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MINIMAL SUBMODULES GRAPH OF MODULES OVER COMMUTATIVE RINGS

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Abstract. Let R be a commutative ring and M be an R-module. In this paper, we define minimal submodules graph of M, denoted by $\Gamma_{min}(M)$, in which the vertex set is the set of nonzero proper submodules of M. Two distinct vertices A and B are adjacent provided that $A \cap B$ is a minimal submodule of M. In this study, we associate some properties of the graph from the properties of module and vice versa. Moreover, if we have an R-module homomorphism from M to M', we compare some invariant numbers and properties of $\Gamma_{min}(M)$ and $\Gamma_{min}(M')$.

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1. Introduction

Associating certain algebraic structure to a certain graph is the most recent research area which combines two different concepts, algebraic structure and graph theory. One of the most often algebraic structures which is involved to a certain graph is module. Modules over rings can be used to construct some graphs. The vertices of the graphs can be the elements of the module or the nonzero submodules, see for example [1], [2], [5], [6] and [17].

Assume that R is a commutative ring. The ring R can be considered as a module over itself and its ideals can be thought as submodules. As a result, we may think of a module over a ring as a ring generalization. There are some researches related to certain graphs of rings which are extended into the graphs of modules. E. Mehdi-Nezhad and A. M. Rahimi in [12] defined comaximal submodule graphs of unitary modules which is a generalization of comaximal ideal graph of a commutative ring. In this paper, E. Mehdi-Nezhad and A. M. Rahimi compared the graph properties of rings and modules. Besides, a generalization of zero divisor graphs of commutative rings, zero divisor graphs for modules over commutative rings are observed in [7]. In this article, it was investigated the relationships between the module and its graph.

In [13], a simple-intersection graph $GS(R)$ of a ring R is defined. The vertex set of $GS(R)$ is $V(GS(R)) = \{I | I \trianglelefteq R, I \neq 0\}$ and two distinct vertices X and Y

are adjacent if $X \cap Y$ is a simple ideal. We develop the simple-intersection graph of rings to modules. If two submodules intersect in a minimal submodule, this minimal submodule becomes a crucial part of the overall structure of the module. In this article, we study minimal submodules graph of modules over commutative rings M (designated with $\Gamma_{min}(M)$). However, we only consider the nonzero proper submodules as vertices of the graph. Two submodules are adjacent if their intersection is minimal. These graphs will help illuminate the structure of the modules. We observe the interplay of properties of module M with the properties of graph $\Gamma_{min}(M)$. We also compare some invariant numbers and decomposition of minimal submodules graph of domain and codomain from given module homomorphism.

2. Preliminary

Some basic concepts which will be used in this study are modules and graph theory.

2.1. Module theory. In this section, we will give some basic theories of modules which are taken from [16]. Let M be an R-module. A nontrivial module M is called a simple module if 0 and M are the only submodules of M. An R-module M is called cyclic if there exists $m \in M$ such that $M = Rm$. A torsion-free module is a module in which 0 is the only element of M which is annihilated by a nonzero element of a ring R. Let $M_1, M_2, \cdots, M_n$ be any submodules of M. The module M is called the direct sum of $M_1, M_2, \cdots, M_n$ if it satisfies the following properties.

(1) $M = \Sigma_{i=1}^{n} M_i$.
(2) $M_i \bigcap \Sigma_{j \neq i} M_j = 0$.

If M is a direct sum of $M_1, M_2, \cdots, M_n$, then it can be denoted by $M = \oplus_{i=1}^{n} M_i$. Furthermore, for any element $m \in M$, it can be uniquely written as $m_1 + m_2 + \cdots + m_n$ where $m_i \in M_i$.

A uniserial module is a module in which any two submodules can be ordered by inclusion [9]. An R-module M is called a multiplication module if for every nonzero submodules N of M, $N = IM$ for some ideal I of R. If we have an R-submodule N of M, we can make an ideal of R, namely $(N : M) = \{a \in R | aM \subseteq N\}$. If M is a multiplication module, then the submodule N can be written as $N = (N : M)M$ [4].

A relatively divisible submodule (RD-submodule) D of an R-module M is a submodule which satisfies $rD = D \cap rM$ for every $r \in R$ [11]. An essential submodule N is an R-submodule of M which meets the condition $N \cap A \neq 0$ for every nonzero submodule A of M [15]. An R-submodule N of M is called a minimal submodule if N is simple as an R-module. Let M be a finite R-module. Then M contains a minimal submodule. In this study, we will only consider finite modules over commutative rings with unity.

There are some lemmas which will be used in the main results. The lemmas are taken from [10] and [14].

Lemma 2.1. *Assume that A, B, N are nonzero proper submodules of M. If $N \subseteq A$ and $N \subseteq B$, then $A/N \cap B/N = (A \cap B)/N$.*

Proof. It is obvious that $(A \cap B)/N \subseteq A/N \cap B/N$. Now take any element $x + N \in A/N \cap B/N$. Then $x + N = a + N = b + N$ for some $a \in A, b \in B$. We can write $(x - a) = n_1, (x - b) = n_2$ for some $n_1, n_2 \in N$. From those equations and the fact that $N \subseteq A, N \subseteq B$, we can make $x = n_1 + a \in A$ and $x = n_2 + b \in B$. It is proved that $x \in A \cap B$ which implies $x + N \in (A \cap B)/N$. Hence $A/N \cap B/N = (A \cap B)/N$. □

Lemma 2.2. *Let N_1, N_2 be any nonzero submodules of an R-module M. Then the following properties hold.*

(1) *If $N_1 \subseteq N_2$, then $(N_1 : M) \subseteq (N_2 : M)$.*
(2) *$(N_1 \cap N_2 : M) = (N_1 : M) \cap (N_2 : M)$.*

Proof. (1) Suppose that $r \in (N_1 : M)$ and $m \in M$. We have $rm \in rM \subseteq N_1 \subseteq N_2$. Thus $r \in (N_2 : M)$.

(2) Let $r \in (N_1 \cap N_2 : M)$. Then $rM \subseteq N_1 \cap N_2$ which means $rM \subseteq N_1$ and $rM \subseteq N_2$. Thus $r \in (N_1 : M) \cap (N_2 : M)$. Now take any element $s \in (N_1 : M) \cap (N_2 : M)$. We have $sM \subseteq N_1$ and $sM \subseteq N_2$ which implies $sM \subseteq N_1 \cap N_2$. Therefore, $(N_1 : M) \cap (N_2 : M) \subseteq (N_1 \cap N_2 : M)$. □

Lemma 2.3. *Suppose that $\beta : M \longrightarrow M'$ is an R-module homomorphism and N_1, N_2 are submodules of M. If β is injective, then $\beta(N_1 \cap N_2) = \beta(N_1) \cap \beta(N_2)$.*

Proof. It is clear that $\beta(N_1 \cap N_2) \subseteq \beta(N_1) \cap \beta(N_2)$. Let $\beta(x)$ be an arbitrary element of $\beta(N_1) \cap \beta(N_2)$. We can write $\beta(x) = \beta(n_1) = \beta(n_2)$ for some $n_1 \in N_1$ and $n_2 \in N_2$. By the injectivity of β, we get $x = n_1 = n_2 \in N_1 \cap N_2$. Therefore, $\beta(N_1 \cap N_2) = \beta(N_1) \cap \beta(N_2)$. □

2.2. Graph theory. There are some concepts of graph theory which will be used in this study and referred from [8]. A graph G is a pair of sets $V = V(G)$ and $E = E(G)$ where V is a nonempty set of objects that we call vertices and E is a set of pair of vertices that we call edge. An edge of a graph G which connects the vertices u and v will be denoted by (u, v), (v, u) or e. A graph G is said to be simple if it does not contain loop and multiple edge. A null graph is a graph with no edge. A bipartite graph is a graph in which the vertex set can be divided into two disjoint sets and the endpoints of every edge belong to those two disjoint sets. A complete bipartite graph is a bipartite graph with every two vertices in different set are adjacent. A complete bipartite graph with n and m vertices is denoted by

$K_{n,m}$. A star graph is a special case of complete bipartite graph and is denoted by $K_{1,n}$.

Walk form vertex u to v is a sequence $u = u_1 - u_2 - u_3 - \cdots - u_k = v$ where u_i and u_{i+1} are adjacent. In this case, the length of walk from u to v is equal to $k-1$. A path is a walk with different vertices. A graph G is said to be connected if there exists a path between any two vertices of the graph G. The distance between two vertices $u, v \in V$, denoted by $d(u, v)$, is defined to be the length of the shortest path between u and v. The number of edges which connect to a vertex u is called degree of u and is denoted by $deg(u)$. The maximum degree of a graph G is denoted by $\Delta(G)$. Let $G_1, G_2, \cdots, G_n$ be subgraphs of G with $E(G_i) \cap E(G_j) = \emptyset$ for $i \neq j$. The collection $G_1, G_2, \cdots, G_n$ is a decomposition of the graph G if every edge of G belongs to one and only one of G_i [3].

3. Main results

We will start by defining a minimal submodules graph of modules over commutative rings and by giving some examples of the graphs.

Definition 3.1. Let M be an R-module. The graph of minimal submodule of M is a graph $\Gamma_{min}(M)$ with vertex set $V(\Gamma_{min}(M)) = \{N | N$ is a submodule of $M, N \neq 0, N \neq M\}$ and two distinct vertices $N_1, N_2 \in V(\Gamma_{min}(M))$ are adjacent if $N_1 \cap N_2$ is a minimal submodule.

By Definition 3.1, the graph of minimal submodule is a simple graph. In this study, we only consider the graph of minimal submodule of non-simple modules. The following are some examples of the graph of minimal submodule of modules.

Example 3.2. Given $\mathbb{Z}$-module $\mathbb{Z}_{12}$. Note that

$$V(\Gamma_{min}(\mathbb{Z}_{12})) = \{\mathbb{Z}\bar{2}, \mathbb{Z}\bar{3}, \mathbb{Z}\bar{4}, \mathbb{Z}\bar{6}\}.$$

The graph of minimal submodule of $\mathbb{Z}_{12}$ is in Figure 1.

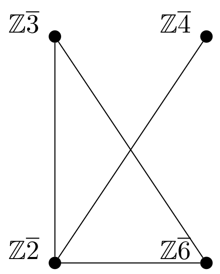

FIGURE 1. $\Gamma_{min}(\mathbb{Z}_{12})$

Example 3.3. Consider the $\mathbb{Z}$-module $\mathbb{Z}_{30}$. The set of vertices is

$$V(\Gamma_{min}(\mathbb{Z}_{30})) = \{\mathbb{Z}\overline{2}, \mathbb{Z}\overline{3}, \mathbb{Z}\overline{5}, \mathbb{Z}\overline{6}, \mathbb{Z}\overline{10}, \mathbb{Z}\overline{15}\}.$$

The graph of minimal submodule of $\mathbb{Z}_{30}$ is in Figure 2.

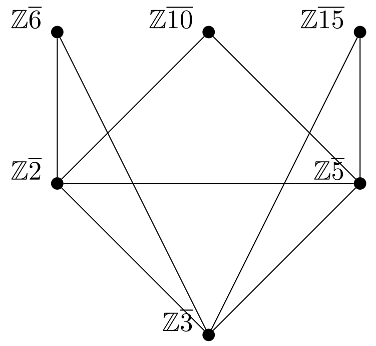

FIGURE 2. $\Gamma_{min}(\mathbb{Z}_{30})$

Example 3.4. Let $\mathbb{Z}_{pq}$ be a $\mathbb{Z}$-module where p and q are distinct prime numbers. Then $\Gamma_{min}(\mathbb{Z}_{pq})$ is a null graph. This is because the only nontrivial submodules of $\mathbb{Z}_{pq}$ are $\mathbb{Z}\overline{p}$ and $\mathbb{Z}\overline{q}$. Therefore, the graph of minimal submodules of $\mathbb{Z}_{pq}$ is a null graph.

Theorem 3.5. *Let M be an R-module and N be a nonzero proper submodule of M. If N is not a minimal submodule, then $\Gamma_{min}(N)$ is a subgraph of $\Gamma_{min}(M)$.*

Proof. Note that every submodule of N is also a submodule of M which implies $V(\Gamma_{min}(N)) \subseteq V(\Gamma_{min}(M))$. Now take an arbitrary edge of $\Gamma_{min}(N)$, namely (A, B). Since $A \cap B$ is a minimal submodule of N, we have that $A \cap B$ is also a minimal submodule of M. Hence $(A, B) \in E(\Gamma_{min}(M))$. □

Lemma 3.6. *Let M be an R-module and S, T be any distinct minimal submodules of M. Then S and T are not adjacent in $\Gamma_{min}(M)$.*

Proof. Assume that S and T are adjacent. Then $S \cap T$ is a minimal submodule satisfying $0 \subset S \cap T \subseteq S$ and $0 \subset S \cap T \subseteq T$. Since S and T are also minimal submodules, we have $S \cap T = S$ and $S \cap T = T$. It implies $S = T$ which is a contradiction. Hence S and T are not adjacent. □

Theorem 3.7. *Let M be an R-module. If $\Gamma_{min}(M)$ is connected, then M has a non minimal proper submodule.*

Proof. Let $\Gamma_{min}(M)$ be a connected graph. It implies $E(\Gamma_{min}(M)) \neq \emptyset$. We can take any edge $(A, B) \in E(\Gamma_{min}(M))$. By Lemma 3.6, either A or B are not minimal submodules. Therefore the result follows. □

Theorem 3.8. *If M is a finite uniserial R-module, then $\Gamma_{min}(M)$ is a star graph.*

Proof. Let $0 \neq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_n = M$ be the chain of all submodules of M. Then M_1 is the unique minimal submodule of M. Hence for $i = 2, 3, \cdots, n$, M_i is adjacent to M_1 and M_i is not adjacent to M_j for all $j = 2, 3, \cdots, n$. Therefore, $\Gamma_{min}(M)$ is a star graph with M_1 as the center. □

Example 3.9. Let $\mathbb{Z}_{p^n}$ be a module over $\mathbb{Z}$ where p is a prime and $n \geq 2$. Note that the only nonzero submodules of $\mathbb{Z}_{p^n}$ are $\mathbb{Z}_{p^n}$ itself, $\mathbb{Z}\overline{p^{n-1}}, \mathbb{Z}\overline{p^{n-2}}, \cdots, \mathbb{Z}\overline{p}$. These submodules form a chain

$$\mathbb{Z}\overline{p^{n-1}} \subset \mathbb{Z}\overline{p^{n-2}} \subset \cdots \subset \mathbb{Z}\overline{p^2} \subset \mathbb{Z}\overline{p^1} \subset \mathbb{Z}\overline{p^0} = \mathbb{Z}_{p^n}.$$

Hence $\mathbb{Z}_{p^n}$ is a uniserial $\mathbb{Z}$-module. Note that the minimal submodule of $\mathbb{Z}_{p^n}$ is unique, namely $\mathbb{Z}\overline{p^n}$. The graph of minimal submodule of $\mathbb{Z}_{p^n}$ is represented on Figure 3.

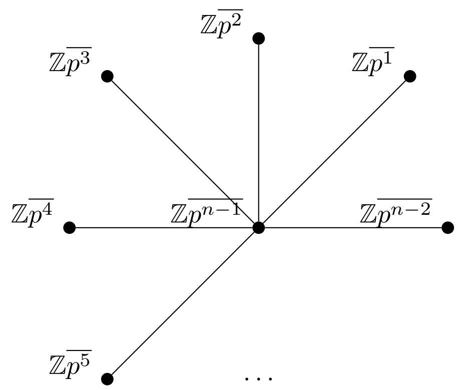

FIGURE 3. $\Gamma_{min}(\mathbb{Z}_{p^n}) = K_{1,n-1}$

Theorem 3.10. *Let N be a nonzero proper R-submodule of M.*

(1) *If $(N, A) \in E(\Gamma_{min}(M))$ for every $A \in V(\Gamma_{min}(M))$, then N is an essential submodule of M.*

(2) *Assume that $A, B \in V(\Gamma_{min}(M))$ where $A \neq B$ and $A \cap B \neq 0$. If $(A + N, B + N) \in E(\Gamma_{min}(M))$, then $(A, B) \in E(\Gamma_{min}(M))$.*

Proof. (1) Assume that N is adjacent to every nonzero proper submodule of M. Then $N \cap A$ is a minimal submodule which means $N \cap A \neq 0$. We have thus proved that N is an essential submodule of M.

(2) Suppose that $(A + N, B + N) \in E(\Gamma_{min}(M))$. It implies $(A + N) \cap (B + N)$ is a minimal submodule. Note that since $0 \subset A \cap B \subseteq (A + N) \cap (B + N)$ and $(A + N) \cap (B + N)$ is minimal, $A \cap B = (A + N) \cap (B + N)$ which is also minimal. Thus $(A, B) \in E(\Gamma_{min}(M))$. □

Theorem 3.11. *Suppose that A, B, N be R-submodules of M where N is minimal and $N \subseteq A, N \subseteq B$. If A/N is adjacent to B/N in $\Gamma_{min}(M/N)$, then A is not adjacent to B in $\Gamma_{min}(M)$.*

Proof. It is given that $(A/N, B/N) \in \Gamma_{min}(M/N)$ which means $(A \cap B)/N = A/N \cap B/N$ is a minimal submodule of M/N. Since $(A \cap B)/N$ is a minimal submodule, $A \cap B \neq N$. Remember that $0 \subset N \subset A \cap B$, which implies that $A \cap B$ cannot be a minimal submodule. Thus A is not adjacent to B in $\Gamma_{min}(M)$. □

Theorem 3.12. *Suppose that M is a torsion-free R-module and $M = Rm$ for some $m \in M, m \neq 0$. Let I and J be proper nonzero ideals of R. Then as submodules, I is adjacent to J in $\Gamma_{min}(R)$ if and only if Im is adjacent to Jm in $\Gamma_{min}(M)$.*

Proof. Let N be any submodule of M satisfying $0 \subset N \subseteq Im \cap Jm$. We can make a nonempty set

$$K = \{r \in R | rm \in N\}$$

which is a nonzero ideal of R. It is evident that $N = Km$. Now we will prove that $K \subseteq I \cap J$. Let $s \in K$. Then $sm \in N \subseteq Im \cap Jm$. We can write $sm = \alpha m$ and $sm = \beta m$ for some $\alpha \in I$ and $\beta \in J$ or equivalently $(s - \alpha)m = 0$ and $(s - \beta)m = 0$. Since M is torsion-free, we have $s = \alpha = \beta \in I \cap J$. We thus have proved that $K \subseteq I \cap J$. As it is known that $I \cap J$ is minimal and $K \neq 0$, we get $K = I \cap J$.
Now take any element $x \in Im \cap Jm$. Then we can write $x = am$ and $x = bm$ for some $a \in I$ and $b \in J$. From those two equations, we get

$$am = bm$$
$$am - bm = 0$$
$$(a - b)m = 0.$$

Since M is torsion-free, we can conclude that $a = b \in I \cap J = K$. It implies $x \in Km$ which means $Km = Im \cap Jm$. Thus $Im \cap Jm$ is a minimal submodule of M or equivalently, $(Im, Jm) \in \Gamma_{min}(M)$.
Conversely, assume that $(Im, Jm) \in E(\Gamma_{min}(M))$. Let L be any nonzero ideal of R such that $0 \subset L \subseteq I \cap J$. Since M is a torsion-free module and $L \neq 0$, we have $Lm \neq 0$. Moreover, it is also satisfied that $Lm \subseteq Im \cap Jm$ since $L \subseteq I \cap J$. By the minimality of $Im \cap Jm$ and the fact that $Lm \neq 0$, we have $Lm = Im \cap Jm$. Now take an arbitrary element $r \in I \cap J$. Note that $rm \in Im \cap Jm = Lm$. It implies that we can write $rm = lm$ for some $l \in L$. Then we get $(r - l)m = 0$. As M is a torsion-free module, we can conclude that $r = l \in L$ which means $L = I \cap J$. Thus I is adjacent to J in $\Gamma_{min}(R)$. □

Theorem 3.13. *Assume that S, T be any nonzero proper R-submodules of torsion-free multiplication module M. If every submodule of M is relatively divisible and*

$(S : M)$ is adjacent to $(T : M)$ in $\Gamma_{min}(R)$, then S is also adjacent to T in $\Gamma_{min}(M)$.

Proof. Let W be any nonzero submodule of M satisfying $0 \subset W \subseteq S \cap T$. Then by Lemma 2.2, $0 \subset (W : M) \subseteq (S \cap T : M) = (S : M) \cap (T : M)$. By the minimality of $(S : M) \cap (T : M)$ and $(W : M) \neq 0$, we get $(W : M) = (S : M) \cap (T : M)$. Assume that x is an arbitrary element of $S \cap T$ and $r \in (W : M) = (S : M) \cap (T : M)$ where $r \neq 0$. Note that $rx \in rM \subseteq W$. Since W is a relatively divisible submodule, we have $rx \in rW$. We can write $rx = rw$ for some $w \in W$ or equivalently, $r(x-w) = 0$. As M is a torsion-free module, we thus have $x = w \in W$ which implies $W = S \cap T$. Therefore, $(S, T) \in E(\Gamma_{min}(M))$. □

Theorem 3.14. *If M is isomorphic to M' as R-modules, then $\Gamma_{min}(M)$ is isomorphic to $\Gamma_{min}(M')$.*

Proof. Let $\alpha : M \longrightarrow M'$ be an R-module isomorphism. We define a map $\tilde{\alpha} : \Gamma_{min}(M) \longrightarrow \Gamma_{min}(M')$ where $\tilde{\alpha}(N) = \alpha(N)$. Assume that N_1 is adjacent to N_2 in $\Gamma_{min}(M)$. It implies that $N_1 \cap N_2$ is a minimal submodule of M. We will show that $\alpha(N_1) \cap \alpha(N_2)$ is a minimal submodule of M'. Note that since $N_1, N_2 \neq 0$ and α is injective, $\alpha(N_1), \alpha(N_2) \neq 0$. Suppose that there exists a nonzero submodule H of M' such that $0 \subset H \subseteq \alpha(N_1) \cap \alpha(N_2)$. By the bijectivity of α, we have a nonzero submodule $\alpha^{-1}(H)$ of M. Let $x \in \alpha^{-1}(H)$. Then $\alpha(x) \in H \subseteq \alpha(N_1) \cap \alpha(N_2)$. We can write $\alpha(x) = \alpha(n_1) = \alpha(n_2)$ for some $n_1 \in N_1$ and $n_2 \in N_2$. By the injectivity of α, we get $x = n_1 = n_2 \in N_1 \cap N_2$. Hence $\alpha^{-1}(H) \subseteq N_1 \cap N_2$. By the minimality of $N_1 \cap N_2$, we can conclude that $\alpha^{-1}(H) = N_1 \cap N_2$ and therefore $H = \alpha(\alpha^{-1})(H) = \alpha(N_1 \cap N_2)$. By Lemma 2.3, we thus have proved that $H = \alpha(N_1) \cap \alpha(N_2)$ which means $\alpha(N_1)$ is adjacent to $\alpha(N_2)$ in $\Gamma_{min}(M')$. Hence $\tilde{\alpha}$ is a graph homomorphism. Since α is bijective, $\tilde{\alpha}$ is also bijective. □

Corollary 3.15. *Let $\theta : M \longrightarrow M'$ be an R-module monomorphism. If $(N_1, N_2) \in E(\Gamma_{min}(M))$, then $(\theta(N_1), \theta(N_2)) \in E(\Gamma_{min}(M'))$.*

Proof. Note that if θ is a monomorphism from M to M', then M is isomorphic to $\theta(M)$. If $(N_1, N_2) \in E(\Gamma_{min}(M))$, then by Theorem 3.14, $(\theta(N_1), \theta(N_2)) \in E(\Gamma_{min}(\theta(M)))$. Furthermore, since $\theta(N_1) \cap \theta(N_2)$ is minimal in $\theta(M)$, we have that $\theta(N_1) \cap \theta(N_2)$ is also minimal in M'. Hence $(\theta(N_1), \theta(N_2)) \in E(\Gamma_{min}(M'))$. □

Theorem 3.16. *Let $\theta : M \longrightarrow M'$ be an R-module monomorphism. If $(L_1, L_2) \in E(\Gamma_{min}(M'))$ and $\theta^{-1}(L_1), \theta^{-1}(L_2)$ are proper nonzero submodules of M, then $(\theta^{-1}(L_1), \theta^{-1}(L_2)) \in E(\Gamma_{min}(M))$.*

Proof. Assume that $(L_1, L_2) \in E(\Gamma_{min}(M'))$. Then by Definition 3.1, $L_1 \cap L_2$ is a minimal submodule in M'. It is clear that $\theta^{-1}(L_1)$ and $\theta^{-1}(L_2)$ are submodules of M. Assume that there exists a nonzero submodule N of M such that $N \subseteq \theta^{-1}(L_1) \cap \theta^{-1}(L_2)$. Since $N \neq 0$ and θ is injective, $\theta(N) \neq 0$ and especially $\theta(N)$ is a proper submodule of M'. Let $\theta(n)$ be an arbitrary element of $\theta(N)$. This means $n \in N \subseteq \theta^{-1}(L_1) \cap \theta^{-1}(L_2)$ which implies $\theta(n) \in L_1 \cap L_2$. Thus $\theta(N) \subseteq L_1 \cap L_2$. Since $L_1 \cap L_2$ is a minimal submodule, $\theta(N) = L_1 \cap L_2$. Now take any element $x \in \theta^{-1}(L_1) \cap \theta^{-1}(L_2)$. It means $\theta(x) \in L_1 \cap L_2 = \theta(N)$. We can write $\theta(x) = \theta(n')$ for some $n' \in N$. By the injectivity of θ, we can conclude that $x = n' \in N$. Hence $N = \theta^{-1}(L_1) \cap \theta^{-1}(L_2)$ which means $\theta^{-1}(L_1) \cap \theta^{-1}(L_2)$ is a minimal submodule. □

Theorem 3.17. *If $\theta : M \longrightarrow M'$ is an R-module monomorphism and N is a nonzero proper R-submodule of M, then $deg(N) \leq deg(\theta(N))$.*

Proof. Note that since $N \neq 0$ and θ is a monomorphism, $\theta(N) \neq 0$ and $\theta(N)$ is a proper submodule of M'. Let $deg(N) = m$ and N be adjacent to distinct proper submodules $L_1, L_2, \cdots, L_m$ of M. Then by Theorem 3.15, $\theta(N)$ is adjacent to $\theta(L_i) \neq 0$ for $i = 1, 2, \cdots, m$. Note that $\theta(L_i) \neq \theta(L_j)$ for $i \neq j$ since θ is a monomorphism. Now let S be any nonzero proper submodule of M' which is adjacent to $\theta(N)$. If $\theta^{-1}(S) = 0$ or $\theta^{-1}(S) = M$, then $\theta^{-1}(S) \notin V(\Gamma_{min}(M))$. It means $\theta^{-1}(S)$ is not adjacent to N. If $\theta^{-1}(S) \neq 0$, by Theorem 3.16 we have $\theta^{-1}(S)$ is adjacent to $\theta^{-1}\theta(N) = N$. Hence $deg(N) \leq \deg(\theta(N))$. □

Theorem 3.18. *Let $\theta : M \longrightarrow M'$ be an R-module monomorphism and T be a nonzero R-submodule of M'. If $\theta^{-1}(T)$ is adjacent to submodule W of M, then T is either adjacent to $\theta(W)$ or $\theta\theta^{-1}(T) \cap \theta(W)$.*

Proof. Assume that $\theta^{-1}(T)$ is adjacent to W. According to Corollary 3.15, $\theta\theta^{-1}(T)$ is adjacent to $\theta(W)$. This means $\theta\theta^{-1}(T) \cap \theta(W)$ is a minimal submodule of M'. Note that $\theta\theta^{-1}(T) \cap \theta(W) \subseteq T \cap \theta(W)$. If $\theta\theta^{-1}(T) \cap \theta(W) = T \cap \theta(W)$, then $(T, \theta(W)) \in E(\Gamma_{min}(M'))$. Now let $\Big(\theta\theta^{-1}(T) \cap \theta(W)\Big) \subset \Big(T \cap \theta(W)\Big)$. We will prove that $\theta\theta^{-1}(T) \cap \theta(W)$ is adjacent to T by showing that $\Big(\theta\theta^{-1}(T) \cap \theta(W)\Big) \cap T = \theta\theta^{-1}(T) \cap \theta(W)$. It is evident that $\Big(\theta\theta^{-1}(T) \cap \theta(W)\Big) \cap T \subseteq \theta\theta^{-1}(T) \cap \theta(W)$. Now take any element $z \in \theta\theta^{-1}(T) \cap \theta(W)$. It implies that $z = \theta(a)$ for some $a \in \theta^{-1}(T)$ which means $z = \theta(a) \in T$. It is proved that $\Big(\theta\theta^{-1}(T) \cap \theta(W)\Big) \cap T = \theta\theta^{-1}(T) \cap \theta(W)$. Thus T is adjacent to $\theta\theta^{-1}(T) \cap \theta(W)$. □

Theorem 3.19. *Suppose that $\theta : M \longrightarrow M'$ is an R-module monomorphism. Then*

(1) $d(\theta(N_1), \theta(N_2)) \leq d(N_1, N_2)$,
(2) $\Delta(\Gamma_{min}(M)) \leq \Delta(\Gamma_{min}(M'))$.

Proof. (1) Assume that $N_1 - A_1 - A_2 - \cdots - A_n - N_2$ is a shortest path from N_1 to N_2. Since θ is a monomorphism, $\theta(N_1) \neq \theta(A_i) \neq \theta(A_j) \neq \theta(N_2)$ for an arbitrary $i, j = 1, 2, \cdots, n$ and $i \neq j$. Then by Corollary 3.15, we have the path $\theta(N_1) - \theta(A_1) - \theta(A_2) - \cdots - \theta(A_n) - \theta(N_2)$. Thus $d(\theta(N_1), \theta(N_2)) \leq d(N_1, N_2)$.

(2) Let $\Delta(\Gamma_{min}(M)) = deg(N) = m$. Note that by Theorem 3.17,

$$deg(N) \leq deg(\theta(N))$$

and we know that

$$deg(\theta(N)) \leq \Delta(\Gamma_{min}(M')).$$

Hence $\Delta(\Gamma_{min}(M)) = deg(N) \leq \Delta(\Gamma_{min}(M'))$. □

Let $\mathcal{X} = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ be a graph decomposition of $\Gamma_{min}(M)$ for an R-module M. Suppose that θ is a module monomorphism from M to M'. Since θ is a monomorphism, for S, T nonzero proper submodules of M, it satisfies $\theta(S), \theta(T) \neq 0$ and $\theta(S), \theta(T) \neq M'$. We define the graph $\theta(\Gamma_i)$ for $i = 1, 2, \cdots, n$ as

$$E(\theta(\Gamma_i)) = \Big\{ \Big(\theta(S), \theta(T)\Big) \Big| (S, T) \in E(\Gamma_i) \Big\}.$$

By Corollary 3.15, the set $E(\theta(\Gamma_i))$ is not empty.

Theorem 3.20. *Let θ be an R-module monomorphism from M to M' and $\mathcal{X} = \{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}$ be a graph decomposition of $\Gamma_{min}(M)$. We define*

$$\mathcal{Y} = \Big\{ \theta(\Gamma_i) \Big| i = 1, 2, \cdots, n \Big\}$$

and subgraph Ω of $\Gamma_{min}(M')$ in which

$$E(\Omega) = E(\Gamma_{min}(M')) - E(\theta(\Gamma_i)) \textit{ for } i = 1, 2, \cdots, n.$$

Then $\mathcal{Y} \cup \{\Omega\}$ is a graph decomposition of $\Gamma_{min}(M')$.

Proof. It is obvious that $E(\theta(\Gamma_i)) \cap E(\Omega) = \emptyset$ for every i. Suppose that

$$\Big(\theta(S), \theta(T)\Big) \in E(\theta(\Gamma_i)) \cap E(\theta(\Gamma_j))$$

for $i \neq j$. Then by definition, $(S, T) \in E(\Gamma_i) \cap E(\Gamma_j)$ which is a contradiction. Thus $E(\theta(\Gamma_i)) \cap E(\theta(\Gamma_j)) = \emptyset$. Now we will prove that every edge of $\Gamma_{min}(M')$ is in one and only one $\theta(\Gamma_i)$ or Ω. Take any edge (A, B) of $\Gamma_{min}(M')$. We will divide into some cases.

(1) If there are no $0 \subset S, T \subset M$ such that $\theta(S) = A$ and $\theta(T) = B$, then $(A, B) \in E(\Omega)$.

(2) Now assume that there are submodules S, T with $0 \subset S, T \subset M$ such that $\theta(S) = A$ and $\theta(T) = B$. By Theorem 3.16, $(S, T) \in E(\Gamma_{min}(M))$. There exists Γ_j such that $(S, T) \in \Gamma_j$. Therefore, $(A, B) = \Big(\theta(S), \theta(T)\Big) \in \theta(\Gamma_j)$.

(3) Assume that there is no nonzero proper submodule S of M such that $\theta(S) = A$. Let $\theta(T) = B$ for a nonzero proper submodule T of M. Then it is clear that $(A, B) \in E(\Omega)$.

Therefore, $\mathcal{Y} \cup \{\Omega\}$ is a graph decomposition of $\Gamma_{min}(M')$. □

Let X be an R-module and Y, Z be R-submodules of X such that $X = Y \oplus Z$. Let A be any nonzero submodule of X. For every $a \in A$, we can write $a = y + z$ uniquely for some $y \in Y, z \in Z$. We define $B = \{y \in Y | y + z \in A, \exists z \in Z\}$ and $C = \{z \in Z | y + z \in A, \exists y \in Y\}$. It is obvious that B is a submodule of Y and C is a submodule of Z. Moreover, we can see clearly that $A = B \oplus C$.

Theorem 3.21. *Let X be an R-module and Y, Z be R-submodules of X such that $X = Y \oplus Z$. Assume that $A_1 = B_1 \oplus C_1, A_2 = B_2 \oplus C_2$ are submodules of X where B_1, B_2 are nonzero submodules of Y and C_1, C_2 are nonzero submodules of Z. If $(A_1, A_2) \in E(\Gamma_{min}(X))$, then $(B_1, B_2) \in E(\Gamma_{min}(Y))$ and $(C_1, C_2) \in E(\Gamma_{min}(Z))$.*

Proof. First, we will prove that $B_1 \cap B_2$ is a minimal submodule of Y. Let D be any submodule of Y in which $0 \subseteq D \subseteq B_1 \cap B_2$. Note that $D \oplus C_1 \subseteq A_1$ and $D \oplus C_2 \subseteq A_2$. Since the sum is direct, we have

$$D \oplus (C_1 \cap C_2) = \Big(D \oplus C_1\Big) \cap \Big(D \oplus C_2\Big)$$

and

$$\Big(B_1 \oplus C_1\Big) \cap \Big(B_2 \oplus C_2\Big) = \Big(B_1 \cap B_2\Big) \oplus \Big(C_1 \cap C_2\Big).$$

Therefore, we have the following conditions

$$D \oplus (C_1 \cap C_2) \subseteq \Big(B_1 \cap B_2\Big) \oplus \Big(C_1 \cap C_2\Big).$$

Since $A_1 \cap A_2 = \Big(B_1 \cap B_2\Big) \oplus \Big(C_1 \cap C_2\Big)$ is a minimal submodule, $D \oplus (C_1 \cap C_2) = 0$ or $D \oplus (C_1 \cap C_2) = \Big(B_1 \cap B_2\Big) \oplus \Big(C_1 \cap C_2\Big)$. Hence $D = 0$ or $D \cong B_1 \cap B_2$, especially, $D = B_1 \cap B_2$. It implies $(B_1, B_2) \in E(\Gamma_{min}(Y))$. By the similar way, we can show that $(C_1, C_2) \in E(\Gamma_{min}(Z))$. □

Corollary 3.22. *If $A_1 = B_1 \oplus C_1$ is adjacent to $B_1' \subseteq B_1$, then B_1 is also adjacent to B_1'.*

Proof. We can write $B_1' = B_1' \oplus 0$. Since $A_1 = B_1 \oplus C_1$ is adjacent to $B_1' = B_1' \oplus 0$, by Theorem 3.21, B_1 is also adjacent to B_1'.

Theorem 3.23. *Assume that $X = Y \oplus Z$ is an R-module, X_1, X_2 are nonzero submodules of X, Y_1, Y_2 are nonzero submodules of Y, and Z_1, Z_2 are nonzero submodules of Z such that $X_1 = Y_1 \oplus Z_1$ and $X_2 = Y_2 \oplus Z_2$. If X_2 is adjacent to Y_1, then X_1 is adjacent to $(Y_1 \cap Y_2)$.*

Proof. It is given that $X_2 = Y_2 \oplus Z_2$ is adjacent to Y_1. This means $(Y_2 \oplus Z_2) \cap Y_1$ is a minimal submodule. Let a be an arbitrary element of $(Y_2 \oplus Z_2) \cap Y_1$. Then we can write $a = y_2 + z_2$ and $a = y_1$ for some $y_2 \in Y_2, z_2 \in Z_2, y_1 \in Y_1$. Note that

$$y_1 = y_2 + z_2$$
$$y_1 - y_2 = z_2 \in Y \cap Z = 0.$$

Thus $z_2 = 0$ and $a = y_1 = y_2 \in Y_1 \cap Y_2$ and we can conclude that $(Y_2 \oplus Z_2) \cap Y_1 \subseteq Y_2 \cap Y_1$. Now suppose that $b \in Y_1 \cap Y_2$. We can consider that $b = b + 0$ where $b \in Y_2$ and $0 \in Z_2$. It implies that $b \in (Y_2 \oplus Z_2) \cap Y_1$. Hence $Y_2 \cap Y_1 = (Y_2 \oplus Z_2) \cap Y_1$ is minimal. Next we will prove that $Y_1 \cap Y_2$ is adjacent to $Y_1 \oplus Z_1$ by showing that $(Y_1 \cap Y_2) \cap (Y_1 \oplus Z_1) = Y_1 \cap Y_2$. It is clear that $\big((Y_1 \cap Y_2) \cap (Y_1 \oplus Z_1)\big) \subseteq Y_1 \cap Y_2$. Now take any element $x \in (Y_1 \cap Y_2)$. Then we can write $x = x + 0$ where $x \in Y_1, 0 \in Z_1$. Hence $Y_1 \cap Y_2 \subseteq \big((Y_1 \cap Y_2) \cap (Y_1 \oplus Z_1)\big)$ and therefore $\big((Y_1 \cap Y_2) \cap (Y_1 \oplus Z_1)\big) = Y_1 \cap Y_2$ which is also a minimal submodule. We can conclude that $Y_1 \cap Y_2$ is adjacent to $Y_1 \oplus Z_1 = X_1$. □

Corollary 3.24. *If a submodule Y_1' of Y_1 is adjacent to Y_1, then Y_1' is also adjacent to $Y_1 \oplus Z_1$.*

Proof. We can assume that $X_2 = Y_1 \oplus 0$. Then by Theorem 3.23, $Y_1' \cap Y_1 = Y_1'$ is adjacent to $Y_1 \oplus Z_1$. □

Theorem 3.25. *Let $X = Y \oplus Z$ be an R-module and A be a nonzero R-submodule of X. If $A = B \oplus C$ where $B \subseteq Y, C \subseteq Z$, then $deg(A) \leq deg(B) + deg(C)$.*

Proof. Assume that $deg(A) = m$ and A is adjacent to $A_i = B_i \oplus C_i$ for $i = 1, 2, \cdots, m$. If $B_i, C_i \neq 0$ for every i, then by Theorem 3.21, we can conclude that B is adjacent to B_i and C is adjacent to C_i for every i. Even though there exists j such that $B_j = 0$, we still have the edge (C, C_j). Therefore $m \leq deg(B) + deg(C)$ and the result follows. □

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