International Electronic Journal of Algebra
Volume 37 (2025) 385-397
DOI: 10.24330/ieja.1571460

ARTINIAN* MODULES

Ajim Uddin Ansari, Hwankoo Kim, Sanjeev Kumar Maurya and Ünsal Tekir

Received: 8 March 2024; Revised: 20 July 2024; Accepted: 26 July 2024
Communicated by A. Çiğdem Özcan

Dedicated to the memory of Professor Syed M. Tariq Rizvi

Abstract. In this paper, we introduce and study Artinian* modules as a generalization of Artinian modules. We transfer several results of Artinian modules to Artinian* modules. We also provide several characterizations of this new class of modules. Furthermore, we investigate the existence of secondary representation for Artinian* modules over commutative regular rings. Finally, we characterize Artinian* modules in the amalgamated module construction.

Mathematics Subject Classification (2020): 13E10, 13C13, 13A15
Keywords: Artinian ring, Artinian module, Artinian* ring, Artinian* module, secondary representable module

1. Introduction

Emil Artin significantly contributed to the advancement of the structure theory of commutative rings by introducing the theory of rings and modules satisfying descending chain conditions. Recall that a module over a ring is called an Artinian module if it satisfies descending chain condition on submodules. A ring is called an Artinian ring if it is an Artinian module over itself. One of the roots of the theory of Artinian rings is Artin's historical article [4] in 1927. Subsequently, there has been ongoing and extensive exploration of Artinian rings and modules, establishing them as pivotal topics in the study of ring and module theory. Because of its importance, many authors attempted to extend and generalize the concept of Artinian rings and modules (see [11], [13], [15], and [16], for example). As one of its crucial generalizations, Sevim et al. [16] introduced the concept of S-Artinian rings and modules. An A-module M is said to be S-Artinian if for each descending chain $\{N_n\}_{n\in\mathbb{N}}$ of submodules of M, there exist an $s \in S$ and an integer $j \geq 1$ such that $sN_j \subseteq N_i$ for every $i \geq j$. A ring A is said to be an S-Artinian ring if it is an S-Artinian module over itself, [16]. They have extended many results on Artinian rings to S-Artinian rings. Moreover, several characterizations of S-Artinian modules are given in [15].

In this paper, we aim to generalize the concept of Artinian modules. Motivated by the notion of S-Artinian rings and modules, we introduce the notion of Artinian* rings and modules as a generalization of Artinian rings and modules. Let M be an A-module. We say that M is Artinian* if M is an S_a-Artinian module for every $0 \neq a \in A \setminus U(A)$, where $S_a = \{a^n : n \geq 0\}$ and $U(A)$ denotes the set of all units of A. A ring A is said to be Artinian* if it is an Artinian* module over itself. It is clear that the class of Artinian* modules is an intermediate class between S-Artinian and Artinian modules. We transfer many results on Artinian rings and modules to Artinian* rings and modules. We show that if A is not a local ring, then the concepts of Artinian* modules and Artinian modules coincide (Proposition 2.3). Also we prove that every prime ideal in an Artinian* ring is maximal (Proposition 2.7). We give several characterizations of Artinian* modules. For example, we prove that if M is a multiplication module over a local ring, then Artinian* modules and zero-dimensional modules coincide (Theorem 2.11). Moreover, we prove that Artinian* modules over regular rings are representable (Theorem 3.4). Finally, we characterize the Artinian* module in the amalgamated module construction (Theorem 4.3).

Throughout this paper, all rings are assumed to be commutative rings with identity unless otherwise stated.

2. Properties and characterizations of Artinian* rings and modules

In this section, we extend the classical notion of Artinian rings and modules to Artinian* rings and modules. We begin this section by introducing their definitions. For a ring A, we denote $A \setminus \{0\}$ by A^* and the set of all units of A by $U(A)$. We also denote the set of all positive integers by $\mathbb{N}$.

Definition 2.1. Let M be an A-module. Then M is said to be an Artinian* A-module if M is an S_a-Artinian A-module for every $a \in A^* \setminus U(A)$, where $S_a = \{a^n : n \geq 0\}$. Also, a ring A is said to be an Artinian* ring if it is an Artinian* module over itself.

It is clear from the definition that an Artinian module is always an Artinian* module. However an Artinian* module need not be Artinian in general. For this, consider the following example.

Example 2.2. Consider the ring $A = F[x_1, x_2, \ldots, x_n, \ldots]/(x_1, x_2^2, \ldots, x_n^n, \ldots)$, where F is a field, and $\mathfrak{p} = (x_1, x_2, \ldots, x_n, \ldots)/(x_1, x_2^2, \ldots, x_n^n, \ldots)$. Then by [6, p.

91], $\mathfrak{p}$ is the unique prime ideal of A. So the nil radical of A is $\mathfrak{p}$. This implies that every element of $\mathfrak{p}$ is nilpotent, and so S_a contains 0 for every $a \in A^* \setminus U(A)$ since $A^* \setminus U(A) \subseteq \mathfrak{p}$. Consequently, A as an A-module is Artinian*. On the other hand, A is not a Noetherian ring since $\mathfrak{p}$ is not finitely generated. This implies that A is not an Artinian ring. Hence A is not an Artinian A-module.

The following result shows that the concepts of Artinian* modules and Artinian modules coincide when the underlying ring is not local.

Proposition 2.3. *Let A be a ring which is not a local ring and M an A-module. Then the following statements are equivalent.*

(1) *M is an Artinian* A-module.*
(2) *M is an Artinian A-module.*

Proof. This follows from [3, Theorem 3.21] for $G = \{e\}$. □

Now, we study the basic properties of the Artinian* modules. Let A be a ring and M an A-module. Following [14], M is called *secondary* if $M \neq 0$ and for every $a \in A$, $aM = M$ or $a^n M = 0$ for some integer $n \geq 1$. In this case $P = \sqrt{Ann(M)}$ is a prime ideal of A and M is called *P-secondary*. Recall that a ring is called *primary* if it has a unique prime ideal.

Proposition 2.4. *The following statements hold for an A-module M.*

(1) *If S is a multiplicatively closed subset of A and M is an Artinian* A-module, then $S^{-1}M$ is an Artinian* $S^{-1}A$-module.*
(2) *If M is an Artinian* A-module, then $S_a^{-1}M$ is an Artinian $S_a^{-1}A$-module for every $a \in A^* \setminus U(A)$.*
(3) *If (A, m) is a local ring and M is an m-secondary A-module, then M is an Artinian* A-module.*
(4) *If A is a primary ring, then M is an Artinian* A-module.*

Proof. (1) Follows easily from the definition of Artinian* module.

(2) Follows from [15, Lemma 1].
(3) Since M is m-secondary, so $\sqrt{Ann(M)} = m$. Therefore for all $a \in A^* \setminus U(A) \subseteq m$, $a^{n_a} M = 0$ for some $n_a \in \mathbb{N}$. This implies that M is an Artinian* A-module.
(4) Let P be the unique prime ideal of A. Then $Nil(A) = P$, where $Nil(A)$ denotes the nil radical of A. Let $a \in A^* \setminus U(A) \subseteq P$. Then $a^{n_a} M = 0$ for some $n_a \in \mathbb{N}$, and so M is an Artinian* A-module. □

In the next example, we conclude that the converse of Proposition 2.4(1) is not true in general.

Example 2.5. Consider $\mathbb{Z}$ as a $\mathbb{Z}$-module and $S = \mathbb{Z} \setminus \{0\}$. Then $S^{-1}\mathbb{Z} = \mathbb{Q}$ is an Artinian* $S^{-1}\mathbb{Z} = \mathbb{Q}$-module. However, $\mathbb{Z}$ is not an Artinian* $\mathbb{Z}$-module. Indeed, let p be a prime in $\mathbb{Z}$ and $S_p = \{p^n : n \geq 0\}$. Let q be a prime in $\mathbb{Z}$ such that $q \neq p$. Consider the following descending chain of submodules $q\mathbb{Z} \supseteq q^2\mathbb{Z} \supseteq \cdots \supseteq q^n\mathbb{Z} \supseteq \cdots$ of $\mathbb{Z}$. Then $p^k q^j \mathbb{Z} \nsubseteq q^i\mathbb{Z}$ for all $i \geq j$ and for all $k, j \in \mathbb{N}$. Thus $\mathbb{Z}$ as a $\mathbb{Z}$-module is not an S_p-Artinian module, so it is not an Artinian* module.

The following theorem gives a characterization of Artinian* modules which generalizes the corresponding characterization of the Artinian modules.

Theorem 2.6. *Let M be an A-module and N be its submodule. Then M is an Artinian* A-module if and only if N and M/N are Artinian* A-modules. In particular, if $0 \to M_1 \to M \to M_2 \to 0$ is an R-exact sequence, then M is an Artinian* module if and only if M_1 and M_2 are Artinian* modules.*

Proof. This follows from [3, Theorem 3.13] for $S = S_a$ and $G = \{e\}$. $\square$

The following result is a generalization of a well-known fact that each prime ideal is maximal in Artinian rings.

Proposition 2.7. *Let A be an Artinian* ring. Then each prime ideal of A is maximal.*

Proof. Let P be a prime ideal of A and $B = A/P$. Then B is an integral domain. On contrary, suppose B is not a field. Then there exists $0 \neq b \in B$ such that b is not unit in B. Consider a descending chain of ideals $(b) \supseteq (b^2) \supseteq (b^3) \supseteq \cdots \supseteq (b^n) \supseteq \cdots$ in B. Since B is an Artinian* ring, B is an S_b-Artinian ring. Then there exist non-negative integers j, k such that $b^j(b^k) \subseteq (b^i)$ for all $i \geq k$. This implies that $b^j(b^k) \subseteq (b^{j+k+1})$, and so there exists $a \in B$ such that $b^{j+k} = ab^{j+k+1}$. Then $b^{j+k}(1 - ab) = 0$. Since B is an integral domain and b is a non-zero element of B, so $ab = 1$ which is a contradiction. Thus B is a field, and we have P is a maximal ideal of A. $\square$

Corollary 2.8. *Let A be a local ring. Then A is Artinian* if and only if $dim(A) = 0$, where $dim(A)$ denotes the Krull dimension of A.*

Proof. Suppose A is an Artinian* ring. Then by Proposition 2.7, $dim(A) = 0$. Conversely, suppose $dim(A) = 0$. Let P be the unique prime ideal of A. Then $Nil(A) = P$, and so S_a contains 0 for all $a \in A^* \setminus U(A) \subseteq P$. Hence A is an Artinian* ring. $\square$

Let $\{I_i : i \in \triangle\}$ be a family of ideals of a ring A, where $\triangle$ is an indexing set. Following [12], a prime ideal P of A is said to be *strongly prime* if $\bigcap_{i\in\triangle} I_i \subseteq P$, then $I_j \subseteq P$ for some $j \in \triangle$. A ring is said to be *strongly 0-dimensional* if all prime ideals of A are strongly prime ideals. The following proposition gives a class of strongly 0-dimensional rings.

Proposition 2.9. *An Artinian* ring is a strongly* 0*-dimensional ring.*

Proof. Let A be an Artinian* ring. Then by Proposition 2.7, A is zero-dimensional. If A is a local ring, then by [12, Lemma 2.14], A is a strongly zero-dimensional ring. If A is not local, then by Proposition 2.3, A is an Artinian ring and so by [12, Lemma 2.17], A is strongly 0-dimensional. $\square$

Corollary 2.10. *A local ring is strongly* 0*-dimensional if and only if it is an Artinian* ring.*

Proof. Follows from Proposition 2.9, Corollary 2.8, and [12, Theorem 2.9]. $\square$

Recall from [7] that an A-module M is said to be a *multiplication module* if every submodule N of M has the form $N = IM$ for some ideal I of A. Note that an A-module M is a multiplication module if and only if $N = (N : M)M$. For more information about multiplication modules, we refer [1] and [9] to the reader. Also, recall that an A module M is said to be *zero-dimensional* if $A/Ann(M)$ is a zero-dimensional ring, i.e., $A/Ann(M)$ has Krull dimension zero.

Theorem 2.11. *A multiplication module M over a local ring A is an Artinian* module if and only if M is a zero-dimensional module.*

Proof. Since M is a multiplication module over a local ring A, by [7, Proposition 4], M is cyclic; whence $M \cong A/I$, where $I = Ann(M)$. If M is an Artinian* A-module, then it is easy to see that A/I is an Artinian* A/I-module, i.e., A/I is an Artinian* ring. Therefore by Corollary 2.8, A/I is a zero-dimensional ring, which implies that M is a zero-dimensional A-module.

Conversely, suppose $M \neq \{0\}$ is a zero-dimensional A-module. Then A/I is a zero-dimensional ring, where $I = Ann(M)$ and so A/I is an Artinian* ring by Corollary 2.8. This implies that $M \cong A/I$ is an Artinian* A/I-module. Now we prove that M is an Artinian* A-module. Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq \cdots$ be a descending chain of submodules of M and $a \in A^* \setminus U(A)$. Then $(N_1 : M)/I \supseteq (N_2 : M)/I \supseteq \cdots \supseteq (N_n : M)/I \supseteq \cdots$ is a descending chain in A/I. Since A/I is an Artinian* ring, by definition A/I is an $\bar{S_a}$-Artinian ring, where $\bar{S_a} = \{a^n + I : n \geq 0\}$. Here we note that A is a local ring with the unique maximal ideal m, so $Jac(A) = m$, therefore if a is a non-zero non-unit element of A, then $a+I$ is non-zero non-unit in A/I. For if $a+I$ is a unit in A/I, then there exists $b+I$ in A/I such that

$ab-1 \in I$ which implies that $abM = M$. Therefore by [7, Proposition 1], $M = 0$ as $ab \in Jac(A)$ which is a contradiction. Now as A/I is an $\bar{S_a}$-Artinian ring, there exist $n_a, k_a \in \mathbb{N}$ such that $(a^{n_a}+I)((N_{k_a} : M)/I) \subseteq (N_n : M)/I$ for all $n \geq k_a$. Since M is a multiplication module, we have $a^{n_a}N_{k_a} = a^{n_a}(N_{k_a} : M)M \subseteq (N_n : M)M = N_n$ for all $n \geq k_a$. Thus M is an S_a-Artinian A-module. Hence M is an Artinian* module. □

It is well known that every injective endomorphism on an Artinian module is bijective. In the case of Artinian* module, it is not true in general. For this, consider the following example.

Example 2.12. Let A be a zero-dimensional local ring and $M = A^{\mathbb{N}} = \prod_{n\geq 1} A$. Then M is an Artinian* A-module since each non-unit element of A is nilpotent. Consider the endomorphism $f : M \to M$ defined by $f((x_1, x_2, \ldots, x_n, \ldots)) = (0, x_1, x_2, \ldots, x_n, \ldots)$. Then f is injective but not surjective since $(1, 0, 0, \ldots, 0, \ldots)$ has no preimage.

However, we have the following.

Proposition 2.13. *Let M be a torsion free Artinian* A-module. Then any injective endomorphism on M is an isomorphism.*

Proof. Let $f : M \to M$ be an injective endomorphism. Then $M \supseteq f(M) \supseteq f^2(M) \supseteq \cdots \supseteq f^n(M) \supseteq \cdots$ is a descending chain of submodules of M. Let $a \in A^* \setminus U(A)$. Since M is an S_a-Artinian module, there exist non-negative integers j, k such that $a^j f^k(M) \subseteq f^n(M)$ for all $n \geq k$. This implies that $f^k(a^jM) \subseteq f^{k+1}(M)$. But $a^jM \subseteq f(M)$ since f is injective. Now consider the chain $M \supseteq aM \supseteq a^2M \supseteq \cdots \supseteq a^nM \supseteq \cdots$ of submodules of M. There exists a non-negative integer t such that $a^tM = a^iM$ for all $i \geq t$ since M is an S_a-Artinian module. Let l be the maximum of j and t. Then $a^iM = a^lM \subseteq f(M)$ for all $i \geq l$. This implies that $a^lM = a^{2l}M \subseteq a^lf(M)$. Now, let $y \in M$. Then there exists $x \in M$ such that $a^ly = a^lf(x)$. This implies that $a^l(y - f(x)) = 0$. Since a is a non-zero element of A and M is a torsion free module, a^l is a non-zero element of A. Consequently, $y = f(x)$ since M is torsion free. Hence f is surjective. □

The following corollary gives a class of examples of secondary modules.

Corollary 2.14. *Let M be a torsion free Artinian* A-module. Then M is a secondary module.*

Proof. Let $a \in A$. Suppose $a^nM \neq 0$ for all integer $n \geq 1$. Define an endomorphism $f_a : M \longrightarrow M$ by $f_a(x) = ax$ for all $x \in M$. Since M is torsion free, f_a is an injective endomorphism . Therefore by Proposition 2.13, f_a is surjective. Hence $aM = M$, and so M is a secondary module. □

3. Secondary representation for Artinian* modules

Recall that an A-module M is said to be *secondary representable* (or *representable*) if M can be written as a finite sum of its secondary submodules, [14]. It is known that an Artinian module is representable (see [14]). Then it is natural to ask when an Artinian* module is representable. In this section, we prove the existence of secondary representation for Artinian* modules.

First, we introduce and study finitely cogenerated* module as a generalization of finitely cogenerated module. Recall [15] that an A-module M is called *finitely cogenerated* if for each nonempty family of submodules $\{N_i\}_{i\in\triangle}$ of M, $\bigcap_{i\in\triangle} N_i = 0$ implies that $\bigcap_{i\in F} N_i = 0$ for some finite subset F of the indexing set $\triangle$.

An A-module M is called *finitely cogenerated** if for each nonempty family of submodules $\{N_i\}_{i\in\triangle}$ of M, $\bigcap_{i\in\triangle} N_i = 0$ implies that $a^{n_a}(\bigcap_{i\in F} N_i) = 0$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$ and a finite subset $F \subseteq \triangle$.

The following example shows that the concept of finitely cogenerated* modules is a proper generalization of finitely cogenerated modules.

Example 3.1. Consider the ring $A = F[x_1, x_2, \ldots, x_n, \ldots]/(x_1, x_2^2, \ldots, x_n^n, \ldots)$, where F is a field, and $\mathfrak{p} = (x_1, x_2, \ldots, x_n, \ldots)/(x_1, x_2^2, \ldots, x_n^n, \ldots)$. Then as in Example 2.2, every element of $\mathfrak{p}$ is nilpotent and $A^* \setminus U(A) \subseteq \mathfrak{p}$, and so for every $a \in A^* \setminus U(A)$, there exists $n_a \in \mathbb{N}$ such that $a^{n_a} = 0$. Consequently, A as an A-module is finitely cogenerated*. On the other hand, A is not a finitely cogenerated A-module. For this, consider the family of ideals $\{I_i : i \in \mathbb{N}, i \geq 2\}$ of A, where $I_i = (\bar{x}_i)$ and $\bar{x}_i = x_i + (x_1, x_2^2, \ldots, x_n^n, \ldots)$. Then $\bigcap_{i\in\mathbb{N}, i\geq 2} I_i = 0$. Now let $F \subset \mathbb{N} \setminus \{1\}$ be a finite subset. Then $F = \{k_1, k_2, \ldots, k_n\}$ for some $k_i \in \mathbb{N}, k_i \geq 2$. Then the non-zero element $\bar{x}_{k_1}\bar{x}_{k_2}\ldots\bar{x}_{k_n} \in \bigcap_{i\in F} I_i$, and so $\bigcap_{i\in F} I_i \neq 0$. Thus A is not a finitely cogenerated A-module.

Let M be an A-module, S be a multiplicatively closed subset of A, and X be a nonempty family of submodules of M. Following [15, Definition 5], X is said to be an *S-cosaturated family* if whenever $sN \subseteq K$ for some $s \in S, N \in X$, and a submodule K of M, then $K \in X$. We say that X is a *cosaturated* family* if whenever $a^{n_a}N \subseteq K$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$, $N \in X$ and a submodule K of M, then $K \in X$. Clearly if X is a cosaturated* family, then X is S_a-cosaturated for every $a \in A^* \setminus U(A)$.

Following [6], $N \in X$ is called a *minimal element* of X if whenever $K \subseteq N$ for some $K \in X$, then $N \subseteq K$. In particular, M satisfies MIN condition if every nonempty family X of submodules of M has a minimal element. We say that $N \in X$ is a *minimal* element* if whenever $K \subseteq N$ for some $K \in X$, then $a^{n_a}N \subseteq K$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. In particular, M satisfies MIN* condition if every nonempty family of submodules X of M has a minimal* element.

The next theorem gives a characterization of Artinian* module in terms of finitely cogenerated* module.

Theorem 3.2. *Let M be an A-module. Then the following statements are equivalent.*

(1) *M is an Artinian* module.*
(2) *Every nonempty cosaturated* family X of M has a minimal element.*
(3) *M satisfies MIN* condition.*
(4) *Every factor module of M is a finitely cogenerated* module.*

Proof. (1) $\Rightarrow$ (2) Let X be a nonempty cosaturated* family of submodules of M. Let $\{N_i\}_{i \in I}$ be an arbitrary chain in X. Take $N = \bigcap_{i \in I} N_i$. Since M is Artinian*, for every $a \in A^* \setminus U(A)$ there exist $n_a \in \mathbb{N}$, $k_a \in I$ such that $a^{n_a}N_{k_a} \subseteq N_i$ for all $i \in I$. This implies that $a^{n_a}N_{k_a} \subseteq N$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in\ I$. Now, since X is a cosaturated* family, it is an S_a-cosaturated family. Therefore $N \in X$, and so by Zorn's lemma, X has a minimal element.

(2) $\Rightarrow$ (3) Let X be a nonempty family of submodules of M. Consider the set $X' = \{N \subseteq M :$ *there exists* $N' \in X$ *such that* $a^{n_a}N' \subseteq N$ *for any* $a \in A^* \setminus U(A)$ *and for some* $n_a \in \mathbb{N}\}$. Clearly, $X \subseteq X'$, and so X' is also nonempty. First, we will show that X' is a cosaturated* family of submodules of M. Take a submodule K of M such that $a^{n_a}N \subseteq K$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$, $N \in X'$. Since $N \in X'$, there exists $N' \in X$ such that $a^{n'_a}N' \subseteq N$ for any $a \in A^* \setminus U(A)$ and for some $n'_a \in \mathbb{N}$. Put $m_a = n_a + n'_a$. Then $a^{m_a}N' = a^{n_a}(a^{n'_a}N') \subseteq a^{n_a}N \subseteq K$ for any $a \in A^* \setminus U(A)$ and some $m_a \in \mathbb{N}$. Consequently, $K \in X'$, and hence X' is a cosaturated* family. So by assumption, X' has a minimal element say $N \in X'$. This implies that there exists $N' \in X$ such that $a^{n_a}N' \subseteq N$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. We claim that N' is a minimal* element of X. Suppose $K \in X$ such that $K \subseteq N'$. Then $a^{n_a}K \subseteq a^{n_a}N' \subseteq N$, and so $a^{n_a}K \subseteq K \cap N$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. This implies that $K \cap N \in X'$. But N is a minimal element of X', so $K \cap N = N$ which implies that $N \subseteq K$. Therefore $a^{n_a}N' \subseteq N \subseteq K$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$ which implies that N' is a minimal* element of X, as required.

(3) $\Rightarrow$ (4) We need to show that M/N is finitely cogenerated* for any submodule N of M. Let $\bigcap_{i \in \triangle}(N_i/N) = 0$, then $\bigcap_{i \in \triangle} N_i = N$. Construct the set $X =$

$\{\bigcap_{i\in F} N_i : F \subseteq \triangle,\ \textit{a finite set}\}$. By hypothesis, X has a minimal* element say $N' = \bigcap_{i\in F} N_i$ for some finite subset F of $\triangle$. Let $k \in \triangle \setminus F$, then $N' \cap N_k \subseteq N'$, where $N' \cap N_k \in X$ being finite intersection. But N' is a minimal* element in X which implies that $a^{n_a} N' \subseteq N' \cap N_k$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. Therefore $a^{n_a} N' \subseteq N_k$ for all $k \in \triangle \setminus F$ which implies that $a^{n_a} N' \subseteq \bigcap_{i\in\triangle} N_i = N$, i.e., $a^{n_a}(\bigcap_{i\in F} N_i) \subseteq N$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. Consequently, $a^{n_a}(\bigcap_{i\in F}(N_i/N)) = 0$ for any $a \in A^* \setminus U(A)$ and for some $n_a \in \mathbb{N}$. Thus M/N is a finitely cogenerated* module.

(4) $\Rightarrow$ (1) Let $N_1 \supseteq N_2 \supseteq \cdots \supseteq N_i \supseteq \cdots$ be a descending chain of submodules of M. Take $N = \bigcap_{i\in\mathbb{N}} N_i$; hence $\bigcap_{i\in\mathbb{N}}(N_i/N) = 0$. By assumption M/N is finitely cogenerated*, so for every $a \in A^* \setminus U(A)$ there exist $n_a \in \mathbb{N}$, and a finite subset F of $\mathbb{N}$ such that $a^{n_a}(\bigcap_{i\in F}(N_i/N)) = 0$. Suppose $F = \{k_1, k_2, \ldots, k_r\} \subset \mathbb{N}$, where $k_1 \leq k_1 \leq \ldots \leq k_r$, then $a^{n_a}(\bigcap_{i=k_1}^{k_r}(N_i/N)) = 0$ implies that $a^{n_a} N_{k_r} \subseteq N \subseteq N_i$ for all $i \geq k_r$. Hence M is an Artinian* module. □

Recall from [2] that a commutative ring A is said to be *regular* if for every $a \in A$ there exists $b \in A$ such that $a = a^2 b$. Also, recall that an A-module M is said to be *sum-irreducible* if $M \neq 0$ and M can not be written as a sum of two proper submodules of M.

We now prove existence of secondary representation for Artinian* modules under a mild condition. First we prove the following lemma.

Lemma 3.3. *An Artinian* sum-irreducible module is secondary.*

Proof. Let M be an Artinian* A-module which is sum-irreducible. Suppose M is not secondary, then there exists $a \in A$ such that $aM \neq M$ and $a^n M \neq 0$ for all $n \geq 1$. This implies that $a \in A^* \setminus U(A)$. Consider the multiplicatively closed set $S_a = \{a^n : n \geq 0\}$ and the descending chain $aM \supseteq a^2M \supseteq \cdots \supseteq a^nM \supseteq \cdots$ in M. Since M is an Artinian* A-module, M is S_a-Artinian which implies that there exist $k, l \in \mathbb{N}$ such that $a^l(a^k M) \subseteq a^i M$ for all $i \geq k$. Take $j = l + k$, then for all $x \in M, a^j x = a^{j+1} x'$ for some $x' \in M$, i.e., $a^j(x - ax') = 0$ which implies that $x - ax' \in (0 :_M a^j)$, where $(0 :_M a^j) = \{y \in M : a^j y = 0\}$. Consequently, $M = aM + (0 :_M a^j)$, where aM and $(0 :_M a^j)$ are proper submodules of M since $aM \neq M$ and $a^n M \neq 0$ for all $n \geq 1$. Hence M is not sum-irreducible, a contradiction. □

Theorem 3.4. *Every Artinian* module over a regular ring is representable.*

Proof. Let A be a regular ring and M be an Artinian* A-module. On contrary, suppose M is not representable. Let X be the set of all non-zero submodules of

M which are not representable. Then X is nonempty as $M \in X$. Let $L \in X$ and K be a submodule of M such that $L \subseteq K$. Since A is regular, by [5, Theorem 2.3], every submodule of a representable module is representable. Then L is not representable implies that K can not be representable. Consequently, $K \in X$. Thus X is a cosaturated family which implies that X is cosaturated*, and so by Theorem 3.2, X has a minimal element, say N. Then N is not representable, in particular N is not secondary. Therefore by Lemma 3.3, N is not sum-irreducible, and so $N = N_1 + N_2$ for some proper submodules N_1 and N_2 of N. Hence by minimality of N, we have $N_1, N_2 \notin X$. Consequently, N_1 and N_2 are representable, so N is representable, a contradiction. Hence M is representable. □

4. Artinian* modules in the amalgamated module construction

Let $f : R \to T$ be a surjective ring homomorphism and M be a T-module. Then it is well known that M is a Noetherian (resp., an Artinian) R-module if and only if M is a Noetherian (resp., an Artinian) T-module. The following result is an analog of this result.

Proposition 4.1. *Let $f : R \to T$ be a surjective ring homomorphism and M be a T-module. Then M is an Artinian* R-module if and only if M is an Artinian* T-module and M is an S_a-Artinian R-module for every $a \in \ker(f) \setminus \{0\}$.*

Proof. We may assume that $T = R/K$, where $K := \ker(f)$. Suppose M is an Artinian* R-module, and let $\bar{r}$ be a non-zero non-unit element of R/K. Then r is a non-zero non-unit element of R such that $r \notin K$. Since M is an S_r-Artinian R-module, M is an $S_{\bar{r}}$-Artinian T-module. Thus M is an Artinian* T-module. Conversely, suppose the conditions hold, and let r be a non-zero non-unit element of R such that $r \notin K$. By Proposition 2.3, we may assume that R is local. Then $\bar{r}$ is a non-zero non-unit element of R/K. Since M is an $S_{\bar{r}}$-Artinian T-module, M is an S_r-Artinian R-module. Thus M is an Artinian* R-module. □

Let $f : R \to T$ be a ring homomorphism and J be an ideal of T. The authors of [8] introduced the definition of amalgamated algebras along an ideal as follows: The following subring of $R \times T$:

$$R \bowtie^f J = \{(r, f(r) + j) \mid r \in R \text{ and } j \in J\},$$

is called the *amalgamation* of R with T along J with respect to f.

The authors of [10] extended the concept of the amalgamated algebra to modules as follows. Let M be an R-module, N be a T-module, and $\varphi : M \to N$ be an R-module homomorphism. They defined the amalgamation of M and N along J with respect to φ by

$$M \bowtie^{\varphi} JN := \{(m, \varphi(m) + n) \mid n \in M \text{ and } n \in JN\}.$$

It can be seen that $M \bowtie^{\varphi} JN$ is an $(R \bowtie^{f} J)$-module by the following scalar product

$$(r, f(r) + j) \cdot (m, \varphi(m) + n) := (rm, \varphi(rm) + f(r)n + j\varphi(m) + jn).$$

Then we have the following pullback construction.

$$\begin{tikzcd}
M \bowtie^{\varphi} JN \arrow[r] \arrow[d] & M \arrow[d, "\pi\circ\varphi"] \\
N \arrow[r, "\pi"] & N/JN
\end{tikzcd}$$

Remark 4.2. [10, Remark 2.1]

(1) $\varphi(M)+JN$ is an $(f(R)+J)$-submodule of N, and so $\varphi(M)+JN$ is an $(R \bowtie^{f} J)$-module via $p_T((r, f(r)+j)) = f(r) + J$, where $\ker(p_T) = f^{-1}(J) \times \{0\}$.

(2) $\pi_N : M \bowtie^{\varphi} JN \to \varphi(M) + JN$ defined by $\pi_N((m, \varphi(m) + n)) = \varphi(m) + n$ is an $(R \bowtie^{f} J)$-epimorphism.

(3) M is an $(R \bowtie^{f} J)$-module via $p_R((r, f(r)+j)) = r$, where $\ker(p_R) = \{0\}\times J$. Also note that $\pi_M : M \bowtie^{\varphi} JN \to M$ defined by $\pi_M((m, \varphi(m) + n)) = m$ is an $(R \bowtie^{f} J)$-homomorphism.

(4) JN is an $(f(R)+J)$-submodule of $\varphi(M)+JN$, and so JN is an $(R \bowtie^{f} J)$-submodule of $\varphi(M) + JN$.

(5) There exists an exact sequence of $(R \bowtie^{f} J)$-modules and $(R \bowtie^{f} J)$-homomorphisms:

$$0 \to JN \xrightarrow{i} M \bowtie^{\varphi} JN \xrightarrow{\pi_M} M \to 0,$$

where $i : JN \to M \bowtie^{\varphi} JN$ defined by $i(n) = (0, n)$.

Now, we characterize Artinian* modules in the amalgamated module construction.

Theorem 4.3. *Let the notation be as above. Then the following statements are equivalent.*

(1) *$M \bowtie^{\varphi} JN$ is an Artinian* $(R \bowtie^{f} J)$-module.*

(2) *M is an Artinian* R-module, JN is an Artinian* $(f(R) + J)$-module, and M is an S_a-Artinian $(R \bowtie^{f} J)$-modules for every $a \in (\{0\} \times J) \setminus \{(0, 0)\}$, and JN is an S_b-Artinian $(R \bowtie^{f} J)$-module for every $b \in (f^{-1}(J) \times \{0\}) \setminus \{(0, 0)\}$.*

(3) *M is an Artinian* R-module, $\varphi(M) + JN$ is an Artinian* $(f(R) + J)$-module, M is an S_a-Artinian $(R \bowtie^f J)$-modules for every $a \in (\{0\} \times J) \setminus \{(0, 0)\}$, and $\varphi(M) + JN$ is an S_b-Artinian $(R \bowtie^f J)$-module for every $b \in (f^{-1}(J) \times \{0\}) \setminus \{(0, 0)\}$.*

Proof. (1) $\Leftrightarrow$ (2) By Remark 4.2 and Theorem 2.6, $M \bowtie^{\varphi} JN$ is an Artinian* $(R \bowtie^f J)$-module if and only if M and JN are Artinian* $(R \bowtie^f J)$-modules. Now the assertion follows from Proposition 4.1 and Remark 4.2.

(1) $\Rightarrow$ (3) Again by Remark 4.2 and Theorem 2.6, M and $\varphi(M) + JN$ are Artinian* $(R \bowtie^f J)$-modules. Now the assertion follows again from Proposition 4.1 and Remark 4.2.

(3) $\Rightarrow$ (2) Consider the inclusion map $i_{JN} : JN \to \varphi(M) + JN$ and apply Theorem 2.6. Then the assertion follows. □

Acknowledgement. The authors sincerely thank the referee for carefully reading, useful suggestions and comments to improve the paper.
Disclosure statement. The authors report that there are no competing interests to declare.

References

[1] D. D. Anderson, T. Arabacı, Ü. Tekir and S. Koç, *On S-multiplication modules*, Comm. Algebra, 48(8) (2020), 3398-3407.

[2] D. F. Anderson and A. Badawi, *Von Neumann regular and related elements in commutative rings*, Algebra Colloq., 19(1) (2012), 1017-1040.

[3] A. U. Ansari and B. K. Sharma, *Graded S-Artinian modules and graded S-secondary representations*, Palest. J. Math., 11(3) (2022), 175-193.

[4] E. Artin, *Zur Theorie der hyperkomplexen Zahlen*, Abh. Math. Sem. Univ. Hamburg, 5(1) (1927), 251-260.

[5] S. E. Atani, *Submodules of secondary modules*, Int. J. Math. Math. Sci., 31(6) (2002), 321-327.

[6] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley Publishing Company, 1969.

[7] A. Barnard, *Multiplication modules*, J. Algebra, 71(1) (1981), 174-178.

[8] M. D'Anna, C. A. Finocchiaro and M. Fontana, *Amalgamated algebras along an ideal*, Commutative Algebra and its Applications: Proceedings of the Fifth International Fez Conference on Commutative Algebra and Applications, Fez, Morocco, June 23-28, 2008, edited by M. Fontana, S. E. Kabbaj, B. Olberding and I. Swanson, Berlin, New York: De Gruyter, 2009, 155-172.

[9] Z. A. El-Bast and P. F. Smith, *Multiplication modules*, Comm. Algebra, 16(4) (1988), 755-779.

[10] R. El Khalfaoui, N. Mahdou, P. Sahandi and N. Shirmohammadi, *Amalgamated modules along an ideal*, Commun. Korean Math. Soc., 36(1) (2021), 1-10.

[11] A. Facchini and Z. Nazemian, *Modules with chain conditions up to isomorphism*, J. Algebra, 453 (2016), 578-601.

[12] C. Jayaram, K. H. Oral and Ü. Tekir, *Strongly 0-dimensional rings*, Comm. Algebra, 41(6) (2013), 2026-2032.

[13] O. Khani-Nasab and A. Hamed, *Weakly S-Artinian modules*, Filomat, 35(15) (2021), 5215-5226.

[14] I. G. Macdonald, *Secondary representation of modules over a commutative rings*, Symposia Mathematica, 11 (1973), 23-43.

[15] M. Özen, O. A. Naji, Ü. Tekir and K. P. Shum, *Characterization theorems of S-Artinian modules*, C. R. Acad. Bulgare Sci., 74(4) (2021), 496-505.

[16] E. S. Sevim, Ü. Tekir and S. Koç, *S-Artinian rings and finitely S-cogenerated rings*, J. Algebra Appl., 19(3) (2020), 2050051 (16 pp).

Ajim Uddin Ansari
Department of Mathematics
CMP Degree College, University of Allahabad
Prayagraj-211002, India
e-mail: ajimmatau@gmail.com

Hwankoo Kim
Division of Computer Engineering, Hoseo University
Asan, Korea
e-mail: hkkim@hoseo.edu

Sanjeev Kumar Maurya (Corresponding Author)
Department of Mathematics, Galgotias University
Greator Noida-203201, India
e-mail: sanjeevm50@gmail.com

Ünsal Tekir
Department of Mathematics, Marmara University
Istanbul, Turkey
e-mail: utekir@marmara.edu.tr