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S-M-CYCLIC SUBMODULES AND SOME APPLICATIONS

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Abstract. In this paper, we introduce the notion of S-M-cyclic submodules, which is a generalization of the notion of M-cyclic submodules. Let M, N be right R-modules and S be a multiplicatively closed subset of a ring R. A submodule A of N is said to be an S-M-cyclic submodule, if there exist $s \in S$ and $f \in Hom_R(M, N)$ such that $As \subseteq f(M) \subseteq A$. Besides giving many properties of S-M-cyclic submodules, we generalize some results on M-cyclic submodules to S-M-cyclic submodules. Furthermore, we generalize some properties of principally injective modules and pseudo-principally injective modules to S-principally injective modules and S-pseudo-principally injective modules, respectively. We study the transfer of this notion to various contexts of these modules.

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1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary right R-modules. Let M be a right R-module. The annihilator of M, denoted by $Ann_R(M)$, is $Ann_R(M) = \{r \in R \mid Mr = 0\}$. A nonempty subset S of R is said to be *multiplicatively closed set of* R, if $0 \notin S, 1 \in S$ and $ss^{'} \in S$ for all $s, s^{'} \in S$. From now on S will always denote a multiplicatively closed set of R. In this paper, we concern with S-M-cyclic submodules which are generalizations of M-cyclic submodules. Let M be a right R-module. Recall from [15], a submodule N of M is called M*-cyclic*, if it is isomorphic to M/L for some submodule L of M. Hence any M-cyclic submodule X of M can be considered as the image of an endomorphism of M. Nguyen Van Sanh et al. in their paper [15] gave the concept of M-cyclic submodules and used them to characterize certain classes of M-principally injective modules. A right R-module N is called *M-principally injective*, if every R-homomorphism from an M-cyclic submodule of M to N can be extended to M. Nguyen Van Sanh et al. give some characterizations and properties of quasi-principally injective modules which generalize results of Nicholson and Yousif ([10]). The notion of M-principally injective module has attracted many researchers and it has been studied in many papers. See, for examples, [8], [11], [12] and [14]. Recall from [5] that a right R-module N is called *pseudo-M-principally injective*, if every monomorphism from an M-cyclic submodule X of M to N can be extended to an R-homomorphism from M to N. They study the structure of the endomorphism ring of a quasi-pseudo-principally injective module M which is a quasi-projective Kasch module (see [5, Theorem 2.5 and Theorem 2.6]). The readers can refer to [4], [6], [13] and [17] for more details on pseudo-M-principally injective modules.

In this paper, we introduce S-M-cyclic submodules, S-M-principally injective modules and S-pseudo-M-principally injective modules which are generalizations of M-cyclic submodules, M-principally injective modules and pseudo-M-principally injective modules, respectively. In Section 2, we give some examples of S-M-cyclic submodules, see Example 2.3. We give the necessary and sufficient conditions for the submodule of a right R-module to be an S-M-cyclic submodule, list in Theorem 2.15 and Theorem 2.16. At the end of Section 2, we give the necessary and sufficient conditions for a simple module to be an S-M-cyclic submodule, list in Proposition 2.16 and Proposition 2.17. In Section 3, we give an example of S-M-principally injective module, see Example 3.2. Several characterizations and some properties of S-M-principally injective modules are given in this section. As the main results, in Section 4, we give the necessary and sufficient conditions for the S-pseudo-M-principally injective module to be an S-M-principally injective module, see Theorem 4.12.

2. S-M-cyclic submodules

We start with the following definitions.

Definition 2.1. Let S be a multiplicatively closed subset of R, M and N be right R-modules.

(1) A submodule A of N is called an S-M-cyclic submodule of N, if there exist $s \in S$ and $f \in Hom_R(M, N)$ such that $As \subseteq f(M) \subseteq A$.

(2) A right R-module N is called an S-M-cyclic module, if every submodule of N is an S-M-cyclic submodule of N.

(3) A right (left) ideal I of R is called an S-R-cyclic right (left) ideal of R, if I_R $({}_RI)$ is an S-R-cyclic submodule of R_R $({}_RR)$ and a ring R is called right (left) S-R cyclic, if R_R $({}_RR)$ is an S-R-cyclic module.

Remark 2.2. (1) Let M be a right R-module and S a multiplicatively closed subset of a ring R. If $ann_R(M) \cap S \neq \phi$, then M is trivially an S-M-cyclic module.

(2) To avoid this trivial case, from now on we assume that all multiplicatively closed subset of a ring R satisfies $ann_R(M) \cap S = \phi$.

(3) Let M be a right R-module. The M-cyclic submodule of M is a special case of S-M-cyclic submodule of M when $S = \{1\}$.

Example 2.3. (1) From [3], for right R-modules M and N, N is called a *fully-M-cyclic module*, if every submodule A of N, there exists $f \in Hom_R(M, N)$ such that $A = f(M)$. It is clear that every fully-M-cyclic module is an S-M-cyclic module.

(2) Let M be a right R-module. We can see that every simple module is an S-M-cyclic module for any multiplicatively closed subset S of R.

(3) Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,

$$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}, M = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}$$

and

$$N = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$

Then

(3.1) R is a ring.

(3.2) M and N are right R-modules.

(3.3) N is an S-M-cyclic module.

Proof. The proof of (3.1) and (3.2) are routine by using definitions of a ring and a right R-module.

(3.3) Note that all nonzero submodules of N are

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_p \right\}, E_k = \left\{ \begin{bmatrix} ak & 0 \\ a & 0 \end{bmatrix} \mid a \in \mathbb{Z}_p \right\} \text{ where } k \in \mathbb{Z}_p \text{ and } N.$$

Let A be a nonzero submodule of N.

Case 1. $A = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_p \right\}$. Define $f : M \to N$ by

$$f\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{for all } \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in M.$$

It is clear that $f \in Hom_R(M, N)$. Choose $s \in S$. We can show that $As \subseteq f(M) \subseteq A$.

Case 2. $A = E_k = \left\{ \begin{bmatrix} ak & 0 \\ a & 0 \end{bmatrix} \mid a \in \mathbb{Z}_p \right\}$ for some $k \in \mathbb{Z}_p$.

Define $f_k : M \to N$ by

$$f_k\left(\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} ak & 0 \\ a & 0 \end{bmatrix} \qquad \text{for all } \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \in M.$$

It is clear that $f_k \in Hom_R(M, N)$. We can choose $s \in S$ and show that $As \subseteq f_k(M) \subseteq A$.

Case 3. $A = N$. It is obvious.

From Case 1, Case 2 and Case 3, we have N is an S-M-cyclic module. □

Proposition 2.4. *Let M and N be right R-modules. Every M-cyclic submodule of N is an S-M-cyclic submodule of N for any multiplicatively closed subset S of R.*

Proof. Let S be a multiplicatively closed subset of R and A be an M-cyclic submodule of N. There exists $f \in Hom_R(M, N)$ such that $A = f(M)$. Choose $s \in S$. Let $as \in As$. Since $a \in A = f(M)$, there exists $m \in M$ such that $a = f(m)$. Then $as = f(m)s = f(ms) \in f(M)$ and thus $As \subseteq f(M)$. So $As \subseteq f(M) \subseteq A$. Therefore A is an S-M-cyclic submodule of N. □

Proposition 2.5. *Let $U(R)$ be the set of all units in a ring R and M, N be right R-modules. If $S \subseteq U(R)$, then every S-M-cyclic submodule of N is an M-cyclic submodule of N.*

Proof. Suppose that $S \subseteq U(R)$. Let A be an S-M-cyclic submodule of N. There exist $s \in S$ and $f \in Hom_R(M, N)$ such that $As \subseteq f(M) \subseteq A$. Then

$$Ass^{-1} \subseteq f(M)s^{-1} \subseteq As^{-1},$$
$$A \subseteq f(M)s^{-1} \subseteq A.$$

So $A = f(M)s^{-1}$. Since $A = f(M)s^{-1} = f(Ms^{-1}) \subseteq f(M) \subseteq A$, $f(M) = A$. Therefore A is an M-cyclic submodule of N. □

Proposition 2.6. *Let M, N be right R-modules and A, B be submodules of N such that $A \subseteq B$. If A is an S-M-cyclic submodule of B, then A is an S-M-cyclic submodule of N.*

Proof. Suppose that A is an S-M-cyclic submodule of B. There exist $s \in S$ and $f \in Hom_R(M, B)$ such that $As \subseteq f(M) \subseteq A$. But $B \subseteq N$, we have $f \in Hom_R(M, N)$ and thus A is an S-M-cyclic submodule of N. □

Proposition 2.7. *Let M be a right R-module, A and B be submodules of M. If A is an S-M-cyclic submodule of M and B is an S-A-cyclic submodule of A, then B is an S-M-cyclic submodule of M.*

Proof. Suppose that A is an S-M-cyclic submodule of M and B is an S-A-cyclic submodule of A. There exist $s_1, s_2 \in S$, $f_1 \in End_R(M)$ and $f_2 \in End_R(A)$ such that $As_1 \subseteq f_1(M) \subseteq A$ and $Bs_2 \subseteq f_2(A) \subseteq B$. Since S is a multiplicatively closed subset of R, $s_2s_1 \in S$ and thus $Bs_2s_1 \subseteq f_2(A)s_1 \subseteq f_2f_1(M) \subseteq f_2(A) \subseteq B$ where $f_2f_1 \in End_R(M)$. Therefore B is an S-M-cyclic submodule of M. □

Proposition 2.8. *Let M and N be right R-modules. Then N is an S-M-cyclic module if and only if every submodule of N is an S-M-cyclic module.*

Proof. First, we suppose that N is an S-M-cyclic module. Let A be a submodule of N and B be a submodule of A. Then B is a submodule of N and by the assumption, there exist $s \in S$ and $f \in Hom_R(M, N)$ such that $Bs \subseteq f(M) \subseteq B$. Since $f(M) \subseteq B$ and $B \subseteq A$, $f \in Hom_R(M, A)$. Hence A is an S-M-cyclic module. The converse of this proposition is obvious. □

We can change from submodules to be essential submodules which is shown in the following result.

Proposition 2.9. *Let M and N be right R-modules. Then N is an S-M-cyclic module if and only if every essential submodule of N is an S-M-cyclic module.*

Proof. ($\Rightarrow$) It follows by Proposition 2.8.
($\Leftarrow$) Since N is an essential submodule of N and by assumption, N is an S-M-cyclic module. □

Proposition 2.10. *Let M, P and Q be right R-modules with $P \cong Q$. If P is an S-M-cyclic module, then Q is an S-M-cyclic module.*

Proof. Suppose that P is an S-M-cyclic module. Let L be a submodule of Q. Since $P \cong Q$, there exists an isomorphism $f : Q \to P$. By assumption, there exist $s \in S$ and $h \in Hom_R(M, P)$ such that $f(L)s \subseteq h(M) \subseteq f(L)$. Then

$$f(Ls) \subseteq h(M) \subseteq f(L),\ f^{-1}f(Ls) \subseteq f^{-1}h(M) \subseteq f^{-1}f(L),\ Ls \subseteq f^{-1}h(M) \subseteq L.$$

But $f^{-1}h \in Hom_R(M, Q)$, we have Q is an S-M-cyclic module. $\square$

Proposition 2.11. *Let $M, M^{'}$ and N be right R-modules which N is an S-M-cyclic module. If M is an R-epimorphic image of $M^{'}$, then N is an S-$M^{'}$-cyclic module.*

Proof. Suppose that M is an R-epimorphic image of $M^{'}$. There exists an R-homomorphism $\alpha : M^{'} \to M$ such that $\alpha(M^{'}) = M$. Let A be a submodule of N. Since N is an S-M-cyclic module, there exist $s \in S$ and $\beta : M \to N$ such that $As \subseteq \beta(M) \subseteq A$. Then $As \subseteq \beta\alpha(M^{'}) \subseteq A$. But $\beta\alpha \in Hom_R(M^{'}, N)$, we have N is an S-$M^{'}$-cyclic module. $\square$

Proposition 2.12. *Let M, N be right R-modules and A, B be submodules of N such that $B \subseteq A$. If A is an S-M-cyclic submodule of N, then A/B is an S-M-cyclic submodule of N/B.*

Proof. Suppose that A is an S-M-cyclic submodule of N. There exist $s \in S$ and $f \in Hom_R(M, N)$ such that $As \subseteq f(M) \subseteq A$. Define $\overline{f} : M \to N/B$ by $\overline{f}(m) = f(m) + B$ for all $m \in M$. It is clear that $\overline{f}$ is well defined and an R-homomorphism. Then $(A/B)s \subseteq \overline{f}(M) \subseteq A/B$. Therefore A/B is an S-M-cyclic submodule of N/B. $\square$

Lemma 2.13. *Let M, N be right R-modules and S_1, S_2 be multiplicatively closed subsets of R such that $S_1 \subseteq S_2$. If N is an S_1-M-cyclic submodule of N, then N is an S_2-M-cyclic submodule of N.*

Proof. This is clear. $\square$

Recall from [1], let S be a multiplicatively closed subset of R. The saturation S^* of S is defined as $S^* = \{x \in R \mid x|y \text{ for some } y \in S\}$. A multiplicatively closed subset S of R is called a *saturated multiplicatively closed set* if $S = S^*$.

Theorem 2.14. *Let M and N be right R-modules and A be a submodule of N. Then A is an S-M-cyclic submodule of N if and only if A is an S^*-M-cyclic submodule of N.*

Proof. ($\Rightarrow$) Since $S \subseteq S^*$ and by Lemma 2.13, we have A is an S^*-M-cyclic submodule of N.
($\Leftarrow$) Suppose that A is an S^*-M-cyclic submodule of N. There exist $x \in S^*$ and $f \in Hom_R(M, N)$ such that $Ax \subseteq f(M) \subseteq A$. Choose $y \in R$ with $xy \in S$. Then

$Axy \subseteq f(M)y = f(My) \subseteq f(M) \subseteq A$. Hence A is an S^*-M-cyclic submodule of N. $\square$

Theorem 2.15. *Let R be a commutative ring, M, N right R-modules and A a submodule of N. Then A is an S-M-cyclic submodule of N if and only if As is an S-M-cyclic submodule of N for all $s \in S$.*

Proof. $(\Rightarrow)$ Let $s \in S$. Since A is an S-M-cyclic submodule of N, there exist $s_1 \in S$ and $f \in Hom_R(M, N)$ such that $As_1 \subseteq f(M) \subseteq A$ and thus $As_1s \subseteq f(M)s \subseteq As$. But R is a commutative ring, $Ass_1 \subseteq f(Ms) \subseteq As$. Define $h : M \to N$ by $h(m) = f(ms)$ for all $m \in M$. It is clear that h is well-defined and an R-homomorphism from M to N. So $Ass_1 \subseteq h(M) \subseteq As$ and hence As is an S-M-cyclic submodule of N.
$(\Leftarrow)$ Since $1 \in S$, A is an S-M-cyclic submodule of N. $\square$

Theorem 2.16. *Let M and N be right R-modules which N is an S-M-cyclic module and A is a submodule of N. Then*

(1) *A is an essential submodule of N if and only if for each $t \in Hom_R(M, N)$-$\{0\}$, $t(M) \cap A \neq \{0\}$.*
(2) *A is a uniform module if and only if for each $t \in Hom_R(M, A)$-$\{0\}$, $t(M)$ is an essential submodule of A.*

Proof.
(1) $(\Rightarrow)$ It is obvious.
$(\Leftarrow)$ Let B be a nonzero submodule of N. Since N is an S-M-cyclic module, there exist $s \in S$ and $f \in Hom_R(M, N)$ such that $Bs \subseteq f(M) \subseteq B$. By assumption, $f(M) \cap A \neq \{0\}$. But $\{0\} \neq f(M) \cap A \subseteq B \cap A$, $B \cap A \neq \{0\}$. Therefore A is an essential submodule of N.
(2) $(\Rightarrow)$ It is obvious.
$(\Leftarrow)$ Let B and C be nonzero submodules of A. Since N is an S-M-cyclic module, there exist $s_1, s_2 \in S$ and $f_1, f_2 \in Hom_R(M, N)$ such that $Bs_1 \subseteq f_1(M) \subseteq B$ and $Cs_1 \subseteq f_2(M) \subseteq C$. But B and C are submodules of A, we have $f_1, f_2 \in Hom_R(M, A)$. By assumption, $f_1(M)$ and $f_2(M)$ are essential submodules of A and thus $f_1(M) \cap f_2(M) \neq \{0\}$. Since $f_1(M) \subseteq B$ and $f_2(M) \subseteq C$, $\{0\} \neq f_1(M) \cap f_2(M) \subseteq B \cap C$ and thus $B \cap C \neq \{0\}$. Therefore A is a uniform module. $\square$

Proposition 2.17. *Let M and N be right R-modules with $Hom_R(M, N) \neq \{0\}$. Then N is a simple module if and only if N is an S-M-cyclic module with every nonzero R-homomorphism from M to N an epimorphism.*

Proof. ($\Rightarrow$) It is obvious.
($\Leftarrow$) Let A be a nonzero submodule of N. Since N is an S-M-cyclic module, there exist $s \in S$ and $f \in Hom_R(M, N)$ such that $As \subseteq f(M) \subseteq A$. By assumption, $f(M) = N$ and thus $A = N$. Hence N is a simple module. □

A right R-module M is said to satisfy $(**)$-property if every non-zero endomorphism of M is an epimorphism (see [16]).

Proposition 2.18. *Let M be a right R-module. Then M is a simple module if and only if M is an S-cyclic module with $(**)$-property.*

Proof. ($\Rightarrow$) It is clear.
($\Leftarrow$) Suppose that M is an S-cyclic module with $(**)$-property. Let N be a non-zero submodule of M. By assumption, there exist $s \in S$ and $f \in End_R(M)$ such that $Ns \subseteq f(M) \subseteq N$. Since M satisfies $(**)$-property, f is an R-epimorphism and thus $f(M) = M$. So we have $M = N$. Hence M is a simple module. □

Corollary 2.19. *If a right R-module M is an S-cyclic module with $(**)$-property, then $End_R(M)$ is a division ring.*

3. S-M-principally injective modules

In this section, we introduce a general form of M-principally injectivity.

Definition 3.1. Let S be a multiplicatively closed subset of a ring R and M be a right R-module. A right R-module N is called an *S-M-principally injective module* (for short S-M-p-injective module) if every R-homomorphism from S-M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. M is called *a quasi S-principally injective module* (for short quasi S-p-injective module), if M is an S-M-principally injective module. In the case of a ring R, R is called *a quasi S-principally injective module* if R_R is a quasi S-principally-injective module. In the case $S = \{1\}$, N is called an *M-principally-injective module* that one refer to [15].

Example 3.2. Let $\mathbb{Z}_p$ be the set of all integers modulo p where p is a prime number,
$$R = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}, \ N = \left\{ \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \mid a \in \mathbb{Z}_p \right\}, \text{ and}$$
$$M = \left\{ \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \mid a, b \in \mathbb{Z}_p \right\}.$$
It is clear that R is a ring under matrix addition and matrix multiplication and M, N are right R-modules. Let S be a multiplicatively closed subset of R. Then

(1) N is an S-R_R-principally injective module.
(2) M is an S-M-principally injective module.

Proof. It is easy to prove. $\square$

Proposition 3.3. *Let M be a right R-module and N be an S-M-cyclic submodule of M. If N is an S-M-principally injective module, then N is a direct summand of M.*

Proof. Suppose that N is an S-M-principally injective module. Consider the short exact sequence $0 \to N \xrightarrow{i_N} M \xrightarrow{\pi_N} M/N \to 0$ where i_N is the inclusion map from N to M and π_N is the canonical projection from M to M/N. Since N is an S-M-principally injective module, there exists an R-homomorphism α from M to N such that $\alpha \circ i_N = i_N$. So the short exact sequence splits. Hence N is a direct summand of M. $\square$

Proposition 3.4. *Let M, N and K be right R-modules with $N \cong K$. If N is an S-M-principally injective module, then K is an S-M-principally injective module.*

Proof. Suppose that N is an S-M-principally injective module. Let A be an S-M-cyclic submodule of M and α be an R-homomorphism from A to K. Since $N \cong K$, there exists an isomorphism f from K to N. But N is an S-M-principally injective module, there exists an R-homomorphism g from M to N such that $g \circ i_A = f \circ \alpha$ where i_A is the inclusion on A. So $f^{-1} \circ g \circ i_A = f^{-1} \circ f \circ \alpha = \alpha$. Therefore K is an S-M-principally injective module. $\square$

Proposition 3.5. *Let M and N be right R-modules and A be a direct summand of N. If N is an S-M-principally injective module, then*

(1) *A is an S-M-principally injective module.*
(2) *N/A is an S-M-principally injective module.*

Proof. Suppose that N is an S-M-principally injective module. Since A is a direct summand of N, there exists a submodule A' of N such that $N = A \oplus A'$.
(1) Let B be an S-M-cyclic submodule of M and α be an R-homomorphism from B to A. Since N is an S-M-principally injective module, there exists an R-homomorphism β from M to N such that $\beta \circ i_B = i_A \circ \alpha$ where i_A and i_B are inclusion maps on A and B, respectively. Let π_A be a canonical projection of $N = A \oplus A'$ to A. Then $\pi_A \circ \beta \circ i_B = \pi_A \circ i_A \circ \alpha = \alpha$. Therefore A is an S-M-principally injective module.
(2) By (1), A' is an S-M-principally injective module. Since $A' \cong N/A$ and by Proposition 3.4, N/A is an S-M-principally injective module. $\square$

Theorem 3.6. *Let A and M be right R-modules. Then A is an S-M-principally injective module if and only if A is an S-X-principally injective module for every S-M-cyclic submodule X of M.*

Proof. ($\Rightarrow$) Suppose that A is an S-M-principally injective module. Let X be an S-M-cyclic submodule of M, B an S-X-cyclic submodule of X and φ an R-homomorphism from B to A. By Proposition 2.8, B is an S-M-cyclic submodule of M. But A is an S-M-principally injective module, there exists $\overline{\varphi} : M \to N$ such that $\overline{\varphi} \circ i_B = \varphi$ where i_B is an inclusion map on B. Hence A is an S-X-principally injective module.
($\Leftarrow$) Clear. $\square$

By A. Haghany and M. R. Vedadi [7], a right R-module M is called *co-Hopfian* (*Hopfian*) if every injective (surjective) endomorphism $f : M \to M$ is an automorphism. According to [9], a right R-module M is called *directly finite*, if it is not isomorphic to a proper direct summand of M.

Lemma 3.7. ([9, Proposition 1.25]) *An R-module M is directly finite if and only if $f \circ g = I$ implies $g \circ f = I$ for any $f, g \in End_R(M)$.*

Proposition 3.8. *Let M be a quasi S-principally injective directly finite module. Then M is a co-Hopfian module.*

Proof. Let $f : M \to M$ be an R-monomorphism. Since M is a quasi S-principally injective module and an S-M-cyclic submodule of M, there exists $g : M \to M$ such that $g \circ f = I_M$ where I_M is an identity map on M. By Lemma 3.7, $f \circ g = I_M$ and thus f is an epimorphism. Therefore M is co-Hopfian. $\square$

Corollary 3.9. *Let M be a quasi S-principally injective and Hopfian module. Then M is a co-Hopfian module.*

4. S-pseudo-M-principally injective modules

In this section, we introduce a general form of pseudo-M-principally injectivity.

Definition 4.1. Let S be a multiplicatively closed subset of a ring R and M be a right R-module. A right R-module N is called *S-pseudo-M-principally injective* (for short S-pseudo-M-p-injective) if every monomorphism from S-M-cyclic submodule of M to N can be extended to an R-homomorphism from M to N. The module M is called *quasi S-pseudo-principally injective* (for short quasi S-pseudo-p-injective) if M is an S-pseudo-M-principally injective module. In the case of a ring R, R

is called *quasi S-pseudo-principally injective* if R_R is a quasi S-pseudo-principally injective module.

In the case $S = \{1\}$, N is called *a pseudo-M-principally injective module* that one refer to [5].

Example 4.2. Let M be a right R-module. Then every S-M-principally injective module is an S-pseudo-M-principally injective module.

Proposition 4.3. *Let M, A and B be right R-modules such that $A \cong B$.*

(1) *If A is an S-pseudo-M-principally injective module, then B is an S-pseudo-M-principally injective module.*
(2) *If M is an S-pseudo-A-principally injective module, then M is an S-pseudo-B-principally injective module.*

Proof. Straightforward. □

Proposition 4.4. *Let A and M be right R-modules. Then A is an S-pseudo-M-principally injective module if and only if A is an S-pseudo-X-principally injective module for every S-M-cyclic submodule X of M.*

Proof. It is similar to the proof of Theorem 3.6. □

Corollary 4.5. *Let M and N be right R-modules. If N is an S-pseudo-M-principally injective module and A is a direct summand of M, then N is an S-pseudo-A-principally injective module.*

Proof. By Proposition 4.4. □

Proposition 4.6. *Let M be a right R-module. Every direct summand of an S-pseudo-M-principally injective module is an S-pseudo-M-principally injective module.*

Proof. Let N be an S-pseudo-M-principally injective module and A be a direct summand of N. Let B be an S-M-cyclic submodule of M and φ be a monomorphism from B to A. Since N is an S-pseudo-M-principally injective module, there exists an R-homomorphism α from M to N such that $\alpha \circ i_B = i_A \circ \varphi$ where i_A and i_B are inclusion maps on A and B, respectively. So $\pi_A \circ \alpha \circ i_B = \pi_A \circ i_A \circ \varphi = \varphi$ where π_A is a canonical projection of N to A. Therefore A is an S-pseudo-M-principally injective module. □

Two right R-modules M_1 and M_2 are relatively (or mutually) S-pseudo principally injective, if M_1 is an S-pseudo-M_2-principally injective module and M_2 is an S-pseudo-M_1-principally injective module.

Proposition 4.7. *Let M_1 and M_2 be right R-modules. If $M_1 \oplus M_2$ is a quasi S-pseudo-principally injective module, then M_1 and M_2 are relatively S-pseudo-principally injective modules.*

Proof. Let A be an S-M_2-cyclic submodule of M_2 and φ a monomorphism from A to M_1. Define $\psi : A \to M_1 \oplus M_2$ by $\psi(a) = (\varphi(a), a)$ for all $a \in A$. It is clear that ψ is well-defined and an R-homomorphism. Since φ is a monomorphism, ψ is a monomorphism from A to $M_1 \oplus M_2$. But $M_1 \oplus M_2$ is a quasi S-pseudo-principally injective module, there exists an R-homomorphism α from $M_1 \oplus M_2$ to $M_1 \oplus M_2$ such that $\alpha \circ i_{M_2} \circ i_A = \psi$ where i_A is an inclusion map on A and i_{M_2} is an injection map on M_2. So $\pi_{M_1} \circ \alpha \circ i_{M_2} \circ i_A = \pi_{M_1} \circ \psi = \varphi$ where π_{M_1} is a projection map from $M_1 \oplus M_2$ to M_1. Hence M_1 is an S-pseudo-M_2-principally injective module. Similarly, we can proved that M_2 is an S-pseudo-M_1-principally injective module. $\square$

Proposition 4.8. *Let M and N_i be right R-modules for all $i = 1, 2, \ldots, n$. If $\bigoplus_{i=1}^{n} N_i$ is an S-pseudo-M-principally injective module, then N_i is an S-pseudo-M-principally injective module for all $i = 1, 2, \ldots, n$.*

Proof. Suppose that $\bigoplus_{i=1}^{n} N_i$ is an S-pseudo-M-principally injective module. Let $i \in \{1, 2, \ldots, n\}$, A be an S-M-cyclic submodule of M and φ be a monomorphism from A to N_i. Since $\bigoplus_{i=1}^{n} N_i$ is an S-pseudo-M-principally injective module and $i_{N_i} \circ \varphi$ is a monomorphism from A to $\bigoplus_{i=1}^{n} N_i$ where i_{N_i} is the $i^{\underline{th}}$ injective map from N_i to $\bigoplus_{i=1}^{n} N_i$, there exists an R-homomorphism α from M to $\bigoplus_{i=1}^{n} N_i$ such that $i_{N_i} \circ \varphi = \alpha \circ i_A$ where i_A is an inclusion map from A to M. So $\pi_{N_i} \circ \alpha \circ i_A = \pi_{N_i} \circ i_{N_i} \circ \varphi = \varphi$ where π_{N_i} is the $i^{\underline{th}}$ projection map from $\bigoplus_{i=1}^{n} N_i$ to N_i. Therefore N_i is an S-pseudo-principally injective module. $\square$

Lemma 4.9. *Let M be a right R-module and A be an S-M-cyclic submodule of M. If A is an S-pseudo-M-principally injective module, then A is a direct summand of M.*

Proof. Suppose that A is an S-pseudo-M-principally injective module. Let $i_A : A \to M$ be an inclusion map and $I_A : A \to A$ be the identity map. By assumption, there exists an R-homomorphism $\varphi : M \to A$ such that $\varphi \circ i_A = I_A$. Thus the short exact sequence $0 \to A \to M$ splits. So $Im(i_A) = A$ is a direct summand of M. $\square$

A right R-module M is called *weakly co-Hopfian* ([7]), if any injective endomorphism f of M is essential i.e., $f(M) \ll_e M$.

Theorem 4.10. *Let M be a quasi S-pseudo-principally injective module.*

(1) *If M is a weakly co-Hopfian module, then M is a co-Hopfian module.*
(2) *Let X be an S-M-cyclic submodule of M. If X is an essential submodule of M and M is a weakly co-Hopfian module, then X is a weakly co-Hopfian module.*

Proof. (1) Suppose that M is a weakly co-Hopfian module. Let $f : M \to M$ be an R-monomorphism. So $f(M) \cong M$ and thus there exists an isomorphism φ from $f(M)$ to M. Let A be an S-M-cyclic submodule of M and $\alpha : A \to f(M)$ be an R-monomorphism. Since M is an quasi S-pseudo-principally injective module and $\varphi \circ \alpha$ is an R-monomorphism, there exists an R-homomorphism $\psi : M \to M$ such that $\varphi \circ \alpha = \psi \circ i_A$ where i_A is an inclusion map from A to M. So $\varphi^{-1} \circ \psi \circ i_A = \varphi^{-1} \circ \varphi \circ \alpha = \alpha$. We have that $f(M)$ is an S-pseudo-M-principally injective module. By Lemma 4.9, $f(M)$ is a direct summand of M. There exists a submodule B of M such that $M = f(M) \oplus B$ and thus $f(M) \cap B = 0$. But M is a weakly co-Hopfian module, $B = 0$. Then $M = f(M) + B = f(M)$. So f is an epimorphism. Therefore M is a co-Hopfian module.

(2) Suppose that X is an essential submodule of M and M is a weakly co-Hopfian module. Let $f : X \to X$ be an R-monomorphism. Since M is an quasi S-pseudo-principally injective module and $i_X \circ f$ is a monomorphism where $i_X : X \to M$ is an inclusion map, there exists an R-homomorphism $\varphi : M \to M$ such that $i_X \circ f \circ i_X = \varphi$. So $Ker(\varphi) \cap X = 0$. But $X \ll_e M$, $Ker(\varphi) = 0$. By [7, Corollary 1.2], $\varphi(X) \ll_e M$. Since $f(X) = \varphi(X)$, we have $f(X) \ll_e M$. But $f(X) \subseteq X \subseteq M$, so $f(X) \ll_e X$. Therefore X is a weakly co-Hopfian module. □

Recall that a right R-module M is said to be *multiplication* if each submodule N of M has the form $N = MI$ for some ideal I of R ([2]).

Proposition 4.11. *Let M be a multiplication quasi S-pseudo-principally injective module. Then every S-M-cyclic submodule of M is quasi S-pseudo-principally injective.*

Proof. Let N be an S-M-cyclic submodule of M, L be an S-N-cyclic submodule of N and φ be a monomorphism from L to N. So L is an S-M-cyclic submodule of M. But M is a quasi S-pseudo-principally injective module, there exists an R-homomorphism α from M to M such that $\alpha \circ i_L = \varphi$ where i_L is an inclusion map on L. Since M is a multiplication module, there exists an ideal I of R with $N = MI$. Then $\alpha(N) = \alpha(MI) = \alpha(M)I \subseteq MI = N$ and thus $\alpha|_N \colon N \to N$. So $\alpha|_N \circ i_L = \varphi$. Therefore N is a quasi S-pseudo-principally injective module. $\square$

Theorem 4.12. *Let M be a uniform module. Then every quasi S-pseudo-principally injective module is a quasi S-principally injective module.*

Proof. Suppose that M is a quasi S-pseudo-principally injective module. Let A be an S-M-cyclic submodule of M and φ an R-homomorphism from A to M.

Case 1. $\ker(\varphi) = 0$. We see that φ is a monomorphism. But M is a quasi S-pseudo-principally injective module, there exists $\overline{\varphi} : M \to M$ such that $\overline{\varphi}|_A = \varphi$.

Case 2. $\ker(\varphi) \neq 0$. Since M is a uniform module, $\ker(\varphi)$ is an essential submodule of M. But $\ker(\varphi) \cap \ker(\varphi + i_A) = 0$ where i_A is the inclusion map from A to M, we have $\ker(\varphi + i_A) = 0$ and thus $\varphi + i_A$ is a monomorphism. Since M is a quasi S-pseudo-principally injective module, there exists an R-homomorphism $\alpha : M \to M$ such that $\alpha(a) = (\varphi + i_A)(a)$ for all $a \in A$. Choose $\overline{\varphi} = \alpha - i_M$ where I_M is an identity map on M. Then $\overline{\varphi}(a) = (\alpha - i_M)(a) = \alpha(a) - i_M(a) = \varphi(a) + i_A(a) - I_M(a) = \varphi(a)$ for all $a \in A$. We have $\overline{\varphi}_A = \varphi$.

From Case 1 and Case 2, we have that M is a quasi S-principally injective module. $\square$

Proposition 4.13. *Let M be a right R-module and A be a submodule of M. If M is a quasi S-pseudo-principally injective module, A is an essential and S-M-cyclic submodule of M, then every monomorphism $\varphi : A \to M$ can be extended to monomorphism in $End_R(M)$.*

Proof. Since M is a quasi S-pseudo-principally injective module, there exists $\overline{\varphi} : M \to M$ such that $\overline{\varphi}|_A = \varphi$. Since $A \cap \ker(\overline{\varphi}) = 0$ and A is an essential submodule of M, $\ker(\overline{\varphi}) = 0$. Thus $\overline{\varphi}$ is a monomorphism in $End_R(M)$. $\square$

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