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RINGS IN WHICH EVERY REGULAR FINITELY GENERATED IDEAL IS S-FINITELY PRESENTED

Salah Eddine Mahdou and Mohamed Chhiti

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Abstract. In this work, we introduce and explore a class of rings where every regular finitely generated ideal is S-finitely presented, called a regular S-coherent ring. This concept represents a weaker version of the S-coherent ring property. It is shown that any S-coherent ring is inherently a regular S-coherent ring, and in the case of domains, the two properties are equivalent. We also investigate how this notion extends to different settings of commutative ring extensions, including direct products, trivial ring extensions, and the amalgamated duplication of a ring along an ideal. The obtained results yield new examples of regular S-coherent rings that are not S-coherent.

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1. Introduction

In this paper, we assume that all rings are commutative with a nonzero identity, and all modules are unital and nonzero. Let R represent such a ring, and let S be a multiplicatively closed subset of R with $0 \notin S$. We denote by $Reg(R)$ the set of regular elements of R, and $Q(R) := R_{Reg(R)}$ refers to the total quotient ring of R. For a positive integer n, a module M is said to be n-presented if there exists an exact sequence of modules:

$$L_n \to L_{n-1} \to \ldots \to L_1 \to L_0 \to M \to 0$$

where each L_i is a finitely generated free module. In particular, 0-presented and 1-presented modules correspond to finitely generated and finitely presented modules, respectively.

The ring R is said to be coherent if every finitely generated ideal in R is finitely presented. Equivalently, this holds if the annihilator $(0 : a)$ and the intersection $I \cap J$ are finitely generated for any element $a \in R$ and any two finitely generated ideals I and J of R. Examples of coherent rings include Noetherian rings, Boolean algebras, von Neumann regular rings, valuation domains, and Prüfer domains (or semihereditary rings). As per [8], we say that R is a regular coherent ring (or reg-coherent for brief) if every regular finitely generated ideal of R is finitely presented. It is clear that any coherent ring is a regular coherent ring, and in the case of domains, the two properties are equivalent.

The concept of coherence originated from the study of coherent sheaves in algebraic geometry and evolved through influences from Noetherian ring theory and homology, ultimately becoming a significant area in algebra. Over the last three decades, several related notions have emerged from coherence, such as finite conductor, quasi-coherent, v-coherent and n-coherent rings. For further reading, refer to [4,8,16,18,19,22].

According to [1], a module M is called S-finite if there exist a finitely generated submodule N of M and an element $s \in S$ such that $sM \subseteq N$. A ring R is referred to as S-Noetherian if every ideal of R is S-finite. As per [4], a module M is S-finitely presented if there exists an exact sequence of modules

$$0 \to N \to L \to M \to 0,$$

where L is a finitely generated free module and N is S-finite. Moreover, a module M is termed S-coherent if it is finitely generated and every finitely generated submodule of M is S-finitely presented. It has been demonstrated that S-coherent rings share a characterization similar to the classical one provided by Chase for coherent rings (see [4, Theorem 3.8] and [24, Theorem 4.4]). Every coherent ring is S-coherent, and any S-Noetherian ring is also S-coherent. For further details, see [1,4].

The trivial extension of a ring R by an R-module M, denoted by $R \propto M$, is constructed as the set $R \times M$, where the operations of addition and multiplication are defined as follows: for elements $(r_1, m_1), (r_2, m_2) \in R \times M$, addition is given by $(r_1, m_1) + (r_2, m_2) = (r_1 + r_2, m_1 + m_2)$, and multiplication by $(r_1, m_1) \cdot (r_2, m_2) = (r_1 r_2, r_1 m_2 + r_2 m_1)$. The foundational properties of trivial ring extensions are well-documented in works such as [16,17]. Trivial extensions have proven to be a valuable tool in addressing various open problems and conjectures in both commutative and non-commutative ring theory (see for instance [2,3,12,13,16,17,20,21,22,23]).

Let R be a ring and I an ideal of R. The amalgamated duplication of R along I, a ring construction introduced by D'Anna in [9], was explored with the goal of applying it to the study of curve singularities over algebraically closed fields. This construction forms a subring of $R \times R$, denoted by $R \bowtie I$, and is defined as:

$$R \bowtie I = \{(a, a+i) \mid a \in A, i \in I\}.$$

If J is an ideal of R, then the set $J \bowtie I := \{(j, j+i) \mid j \in J, i \in I\}$ is an ideal in the ring $R \bowtie I$. Additionally, there exists an isomorphism between the quotient rings $\frac{R \bowtie I}{J \bowtie I}$ and $\frac{R}{J}$. In particular, when $J = 0$, we have $\frac{R \bowtie I}{(0) \bowtie I} \cong R$. Furthermore, R can be identified as a subring of $R \bowtie I$ via the natural injection $R \hookrightarrow R \bowtie I$, where each element $r \in R$ is mapped to (r, r). For further discussions on this topic, refer to [5,9,10,11,14,15].

In this paper, we explore how the regular S-coherent property may be transferred to various ring constructions, including direct products and different types of trivial extensions. Additionally, we examine how this property can be transferred to specific pullbacks and to the amalgamated duplication of a ring along an ideal. Based on these findings, we construct several families of examples of rings that are regular S-coherent but not S-coherent.

2. Main results

We begin with the following definition of a regular S-coherent ring.

Definition 2.1. Let R be a ring and S a multiplicatively closed subset of R. We say that R is a regular S-coherent ring (or reg-S-coherent for brief) if every regular finitely generated ideal of R is S-finitely presented.

Remark 2.2. (1) Every coherent (resp., S-coherent) ring is regular coherent (resp., reg-S-coherent). The equivalence holds for integral domains.

(2) Every coherent (resp., regular coherent) ring is S-coherent (resp., reg-S-coherent). The equivalence holds if $S \subseteq U(R)$.

(3) Every total ring of quotients is reg-S-coherent.

Let J be an S-finite ideal of R such that $J \cap S = \emptyset$. Define $T = \{s + J \mid s \in S\}$, which is a multiplicative subset of R/J. If R is S-coherent, then R/J is a T-coherent ring, as stated in [4, Proposition 3.9(1)]. However, this result does not hold in the context of reg-S-coherent rings, as illustrated by the following example.

Example 2.3. Let (A, M) be a non-S_0-coherent local integral domain (for instance, take $A = K[[X_1, X_2, \ldots]]$, the power series ring in infinitely many indeterminates over a field K, and $S_0 = \{1\}$). Define $R := A \propto (A/M)$, a trivial ring extension of A by A/M, and let $S = \{(s, 0) \mid s \in S_0\}$ which is a multiplicatively closed subset of R. Then:

(1) R is reg-S-coherent, as it is a total ring of quotients.

(2) $R/(0 \propto (A/M)) \cong A$ is a non-reg-S_0-coherent ring since A is a non-S_0-coherent integral domain, $S_0 \cong T$, and $S \cap (0 \propto (A/M)) = \emptyset$.

Now, we examine the transfer of the reg-S-coherent property to the direct product of rings.

Theorem 2.4. *Let S_i be a multiplicatively closed subset of a ring R_i for each $i = 1, \ldots, n$. Set $R = \prod_{i=1}^n R_i$ and $S = \prod_{i=1}^n S_i$. Then R is reg-S-coherent if and only if each R_i is reg-S_i-coherent for $i = 1, \ldots, n$.*

Proof. By induction, it suffices to establish the result for $n = 2$. Suppose $R := R_1 \times R_2$ is reg-S-coherent, where $S := S_1 \times S_2$. Let I_i be a regular finitely generated ideal of R_i for $i = 1, 2$. Then $I_1 \times I_2$ is a regular finitely generated ideal of R, and hence $I_1 \times I_2$ is S-finitely presented.

Assume without loss of generality that $I_1 = \sum\limits_{i=1}^{n} R_1 x_i$ and $I_2 = \sum\limits_{i=1}^{n} R_2 y_i$ for some $x_i \in R_1$, $y_i \in R_2$, and a positive integer n. Consider the exact sequence of R_1-modules:

$$0 \longrightarrow \ker(u_1) \longrightarrow R_1^n \xrightarrow{u_1} I_1 \longrightarrow 0,$$

where $u_1((a_i)_{i=1,\ldots,n}) = \sum_{i=1}^n a_i x_i$. Similarly, consider the exact sequence of R_2-modules:

$$0 \longrightarrow \ker(u_2) \longrightarrow R_2^n \xrightarrow{u_2} I_2 \longrightarrow 0,$$

where $u_2((b_i)_{i=1,\ldots,n}) = \sum_{i=1}^n b_i y_i$.

Now, consider the exact sequence of $R := R_1 \times R_2$-modules:

$$0 \longrightarrow \ker(u) \longrightarrow R^n \xrightarrow{u} I_1 \times I_2 \longrightarrow 0,$$

where $u((a_i, b_i)_{i=1,\ldots,n}) = \sum_{i=1}^n (a_i b_i)(x_i, y_i)$. It is evident that $\ker(u) = \ker(u_1) \times \ker(u_2)$.

On the other hand, $\ker(u)$ is S-finite since $I_1 \times I_2$ is S-finitely presented. Therefore, there exist $(s_1, s_2) \in S$ and a finitely generated $(R_1 \times R_2)$-module $L := L_1 \times L_2$, where L_i is a finitely generated R_i-module, such that:

$$(s_1, s_2) \ker(u) \subseteq L = L_1 \times L_2 \subseteq \ker(u).$$

Thus, we have

$$s_i \ker(u_i) \subseteq L_i \subseteq \ker(u_i)$$

for $i = 1, 2$. Therefore, $\ker(u_i)$ is S_i-finite, and consequently, I_i is S_i-finitely presented for $i = 1, 2$.

Conversely, assume that each R_i is reg-S_i-coherent. Let $I_1 \times I_2$ be a regular finitely generated ideal of $R_1 \times R_2$. Then I_i is a regular finitely generated ideal of

R_i for $i = 1, 2$, and since R_i is regular S_i-coherent, I_i is an S_i-finitely presented ideal of R_i.

Applying similar reasoning as before, we can show that $I_1 \times I_2$ is an S-finitely presented ideal of R, where $S = S_1 \times S_2$. This completes the proof. □

Using Theorem 2.4 in the case when S consists of unit elements, we recover the result stated in [8, Proposition 2.6].

Corollary 2.5. [8, Proposition 2.6] *Let* $R := \prod_{i=1}^{n} R_i$ *be the direct product of rings* R_i. *Then* R *is regular coherent if and only if so is* R_i, *for every* $i = 1, \ldots, n$.

We now examine whether the regular S-coherent property is preserved in the trivial ring extension of a domain D by a K-vector space V, where K is the field of quotients of D.

Theorem 2.6. *Let* $R = D \propto V$, *where* D *is an integral domain that is not a field,* $K = qf(D)$ *is its quotient field, and* V *is a* K*-vector space. Let* S' *be a multiplicatively closed subset of* R, *and define* S *as* $\{s \in D \mid (s, v) \in S'$ *for some* $v \in V\}$, *which is a multiplicatively closed subset of* D. *Then,* R *is regular* S'*-coherent if and only if* D *is (regular)* S*-coherent.*

Proof. Since D is a domain, (regular)-S-coherent and S-coherent are the same notion.

Assume that D is S-coherent and let J be a proper regular finitely generated ideal of R. There exists $(d, v) \in J$ such that $d \neq 0$ (since $(0, v)(0 \propto V) = 0_R$). Since $(d, v)R = dD \propto V$, we have $J = I \propto V$ for some proper finitely generated ideal I of D. Hence, $I = \sum_{i=1}^{n} Dx_i$ for some $x_i \in D\backslash\{0\}$ and some positive integer n and I is S-finitely presented since D is S-coherent. Hence, $J = I \otimes_D R = IR = I \propto V = \sum_{i=1}^{n} R(x_i, 0)$ since R is a D-flat module. Our aim is to show that J is S-finitely presented.

Consider the exact sequence of D-modules:

$$0 \longrightarrow \ker(v) \longrightarrow D^n \xrightarrow{v} I \longrightarrow 0$$

where $v((d_i)_{i=1,\ldots,n}) = \sum_{i=1}^{n} d_i x_i$. Also, since I is an S-finitely presented ideal of D, there exist $s \in S$ and $y_i \in D^n\backslash\{0\}$ such that:

$$s \ker(v) \subseteq \sum_{i=1}^{m} Dy_i \subseteq \ker(v) \qquad (1)$$

for some positive integer m. Now, consider the exact sequence of R-modules:

$$0 \longrightarrow \ker(u) \longrightarrow R^n \xrightarrow{u} J \longrightarrow 0$$

where $u((d_i, v_i)_{i=1,\dots,n} = \sum_{i=1}^{n}(d_i, v_i)(x_i, 0)$. Hence,

$$\begin{aligned}\ker(u) &= \left\{(d_i, v_i)_{i=1,\dots,n} \in R^n \mid \sum_{i=1}^{n}(d_i, v_i)(x_i, 0) = 0\right\} \\ &= \ker(v) \propto G\end{aligned}$$

where $G := \left\{(v_i)_{i=1,\dots,n} \in D^n \mid \sum_{i=1}^{n} x_i v_i = 0\right\} = K\ker(v)$. Hence, by (1), we have

$$(s, 0)\ker(u) \subseteq \sum_{i=1}^{n} R(x_i, 0) \subseteq \ker(u)$$

(since $sV = V$) and so $\ker(u)$ is an S-finite R-module. Therefore, J is S-finitely presented, as desired.

Conversely, assume that R is regular S-coherent and let $I := \sum_{1=i}^{n} Dx_i$ be a finitely generated proper ideal of D, where n is a positive integer and $x_i \in D$. Hence, $J := I \otimes_D R = IR = \sum_{i=1}^{n} R(x_i, 0)$ is a finitely generated regular ideal of R. Hence, J is an S-finitely presented ideal of R. By the same argument as above, we have $\ker(v)$ is an S-finite D-module. Hence I is an S-finitely presented ideal of D which completes the proof. □

Using Theorem 2.4 in the case when S consists of unit elements, we recover the result stated in [8, Theorem 2.8(1)].

Corollary 2.7. [8, Theorem 2.8(1)] *Let $R = D \propto V$, where D is an integral domain that is not a field, $K = qf(D)$ is its quotient field, and V is a K-vector space. Then, R is a regular coherent ring if and only if D is coherent.*

Now, we construct an example of a regular S-coherent ring that is neither S-coherent nor regular coherent.

Example 2.8. Let $R = D \propto V$, where D is an integral domain that is not a field, $K = \text{qf}(D)$ is its quotient field, and V is a K-vector space. Set $S = D\backslash\{0\}$ and $S' = S \propto 0$. Then:

(1) R is reg-S'-coherent.
(2) R is not S'-coherent.
(3) Assume that D is a not coherent. Then R is a non-regular coherent ring.

Proof. (1) Follows from Theorem 2.6 since D is S-coherent (by [6, Example 2.1]).

(2) Follows from [6, Theorem 2.2].

(3) Assume that D is not coherent and consider a finitely generated proper ideal I which is not finitely presented. Hence, $J := I \propto V$ is a regular finitely generated proper ideal which is not finitely presented since R is a faithfully flat D-module. □

Next, we are interesting to the trivial ring extension of a local ring (A, M) by an A-module E where $ME = 0$.

Proposition 2.9. *Consider a local ring (A, M) and an A-module E such that $ME = 0$. Then, $R := A \propto E$ is a reg-S-coherent ring for any multiplicative subset S of R.*

Proof. Straightforward since R is a total ring. □

Let's construct an example of a reg-S-coherent ring that is not S-coherent.

Example 2.10. Let (D, M) be a non-S_0-coherent local domain for some multiplicatively closed subset S_0 of D (for instance, take $D = \mathbb{Z}_{2\mathbb{Z}} + X\mathbb{R}[[X]]$ and $S_0 = \{1\}$), $E = D/M$, and $R := D \propto E$, and set $S := S_0 \propto 0$ which is a multiplicatively closed subset of R. Then:

(1) R is a reg-S-coherent ring (by Proposition 2.9).
(2) R is not S-coherent (by [6, Theorem 2.4(1)]) since D is not S_0-coherent.

Next, we investigate how the reg-S-coherent property is preserved in a specific case of pullbacks.

Theorem 2.11. *Consider a ring $T = K + M$, where K is a field and M is a maximal ideal of T. Let $R = D + M$, with D a subring of K. Suppose S is a multiplicatively closed subset of D. Then:*

(1) *If R is reg-S-coherent, then so is D.*
(2) *Assume that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (for example, if $M^n = 0$ for some positive integer n). Then, R is a reg-S-coherent ring if and only if so is D.*

Proof. (1) Let $I = (d_1, \cdots, d_n)$ be a finitely generated proper ideal of D. Since R is a flat D-module, $J := I \otimes_D R = IR = I + M = \sum_{i=1}^{n} Rd_i$ is a finitely generated ideal of R. It is clear that J is a regular ideal of R (since $0 \neq I \subseteq J$). Hence, J is an S-finitely presented ideal of R. Consider the exact sequence of R-modules:

$$0 \longrightarrow \ker(u) \longrightarrow R^n \xrightarrow{u} J \longrightarrow 0,$$

where $u\left((a_i + m_i)_{i=1,\dots,n}\right) = \sum_{i=1}^{n}(a_i + m_i)d_i = \sum_{i=1}^{n} a_i d_i + \sum_{i=1}^{n} m_i d_i$. By tensorizing the exact sequence of D-modules:

$$0 \longrightarrow \ker(v) \longrightarrow D^n \longrightarrow I \longrightarrow 0,$$

where $v((a_i)_{i=1,\dots,n}) = \sum_{i=1}^{n} a_i d_i$ by the D-flat module R, we get:

$$\begin{aligned}\ker(u) &= R\ker(v)\\ &= \ker(v) + M\ker(v).\end{aligned}$$

But $\ker(u)$ is S-finite since J is finitely generated regular and R is reg-S-coherent. Hence, there exists $s \in S$ such that:

$$s\ker(u) \subseteq \sum_{i=1}^{n} R(x_i + y_i) \subseteq \ker(u)$$

where $x_i \in D^n$, $y_i \in M^n$ and n is a positive integer. Hence, we have:

$$s\ker(v) \subseteq \sum_{i=1}^{n} Rx_i \subseteq \ker(v)$$

since $\ker(u) = \ker(v) + M\ker(v)$ and so $\ker(v)$ is an S-finite D-module. Hence, I is an S-finitely presented ideal and so D is (regular) S-coherent.

(2) Assume that for each $m \in M$, there exists $n \in M$ such that $mn = 0$. We only have to prove that if D is reg-S-coherent, then so is R. Hence, assume that D is (regular) S-coherent and let J be a regular finitely generated proper ideal of R. Hence, $J \nsubseteq M$ since M consists entirely of zero-divisors and so there exist $d \in D\backslash\{0\}$ and $m \in M$ such that $d + m \in J$. Therefore, $Dd + M = R(d + m) \subseteq J$ and so $J = I + M$ for some finitely generated proper ideal I of D. Then, I is an S-finitely presented ideal since D is a (regular) S-coherent domain and so J is an S-finitely presented ideal of R by the same argument as in (1), we may show that $\ker(u)$ is an S-finite R-module since $\ker(v)$ is an S-finite D-module and so J is an S-finitely presented ideal of R. □

Corollary 2.12. [8, Theorem 2.10] *Consider a ring $T = K + M$, where K is a field and M is a maximal ideal of T. Let $R = D + M$, with D a subring of K. Suppose that for each $m \in M$, there exists $n \in M$ such that $mn = 0$ (take for instance $M^n = 0$ for some positive integer n). Then, R is reg-coherent if and only if D is coherent.*

Let's construct an example of a reg-S-coherent ring that is neither S-coherent nor regular coherent.

Example 2.13. Let D be a non-coherent domain which is not a field, $K := qf(D)$, L be a subfield of K such that $[L : K] = \infty$, $T = \frac{L[[X]]}{<X^n>} = L + XT$ for some positive integer $n \in \mathbb{N}^*$. Set $S = D \setminus \{0\}$ and $R = D + XT$. Then:

(1) R is a regular S-coherent ring.
(2) R is not a reg-coherent ring.

(3) R is not an S-coherent ring.

Proof. (1) By [6, Example 2.1], D is (regular) S-coherent. Hence, using Theorem 2.11, R is reg-S-coherent.

(2) Since D is not coherent, R is not reg-coherent (by [8, Theorem 2.10]).

(3) If R is S-coherent, then $S^{-1}R$ is coherent. Hence, it suffices to show that $S^{-1}R$ is not coherent. But, $S^{-1}R = L + XT$ and

$$\mathrm{Ann}_{S^{-1}R}(X^{n-1}(S^{-1}R)) = \mathrm{Ann}_{S^{-1}R}(X^{n-1}(L+XT)) = XT$$

which is a non-finitely generated ideal of $S^{-1}R$. Hence, $S^{-1}R := L + XT$ is non-coherent, and so R is non-S-coherent. □

Now, we turn to the last main result of this paper, where we examine the transfer of the reg-S-coherent property to the amalgamation duplication of a ring along an ideal.

Theorem 2.14. *Let A be a ring, I an ideal of A, and $R = A \bowtie I$. Let S_0 be a multiplicatively closed subset of A, and define $S := \{(s,s) \mid s \in S_0\}$, which is a multiplicatively closed subset of R. Then the following statements hold:*

(1) *If R is reg-S-coherent, then A is reg-S_0-coherent.*
(2) *If there exists $s \in S_0$ such that $sI = 0$, then A is reg-S_0-coherent if and only if R is reg-S-coherent.*
(3) *If A is a total ring of fractions and $I \subseteq J(A)$, where $J(A)$ denotes the Jacobson radical of A, then R is a reg-S-coherent ring.*

Before proceeding with the proof of the above theorem, we first establish the following lemma.

Lemma 2.15. [7, Lemma 2.13] *Let A be a ring, I an ideal of A, and $R = A \bowtie I$.*

(1) *Let $a \in A$ and $i \in I$. If $(a, a+i) \in \mathrm{Reg}(R)$, then $a \in \mathrm{Reg}(A)$.*
(2) *Let $a \in A$. Then, $(a,a) \in \mathrm{Reg}(R)$ if and only if $a \in \mathrm{Reg}(A)$.*

Proof of Theorem 2.14.

(1) Let $J_0 := \sum_{i=1}^{n} Aa_i$ be a regular finitely generated proper ideal of A, and define $J := \sum_{i=1}^{n}(A \bowtie I)(a_i, a_i)$. Note that J is a regular ideal by Lemma 2.15. Now, the proof is similar to the proof of [6, Theorem 3.1(1)].

(2) By (1), if R is reg-S-coherent, then A is reg-S_0-coherent. Conversely, assume that A is reg-S_0-coherent and let $J := \sum_{i=1}^{n} R(b_i, b_i+j_i)$ be a regular finitely generated

ideal of R. Set

$$I_0 := \{b \in A \mid (b, b+j) \in J \text{ for some } j \in I\} = \sum_{i=1}^{n} Ab_i.$$

By Lemma 2.15 and since J is a regular finitely generated ideal of R, I_0 is a finitely generated regular ideal of A. Hence, I_0 is an S_0-finitely presented ideal of A. Consider, the exact sequence of A-modules:

$$0 \longrightarrow \ker(f) \longrightarrow A^n \xrightarrow{f} I_0 \longrightarrow 0,$$

where $f((a_i)_{i=1,\dots,n}) = \sum\limits_{i=1}^{n} a_i b_i$. Then, $\ker(f)$ is S_0-finite. Now, consider the exact sequence of R-modules,

$$0 \longrightarrow \ker(g) \longrightarrow R^n \xrightarrow{g} J \longrightarrow 0,$$

where $g((a_i, a_i + k_i)_{i=1,\dots,n}) = \sum\limits_{i=1}^{n} (a_i, a_i + k_i)(b_i, b_i + j_i)$. We have

$$\begin{aligned}
\ker(g) &= \left\{ (a_i, a_i + k_i)_{i=1,\dots,n} \in R^n \mid \sum_{i=1}^{n} (a_i, a_i + k_i)(b_i, b_i + j_i) = (0,0) \right\} \\
&= \left\{ (a_i, a_i + k_i)_{i=1,\dots,n} \in R^n \mid \sum_{i=1}^{n} a_i b_i = 0 \text{ and } \sum_{i=1}^{n} a_i j_i + b_i k_i + k_i j_i = 0 \right\} \\
&= \ker(f) \propto G.
\end{aligned}$$

But $\ker(f)$ is S_0-finite. Hence there exists $s_0 \in S_0$ such that:

$$s_0 \ker(f) \subseteq \sum_{i=1}^{n} Ax_i \subseteq \ker(f)$$

for some $x_i \in A^m$ and a positive integer m. Also, there exists $s \in S$ such that $sI = 0$. Since $x_i \in \ker(f)$, there exists $y_i \in I^n$ such that $(x_i, x_i + y_i) \in \ker(g)$. Therefore,

$$(s_0 s, s_0 s) \ker(g) \subseteq \sum_{i=1}^{n} R(x_i, x_i + y_i) \subseteq \ker(g)$$

since $s_0 sG = 0$ and so $\ker(g)$ is an S-finite module. Hence, J is S-finitely presented, and so R is reg-S-coherent, as desired.

(3) Assume that A is a total ring of fractions and $I \subseteq J(A)$. To show that $R := A \bowtie I$ is reg-S-coherent, it suffices to show that R is a total ring of fractions. The rest of the proof is like the proof of [7, Theorem 2.14(3)].

Corollary 2.16. *Let A be a ring of characteristic 6, $I = 2A$, $R := A \bowtie I$, $S_0 = \{3^n / n \in \mathbb{N}\}$ which is a multiplicative closed subset of A, and set $S = \{(s,s)/s \in S_0\}$ which is a multiplicatively closed subset of R. Then R is a reg-S-coherent ring if and only if A is a reg-S_0-coherent ring.*

Now, we present a new example of a ring that is not S-coherent but is regular S-coherent.

Example 2.17. Let $A_0 = \mathbb{Z}/6\mathbb{Z}$, $I_0 := 2A_0 (= 2\mathbb{Z}/6\mathbb{Z})$, $A = A_0 \propto (\frac{A_0}{I_0})^{\infty}$, $S_0 = \{(\bar{3},\bar{0})^n/n \in \mathbb{N}\} = \{(\bar{1},\bar{0}),(\bar{3},\bar{0})\}$ a multiplicatively closed subset of A. Set $I := (\bar{2},0)A$ (:=$2A_0 \propto 0$), $R := A \bowtie I$, and $S = \{(s,s)/s \in S_0\}$ which is a multiplicative closed subset of R. Then:

(1) R is reg-S-coherent.

(2) R is not S-coherent.

Proof. (1) By Theorem 2.14(2), it remains to show that A is reg-S_0-coherent since $(\bar{3},\bar{0})I = 0_{A_0}$. But, it is easy to see that A is a total ring of quotient since $(\bar{0},e)(0,f) = (\bar{2},e)(0,f) = (\bar{3},e)(\bar{2},0) = (\bar{4},e)(0,f) = 0_A$, $(\bar{1},e)$ and $(\bar{5},e)$ are invertibles in A for every $e,f \in (\frac{A_0}{I_0})^{\infty}$. Hence, A is reg-S_0-coherent and so R is reg-S-coherent, as desired.

(2) Our aim is to show that R is not an S-coherent ring. By [6, Theorem 3.1], it suffices to show that A is not an S_0-coherent ring.

We claim that $Ann((\bar{0},e))$ is not an S_0-finite ideal of A, where $e = (\bar{1},\bar{0},\dots,\bar{0},\dots) \in (\frac{A_0}{I_0})^{\infty}$. Deny. Hence, $Ann((\bar{0},e))$ is an S_0-finite ideal of A. But,

$$Ann((\bar{0},e)) = \{(a,f) \in A/(0,0) = (\bar{0},e)(a,f)\} = I_0 \propto (\frac{A_0}{I_0})^{\infty}.$$

Hence, there exists $n = 0,1$ such that:

$$(3^n,0)(I_0 \propto (\frac{A_0}{I_0})^{\infty}) \subseteq \sum_{i=1}^{m}(A_0 \propto (\frac{A_0}{I_0})^{\infty})(x_i,f_i) \subseteq I_0 \propto (\frac{A_0}{I_0})^{\infty}$$

for some positive integer m, $x_i \in I_0$ and $f_i \in (\frac{A_0}{I_0})^{\infty}$. Hence, since $3I_0 = 0$ and $3(\frac{A_0}{I_0}) = \frac{A_0}{I_0}$, we have:

$$0 \propto (\frac{A_0}{I_0})^{\infty} \subseteq \sum_{i=1}^{m}(A_0 \propto (\frac{A_0}{I_0})^{\infty})(x_i,f_i) \subseteq I_0 \propto (\frac{A_0}{I_0})^{\infty}$$

and so

$$(\frac{A_0}{I_0})^{\infty} \subseteq \sum_{i=1}^{m} A_0 f_i \subseteq (\frac{A_0}{I_0})^{\infty}$$

which means that $(\frac{A_0}{I_0})^{\infty} = \sum_{i=1}^{m} A_0 f_i$ is a finitely generated A_0-module, a desired contradiction.

Therefore, $Ann((\bar{0},e))$ is not an S_0-finite ideal of A and so A is not an S_0-coherent ring. Hence, R is not an S-coherent ring and this completes the proof of Example 2.17. □

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Salah Eddine Mahdou
Laboratory of Modelling and Mathematical Structures
Faculty of Science and Technology of Fez, Box 2202
University S. M. Ben Abdellah
Fez, Morocco
e-mail: salahmahdoulmtiri@gmail.com

Mohamed Chhiti (Corresponding Author)
Laboratory of Modelling and Mathematical Structures
Faculty of Economics and Social Sciences of Fez
University S. M. Ben Abdellah
Fez, Morocco
e-mail: chhiti.med@hotmail.com