is not an automorphism maps the group into its center, that is to the identity.

We denote by

$$g^{\mathrm{Aut}(G)} = \{\sigma(g) : \sigma \in \mathrm{Aut}(G)\}, \quad g \in G,$$

the orbit of g under the natural action $\mathrm{Aut}(G)$ on G, and call it the *automorphic class* of g. Let $\mathrm{n}(G)$ denote the number of automorphic classes in G. Similarly, we define the *endomorphic relator* of $g \in G^{\#}$ by

$$R(g)^{\mathrm{End}(G)} = \{\sigma(g) : \sigma \in \mathrm{End}(G)\}.$$

Note that for a nonabelian simple group S and $g \in S^{\#}$, we have

$$R(g)^{\mathrm{End}(S)} = g^{\mathrm{Aut}(S)} \cup \{1\}, \text{ and } \cup_{g\in S^{\#}} R(g)^{\mathrm{End}(S)} = S,$$

and so the number of nontrivial endomorphic relators is equal to $\mathrm{n}(S) - 1$.

Theorem 3.8. *Let G be a group and let $n \geq \mathrm{n}(G) + 1$ be an integer. Then the n^{th} Cartesian power G^n has an equal covering. Furthermore, if S is a nonabelian simple group, then $S^{\mathrm{n}(S)}$ has an equal covering, too.*

Proof. Let $A = \mathrm{Aut}(G)$. Since $n > \mathrm{n}(G)$, by the pigeonhole principle it follows that every n-tuple in G^n is an ij-automorphic pair diagonal for some $i \neq j$. Now fix $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$ and let σ be an element of A. We define $H_{i,j,\sigma}$ to be the set of all n-tuples, which are ij-automorphic pair diagonals with $x_j = \sigma(x_i)$. Note that $H_{i,j,\sigma}$ is a proper subgroup of G^n. Let

$$\Pi = \{H_{i,j,\sigma} : 1 \leq i, j \leq n, \ i \neq j, \ \sigma \in A\}.$$

It is routine to check that Π gives an equal covering of G^n.

Now suppose that S is a nonabelian simple group and let $E = \mathrm{End}(S)$. Write $n = \mathrm{n}(S)$. As in the preceding case, we observe that every n-tuple $(x_1, x_2, \ldots, x_n)$ in S^n is an ij-endomorphic pair diagonal for some $i \neq j$, because $n > \mathrm{n}(S) - 1$. Now fix $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$ and let σ be an element of E. We define $H_{i,j,\sigma}$ to be the set of all n-tuples, which are ij-endomorphic pair diagonals with $x_j = \sigma(x_i)$, which is a proper subgroup of S^n. Now, the collection

$$\Pi = \{H_{i,j,\sigma} : 1 \leq i, j \leq n, \ i \neq j, \ \sigma \in E\},$$

forms an equal covering of S^n, and the proof is complete. □

As an immediate consequence of Theorem 3.8, we have the following result.

Corollary 3.9. *Let G be a group. Then $\xi(G) \leq \mathrm{n}(G) + 1$ and, in particular, $\xi(S) \leq \mathrm{n}(S)$ in the case that S is a nonabelian simple group.*

For instance, we have $\xi(\mathbb{A}_5) = 2 < 4 = \mathrm{n}(\mathbb{A}_5)$.

Corollary 3.10. *If G is a perfect group, then we have*

$$\begin{aligned}\xi(G) &\leq \min\{\xi(G/N) : N \textit{ is a maximal normal subgroup of } G\}\\ &\leq \min\{\mathrm{n}(G/N) : N \textit{ is a maximal normal subgroup of } G\}.\end{aligned}$$

Proof. It follows immediately from Lemma 2.5 and Theorem 3.8. □

Remark 3.11. Note that there are perfect groups G with

$$\xi(G) < \min\{\mathrm{n}(G/N) : N \text{ is a maximal normal subgroup of } G\}.$$

For example, using `GAP` [12] we found out that Perfect-10752-1 = $\mathrm{PSL}_2(7)$ extended downwards by a module $2^3 \times 2^3$, has an equal cover while $\xi(\mathrm{PSL}_2(7)) = 3$. Also, using `GAP` [12] we found that SmallGroup$(288, 409) = (\mathbb{Z}_3 : \mathbb{Z}_4) \times \mathrm{SL}_2(3)$ is equal covered by its subgroups of order 12, but neither of its direct factors is equal covered by proper subgroups.

In the forthcoming lemmas, we will examine specific scenarios where Cartesian products of groups result in a group with an equal covering.

Lemma 3.12. *If G and H are nilpotent groups with $\pi(G) \cap \pi(H) \neq \emptyset$, then $G \times H$ has an equal covering.*

Proof. Since $G \times H$ is a noncyclic nilpotent group, the assertion is immediate from Lemma 2.8. □

Lemma 3.13. *If G and H are supersolvable groups and p is the smallest prime divisor of both $|G|$ and $|H|$, then $G \times H$ has an equal covering.*

Proof. This is immediate from Lemma 2.7 and the fact that $G \times H$ is a supersolvable group with a noncyclic Sylow p-subgroup, where p is the smallest prime divisor of $|G|$. □

Note that Lemma 3.13 cannot be improved, because the supersolvable group $\mathbb{Z}_3 \times \mathbb{S}_3$ of order 18 has a noncyclic Sylow 3-subgroup, but does not have an equal covering.

4. The finite simple groups

In this section, we establish that finite simple groups do not possess an equal covering. Since the only abelian simple groups are the cyclic groups of prime order, they inherently lack any form of covering, including an equal covering. As a result, our attention will be directed solely towards nonabelian simple groups in the ensuing discourse. According to the classification of finite simple groups, each nonabelian finite simple group falls into one of the following categories: alternating groups, sporadic groups, classical groups, or exceptional groups of Lie type.

Theorem 4.1. *The alternating and symmetric groups, $\mathbb{A}_n$ and $\mathbb{S}_n$, have no equal covering.*

Proof. Using `GAP` [12], we may assume that $n \geq 7$. Note first of all that if H is a subgroup of $\mathbb{S}_n$, then $|H \cap \mathbb{A}_n| = |H|$ or $|H|/2$. Therefore, if $|H|$ is divisible by 4, then $\pi(H) = \pi(H \cap \mathbb{A}_n)$. We also notice that by [18] all the maximal subgroups of $\mathbb{S}_n$ for $n \geq 7$ have order divisible by 4. Now, if $n \geq 7$ and $K < \mathbb{S}_n$ is maximal with $\pi(K) = \pi(\mathbb{S}_n)$, then by Table 1 in [21], $K \cap \mathbb{A}_n \leq M$, where $M \leq \mathbb{S}_k \times \mathbb{S}_{n-k}$, since $\pi(K) = \pi(K \cap \mathbb{A}_n) = \pi(\mathbb{A}_n)$. Thus, K must be a group of type (a) as in [18]. On the other hand, it follows from [18] that if $G \in \{\mathbb{A}_n, \mathbb{S}_n\}$, then the only maximal subgroups M of G with $\pi(M) = \pi(G)$ are

$$\mathbb{A}_k \trianglelefteq M \leq \mathbb{S}_k \times \mathbb{S}_{n-k},$$

where $1 < k < n$ and $p \leq k$ for every prime number $p \leq n$, which are intransitive. Thus, such maximal subgroups M of G do not contain an n-cycle. Also, according to [18], when $n = 2k$ the subgroup $(\mathbb{S}_k \times \mathbb{S}_k) \cap G$ is not maximal.

Now suppose $G \in \{\mathbb{A}_n, \mathbb{S}_n\}$ has an equal covering $\Pi = \{H_i\}$. Then G has a covering $\Psi = \{M_i\}$ with $H_i \leq M_i$ and for each i, $\mathbb{A}_k \trianglelefteq M_i \leq \mathbb{S}_k \times \mathbb{S}_{n-k}$ with $k < n$. We shall treat the cases n odd and n even, separately.

(a) First assume that n is *odd*. In this case, we see that Ψ can not cover G, because there is an n-cycle in G which is an even permutation, while each M_i is an intransitive subgroup of G, a contradiction.

(b) Next assume that $n = 2k$ is *even*. Again, in this case Ψ can not cover G, because a permutation σ which is the product of two disjoint k-cycles lies in G, while σ cannot belong to any M_i, a contradiction.

Thus, in both cases, we arrive at contradictions, and so G has no equal covering. □

Theorem 4.2. *A nonabelian simple group has no equal covering.*

Proof. Suppose a nonabelian simple group S has an equal covering, say

$$\Pi = \{H_1, H_2, \ldots, H_n\}, \text{ with } |H_1| = |H_2| = \cdots = |H_n|.$$

Then by Lemma 2.1, we have $\pi(H_i) = \pi(S)$ for all $H_i \in \Pi$. Since every proper subgroup lies in a maximal subgroup, we may assume that for each i, $H_i \leq M_i$, where M_i is a maximal subgroup of S. Clearly, $\pi(M_i) = \pi(S)$. Table 1 below contains a list of all the possible pairs (S, M), where M is a maximal subgroup of the simple group S with $\pi(M) = \pi(S)$ [19] (see also [20, Lemma 3] and [7, Table 2]).

S	Type of M	Remarks
$\mathbb{A}_n$	$\mathbb{A}_n \cap (\mathbb{S}_k \times \mathbb{S}_{n-k})$	p prime, $p \leq n \Longrightarrow\ p \leq k$
$\mathrm{Sp}_{2m}(q)$	$O^-_{2m}(q)$	m, q even
$\Omega_{2m+1}(q)$	$O^-_{2m}(q)$	m even, q odd
$\mathrm{P}\Omega^+_{2m}(q)$	$O_{2m-1}(q)$	m even, q odd
$\mathrm{P}\Omega^+_{2m}(q)$	$\mathrm{Sp}_{2m-2}(q)$	m, q even
$\mathrm{PSp}_4(q)$	$\mathrm{Sp}_2(q^2)$	
$\mathbb{A}_6$	$\mathrm{L}_2(5)$	
$\mathrm{L}_6(2)$	P_1, P_5	
$\mathrm{U}_3(3)$	$\mathrm{L}_2(7)$	
$\mathrm{U}_3(5)$	$\mathbb{A}_7$	
$\mathrm{U}_4(2)$	$P_2, \mathrm{Sp}_4(2)$	
$\mathrm{U}_4(3)$	$\mathrm{L}_3(4), \mathbb{A}_7$	
$\mathrm{U}_5(2)$	$\mathrm{L}_2(11)$	
$\mathrm{U}_6(2)$	M_{22}	
$\mathrm{PSp}_4(7)$	$\mathbb{A}_7$	
$\mathrm{Sp}_6(2)$	$O^+_6(2)$	
$\Omega^+_8(2)$	$P_1, P_3, P_4, \mathbb{A}_9$	
$G_2(3)$	$\mathrm{L}_2(13)$	
${}^2F_4(2)'$	$\mathrm{L}_2(25)$	
M_{11}	$\mathrm{L}_2(11)$	
M_{12}	$\mathrm{M}_{11}, \mathrm{L}_2(11)$	
M_{24}	M_{23}	
HS	M_{22}	
McL	M_{22}	
Co_2	M_{23}	
Co_3	M_{23}	

Table 1: Pairs (S, M) with $\pi(M) = \pi(S)$.

However, applying the results that we have so far established and using $\mathbb{ATLAS}$ [9], `GAP` [12] and `MAGMA` [5], we can exclude "most" finite simple groups in Table 1. More precisely, observe that

(a) Using Theorem 4.1, we can eliminate the alternating groups.

(b) Using the results in Table 3 of [24], we can eliminate the following simple groups:

$$\mathrm{M}_{11},\ \mathrm{M}_{12},\ U_3(3) \text{ and } U_4(2).$$

(c) Using Lemma 2.2 and 𝔸𝕋𝕃𝔸𝕊 [9], we can eliminate the following simple groups:

$$^2F_4(2)', \text{ HS}, \text{ M}_{24}, \text{ Co}_2 \text{ and Co}_3.$$

(d) Using Corollary 2.4 and [21, Theorem], we can eliminate the following simple groups:

$$\text{U}_3(5), \text{ U}_5(2), \text{ U}_6(2), \text{ Sp}_6(2), \ G_2(3) \text{ and McL}.$$

Note that in view of [21] non of these groups have a maximal subgroup with the same exponent as the group.

(e) Using `GAP` [12] and `MAGMA` [5], we can eliminate the following groups:

$$\text{L}_6(2), \text{ U}_4(3), \text{ U}_6(2), \text{ Sp}_6(2), \text{ PSp}_4(7) \text{ and P}\Omega_8^+(2).$$

Therefore, in order to complete our analysis of simple groups in Table 1, we will examine the remaining simple groups S, which are listed below:

S	Type of M	Remarks
$\text{Sp}_{2m}(q)$	$O^-_{2m}(q)$	m, q even
$\Omega_{2m+1}(q)$	$O^-_{2m}(q)$	m even, q odd
$\text{P}\Omega^+_{2m}(q)$	$O_{2m-1}(q)$	m even, q odd
$\text{P}\Omega^+_{2m}(q)$	$\text{Sp}_{2m-2}(q)$	m, q even
$\text{PSp}_4(q)$	$\text{Sp}_2(q^2)$	

From now on, we assume that S is one of these simple groups. Since for each i, M_i is a maximal subgroup of S with $\pi(M_i) = \pi(S)$, it follows that all of the M_i are isomorphic. On the other hand, by Lemma 2.3, for each i, M_i is isospectral to S. In particular, the prime graphs $\text{GK}(M)$ and $\text{GK}(S)$ coincide, and in view of [20, Theorem] (see also [7, Table 1]) we observe that the pair (S, M) of a simple group with a maximal subgroup that have the same prime graph must be in the following table:

S	M	Conditions	Equal-covering?
$\text{Sp}_8(q)$	$O_8^-(q)$	q even	No, Lemma 2.2
$\Omega_8^+(q)$	$\text{Sp}_6(q)$	q even	No, Lemma 2.2
$\text{P}\Omega_8^+(q)$	$\Omega_7(q)$	q odd	No, Corollary 2.4
$\text{Sp}_4(q)$	$O_4^-(q)$	$q > 2$ even	No, diagonal element

The case of $S = \text{Sp}_8(q)$ for $q = 2^f$ is handled by Lemma 2.2, since S has only one conjugacy class of maximal subgroups M with $\pi(M) = \pi(S)$, namely $M \cong O_8^-(q)$.

Similarly, the case of $S = \Omega_8^+(q)$ for $q > 2$ even is handled by Lemma 2.2, since S has only one conjugacy class of maximal subgroups M with $\pi(M) = \pi(S)$, namely $M \cong \mathrm{Sp}_6(q)$. (For $q = 2$, there are multiple classes and isomorphism classes, but this was handled earlier in (e).)

The case of $\mathrm{P}\Omega_8^+(q)$ for $q > 3$ odd is handled by Corollary 2.4 and [13, Theorem 9]. Though this group has more than one conjugacy class of maximal subgroups M with $\pi(M) = \pi(S)$, all such M are isomorphic to $O_7(q)$, but these subgroups are not isospectral with S by [13, Theorem 9 (ii)].

The case of $S = \mathrm{Sp}_4(q)$ for $q > 4$ even can be handled by exhibiting an element g not contained in any maximal subgroup M_i with $\pi(M_i) = \pi(S)$. There are two conjugacy classes of M_i. The first is of type $\mathcal{C}_8$ and is the natural copy of $M_8 = \mathrm{SO}_4^-(q) \leq \mathrm{Sp}_4(q)$, for q even. The second is of type $\mathcal{C}_3$, the so-called 'semilinear' case with $M_3 \leq \Gamma\mathrm{Sp}_2(q^2)$; most relevant for us is that M_3 has an index 2 subgroup M_3' isomorphic to $\mathrm{Sp}_2(q^2)$, and the action of M_3' on the natural module V of dimension 4 is induced by the natural action on a 2-dimensional vector space over $\mathrm{GF}(q^2)$; the Galois field with q^2 elements.

For any element $\omega \neq 0, 1$ of $\mathrm{GF}(q)$; the Galois field with q elements, the element $g = h_1(\omega) \cdot h_2(\omega^3) \in S$, which is equal to the diagonal matrix:

$$g = \begin{bmatrix} \omega & & & \\ & \omega^2 & & \\ & & \omega^{-2} & \\ & & & \omega^{-1} \end{bmatrix},$$

with respect to a standard representation of S with respect to the anti-diagonal symplectic form, is diagonalizable with no eigenvalues equal to 1. By [11, Theorem 3.1.7], this element is not $\mathrm{GL}_4(q)$-conjugate to any element in $SO_4^i(q)$. Thus, g is not contained in the union of the M_8s. For any ω not contained in a subfield of 4 elements, $\omega^3 \neq 1$, so the element g has four distinct eigenvalues over $\mathrm{GF}(q)$. Since an element of $\mathrm{Sp}_2(q^2)$ can only have 2 distinct eigenvalues, g cannot be contained in $[M_3, M_3] \cong \mathrm{Sp}_2(q^2)$. Since $M_3/[M_3, M_3]$ is cyclic of order 2, for any $h \in M_3$, $h^2 \in [M_3, M_3]$, but since g^2 is also diagonalizable with four distinct eigenvalues (raising to the 2^{nd} power is an automorphism of the group of nonzero elements of $\mathrm{GF}(q)$), no conjugate of g^2 is in $[M_3, M_3]$, and so no conjugate of g is in M_3.

In particular, for any even $q > 4$ and any ω in $\mathrm{GF}(q)$, but not in any subfield of order 4, the element g in the previous paragraph is not contained in any conjugate of M_8 or M_3, so $S = \mathrm{Sp}_4(q)$ is not the union of its subgroups H_i with $\pi(H_i) = \pi(S)$, and so S has no equal covering. The group $\mathrm{Sp}_4(4)$ can be handled by a quick calculation in GAP or Magma (two conjugacy classes of S are missed by the union of the conjugates of M_3 and M_8). The proof is complete. □

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