

## G-OUTER INVERSES AND PARTIAL ORDERS IN RINGS

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**ABSTRACT.** The notions of G-outer inverse and its one-sided versions for rectangular complex matrices are generalized to elements of a ring based on the  $(b, c)$ -inverse. Various properties and characterizations of G-outer inverse and its one-sided kinds are developed. We consider partial orders applying G-outer inverse and its one-sided types in a ring. Consequently, one-sided G-Drazin inverses are introduced for elements of a ring as well as partial orders induced by them.

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### 1. Introduction

Let  $\mathcal{R}$  be an associative ring with the unit 1. If  $a \in \mathcal{R}$ , we have the next image ideals  $a\mathcal{R} = \{ax : x \in \mathcal{R}\}$ ,  $\mathcal{R}a = \{xa : x \in \mathcal{R}\}$ ; and the kernel ideals  $a^\circ = \{x \in \mathcal{R} : ax = 0\}$ ,  ${}^\circ a = \{x \in \mathcal{R} : xa = 0\}$ . The symbol  $a_{\mathcal{B}}^{-1}$  denotes the inverse of  $a$  in a subring  $\mathcal{B}$  of  $\mathcal{R}$ .

An element  $a \in \mathcal{R}$  is regular if there is  $x \in \mathcal{R}$  such that  $axa = a$ . In this case,  $x$  is an inner inverse of  $a$ . The set of all regular elements of  $\mathcal{R}$  is denoted by  $\mathcal{R}^-$  and the set of all inner inverses of  $a \in \mathcal{R}^-$  will be marked as  $a\{1\}$ . For  $a \in \mathcal{R}$ , if  $xax = x$  holds for some  $x \in \mathcal{R} \setminus \{0\}$ , then  $x$  is an outer inverse of  $a$ .

An element  $a \in \mathcal{R}$  has a Drazin inverse  $x = a^d \in \mathcal{R}$  if

$$ax = xa, \quad x = ax^2, \quad a^k = a^{k+1}x,$$

for some non-negative integer  $k$ . If there is a Drazin inverse  $a^d$ , then  $a$  is Drazin invertible and the smallest non-negative integer  $k$  in equality  $a^k = a^{k+1}x$  is called the Drazin index  $\text{ind}(a)$  of  $a$ . The set of all Drazin invertible elements of  $\mathcal{R}$  is denoted by  $\mathcal{R}^d$ .

The special type of outer inverse was defined by Drazin [3] and called the  $(b, c)$ -inverse. Let  $b, c \in \mathcal{R}$ . An element  $a \in \mathcal{R}$  is  $(b, c)$ -invertible if there exists  $x \in \mathcal{R}$

such that

$$x \in (b\mathcal{R}x) \cap (x\mathcal{R}c), \quad xab = b \quad \text{and} \quad cax = c.$$

The  $(b, c)$ -inverse  $x$  of  $a$  satisfies  $xax = x$ , it is unique (if exists) and denoted by  $a^{\parallel(b,c)}$  [3]. We will use  $\mathcal{R}^{\parallel(b,c)}$  to denote the set of all  $(b, c)$ -invertible elements of  $\mathcal{R}$ . Notice that  $b\mathcal{R} = a^{\parallel(b,c)}\mathcal{R}$  and  $c^\circ = (a^{\parallel(b,c)})^\circ$ . Some interesting results about the  $(b, c)$ -inverse were given in [13,14,15,16,17].

In the rest of this paper, we will use the term minus partial order. For  $a, b \in \mathcal{R}$ , we say that  $a$  is below to  $b$  under the minus partial order (denoted by  $a \leq^- b$ ) [6] if  $a^-a = a^-b$  and  $aa^- = ba^-$ , for some  $a^- \in a\{1\}$ . This relation is a partial order on  $\mathcal{R}^-$ .

The notation of the G-Drazin inverse was defined for square matrices in [15] and extended for elements of a ring in [2]. For  $a \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ , an element  $z \in \mathcal{R}$  is a left G-Drazin inverse of  $a$  if

$$aza = a, \quad za^{k+1} = a^k \quad \text{and} \quad a^{k+1}z = a^k.$$

It is known that the G-Drazin inverse is not unique (when it exists) in general.

Let  $\mathbb{C}^{m \times n}$  be the set of all  $m \times n$  complex matrices. We mark  $A^*$ ,  $\text{rank}(A)$ ,  $\mathcal{N}(A)$  and  $\mathcal{R}(A)$ , respectively, conjugate transpose, rank, null space and range (column space) of  $A \in \mathbb{C}^{m \times n}$ . The uniquely determined outer inverse of  $A$  with prescribed range  $T$  and null space  $S$  is a matrix  $X \in \mathbb{C}^{n \times m}$  (represented by  $A_{T,S}^{(2)}$ ) for which

$$XAX = X, \quad \mathcal{R}(X) = T, \quad \mathcal{N}(X) = S,$$

where  $T$  is a subspace of  $\mathbb{C}^n$  dimension  $s \leq r = \text{rank}(A)$ , and  $S$  is a subspace of  $\mathbb{C}^m$  dimension  $m - s$ . Note that  $AT \oplus S = \mathbb{C}^m$  and only if  $A_{T,S}^{(2)}$  exists. For a given  $A \in \mathbb{C}^{m \times n}$ , appropriately chosen  $T, S$ , the notation

$$\mathbb{C}_{T,S}^{m \times n} = \{A \in \mathbb{C}^{m \times n} : AT \oplus S = \mathbb{C}^m\} \subseteq \mathbb{C}^{m \times n}$$

will be used.

The concept of  $G$ -outer inverse was introduced as a generalization of the G-Drazin inverse using outer inverse with prescribed range and null space in [9]. A matrix  $Z \in \mathbb{C}^{n \times m}$  is  $G$ -outer  $(T, S)$ -inverse of  $A \in \mathbb{C}_{T,S}^{m \times n}$  if

$$AZA = A, \quad ZAA_{T,S}^{(2)} = A_{T,S}^{(2)} \quad \text{and} \quad A_{T,S}^{(2)}AZ = A_{T,S}^{(2)}.$$

The definitions of both left and right versions of  $G$ -outer inverse were given in [11]. A matrix  $Z \in \mathbb{C}^{n \times m}$  is a left  $G$ -outer  $(T, S)$ -inverse of  $A \in \mathbb{C}_{T,S}^{m \times n}$  if it satisfies

$$AZA = A \quad \text{and} \quad ZAA_{T,S}^{(2)} = A_{T,S}^{(2)}.$$

A matrix  $Z \in \mathbb{C}^{n \times m}$  is a right  $G$ -outer  $(T, S)$ -inverse of  $A \in \mathbb{C}_{T,S}^{m \times n}$  if it satisfies

$$AZA = A \quad \text{and} \quad A_{T,S}^{(2)}AZ = A_{T,S}^{(2)}.$$

G-Drazin inverses, G-outer inverses and its one-sided versions are significant in solving some matrix equations and studying partial orders [1,4,5,8,10]. A zeroing neural network (ZNN) approach for calculating time-varying G-outer inverse can be found in [12]. Recent results related to the G-Drazin inverse were stated in [7].

In this paper, we generalize the notions of G-outer inverse for rectangular complex matrices to elements of a ring, using the  $(b, c)$ -inverse. One-sided G-outer inverses in a ring are defined and characterized as weaker versions of G-outer inverse. By means of G-outer inverse and its one-sided kinds, we study partial orders in a ring. As special cases of one-sided G-outer inverses, one-sided G-Drazin inverses are introduced and investigated for elements of a ring. Also, we consider partial orders induced by one-sided G-Drazin inverses. Thus, we extend some results for complex matrices in a more general setting.

The paper is organized as follows. In Section 2, we introduce the G-outer inverse in a ring and discuss its properties. Section 3 is devoted to the definition and investigation of one-sided G-outer inverses in a ring. In Section 4, we define the left and right G-outer partial orders, as well as the G-outer partial order, and provide characterizations of these relations. Section 5 presents left and right G-Drazin inverses for elements of a ring, together with their characterizations. Finally, in Section 6, we define and examine the left and right G-Drazin partial orders.

## 2. G-outer inverse in a ring

In this section, we define a G-outer  $(b, c)$ -inverse for elements in a ring and establish its properties. Thus, we extend the concept of  $G$ -outer  $(T, S)$ -inverse in settings of a ring.

**Definition 2.1.** Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . An element  $z \in \mathcal{R}$  is a G-outer  $(b, c)$ -inverse of  $a$  if

$$aza = a \quad \text{and} \quad a^{||(b,c)}az = zaa^{||(b,c)}.$$

The G-outer  $(b, c)$ -inverse of  $a$  is not unique. We introduce a very useful notation. The set of all G-outer  $(b, c)$ -inverses of  $a$  is denoted by  $a\{go, b, c\}$ . The next inclusion applies  $a\{go, b, c\} \subseteq a\{1\}$ .

We characterize the G-outer  $(b, c)$ -inverse in the following manner.

**Theorem 2.2.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . For  $z \in \mathcal{R}$ , the following conditions are equivalent:*

- (i)  $z \in a\{go, b, c\}$ ;
- (ii)  $aza = a$  and  $a^{||(b,c)}az = a^{||(b,c)} = zaa^{||(b,c)}$ ;
- (iii)  $aza = a$ ,  $aa^{||(b,c)}az = aa^{||(b,c)}$  and  $zaa^{||(b,c)}a = a^{||(b,c)}a$ ;
- (iv)  $aza = a$ ,  $caz = c$  and  $zab = b$ ;
- (v)  $aza = a$ ,  $(az)^\circ \subseteq c^\circ$  and  $b\mathcal{R} \subseteq za\mathcal{R}$ .

**Proof.** (i)  $\Rightarrow$  (ii): For  $z \in \mathcal{R}$  which is a G-outer  $(b, c)$ -inverse of  $a$ , by Definition 2.1, we have  $aza = a$  and  $a^{||(b,c)}az = zaa^{||(b,c)}$ . We get

$$a^{||(b,c)}aza^{||(b,c)} = a^{||(b,c)}aa^{||(b,c)} = a^{||(b,c)}$$

and further

$$a^{||(b,c)} = a^{||(b,c)}aza^{||(b,c)} = zaa^{||(b,c)}aa^{||(b,c)} = zaa^{||(b,c)}.$$

In a similar way, we prove that  $a^{||(b,c)} = a^{||(b,c)}az$ .

(ii)  $\Rightarrow$  (iii): This implication is clear.

(iii)  $\Rightarrow$  (iv): Using the fact  $caa^{||(b,c)} = c$  and the assumption  $aa^{||(b,c)} = aa^{||(b,c)}az$ , we conclude that  $caz = caa^{||(b,c)}az = caa^{||(b,c)} = c$ . Since  $a^{||(b,c)}ab = b$ , multiplying  $a^{||(b,c)}a = zaa^{||(b,c)}a$  by  $b$  from the right side, we get  $b = zab$ .

(iv)  $\Rightarrow$  (v): It is evident.

(v)  $\Rightarrow$  (i): The hypothesis  $aza = a$  gives  $azaz = az$ . Now,  $1 - az \in (az)^\circ \subseteq c^\circ = (a^{||(b,c)})^\circ$  and so  $a^{||(b,c)}az = a^{||(b,c)}$ . From  $a^{||(b,c)} \in b\mathcal{R} \subseteq za\mathcal{R} = (1 - za)^\circ$ , we deduce that  $a^{||(b,c)} = zaa^{||(b,c)}$  and thus  $a^{||(b,c)}az = zaa^{||(b,c)}$ .  $\square$

Based on idempotents, we obtain the next characterizations of G-outer  $(b, c)$ -invertibility.

**Theorem 2.3.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)}$ . The following conditions are equivalent:*

- (i)  $a\{go, b, c\} \neq \emptyset$ ;
- (ii) there are idempotents  $p, q \in \mathcal{R}$ , such that

$$a^{||(b,c)}p = qa^{||(b,c)}, \quad p\mathcal{R} = a\mathcal{R} \quad \text{and} \quad \mathcal{R}q = \mathcal{R}a.$$

Additionally, for arbitrary  $a^- \in a\{1\}$ , it is valid  $qa^-p \in a\{go, b, c\}$  and

$$q \cdot a\{1\} \cdot p \subseteq a\{go, b, c\}.$$

**Proof.** (i)  $\Rightarrow$  (ii): Assume that  $a \in \mathcal{R}^-$  and  $z \in a\{go, b, c\}$ . Let  $p = az$  and  $q = za$ . Since  $z \in a\{1\}$ , we get  $p^2 = azaz = az = p$  and  $q^2 = zaza = za = q$ . So,  $p$  and

$q$  are idempotents. Notice that  $p\mathcal{R} = az\mathcal{R} \subseteq a\mathcal{R}$  and  $a\mathcal{R} = aza\mathcal{R} = pa\mathcal{R} \subseteq p\mathcal{R}$ , which imply  $p\mathcal{R} = a\mathcal{R}$ . It also applies  $\mathcal{R}q = \mathcal{R}za = \mathcal{R}a$  and  $a^{|| (b,c)} p = a^{|| (b,c)} az = zaa^{|| (b,c)} = qa^{|| (b,c)}$ . Thus, (ii) holds.

(ii)  $\Rightarrow$  (i): Let  $p, q \in \mathcal{R}$  be idempotents such that  $p\mathcal{R} = a\mathcal{R}$ ,  $\mathcal{R}q = \mathcal{R}a$  and  $a^{|| (b,c)} p = qa^{|| (b,c)}$ . We prove the existence of a G-outer  $(b, c)$ -inverse of  $a$ . Because  $p\mathcal{R} = a\mathcal{R}$ , it is valid  $p = ar$  and  $a = pu = p^2u = pa$ , for some  $r, u \in \mathcal{R}$ . Furthermore,  $a = pa = ara$ , which yields that  $a$  is regular and  $p = ar = aa^-ar = aa^-p$ , where  $a^- \in a\{1\}$  is arbitrary. From  $\mathcal{R}q = \mathcal{R}a$ , we have  $q = qa^-a$  and  $a = aq$ . If  $z = qa^-p$ , then  $aza = aqa^-pa = (aq)a^-(pa) = aa^-a = a$  and  $a^{|| (b,c)} az = a^{|| (b,c)} (aq)a^-p = a^{|| (b,c)} (aa^-p) = a^{|| (b,c)} p = qa^{|| (b,c)} = qa^-aa^{|| (b,c)} = (qa^-p)aa^{|| (b,c)} = zaa^{|| (b,c)}$ , by which we proved that  $z$  is the G-outer  $(b, c)$ -inverse of  $a$ .  $\square$

Note that the product  $zaz'$  is a G-outer  $(b, c)$ -inverse of  $a$  in the case that  $z$  and  $z'$  are G-outer  $(b, c)$ -inverses of  $a$ .

**Theorem 2.4.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{|| (b,c)} \cap \mathcal{R}^-$ . Then*

$$a\{go, b, c\} \cdot a \cdot a\{go, b, c\} \subseteq a\{go, b, c\}.$$

**Proof.** We can verify this result similarly as [9, Theorem 2.4].  $\square$

### 3. One-sided G-outer inverses in a ring

Removing the second or the third equation in the definition of the G-outer  $(b, c)$ -inverse, we introduce one-sided G-outer  $(b, c)$ -inverses as weak kinds of generalized inverses. In this way, we extend definitions of left and right G-outer  $(T, S)$ -inverses to elements of a ring.

**Definition 3.1.** Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{|| (b,c)} \cap \mathcal{R}^-$ . An element  $z \in \mathcal{R}$  is

(a) a left G-outer  $(b, c)$ -inverse of  $a$  if

$$aza = a \quad \text{and} \quad a^{|| (b,c)} = zaa^{|| (b,c)};$$

(b) a right G-outer  $(b, c)$ -inverse of  $a$  if

$$aza = a \quad \text{and} \quad a^{|| (b,c)} = a^{|| (b,c)} az.$$

Let us introduce the following useful tags. We denote the set of all left (or right) G-outer  $(b, c)$ -inverses of  $a$  with  $a\{l, go, b, c\}$  (or  $a\{r, go, b, c\}$ ). The following relations apply  $a\{l, go, b, c\} \subseteq a\{1\}$  and  $a\{r, go, b, c\} \subseteq a\{1\}$ .

By Theorem 2.2, we have the next characterizations of left and right G-outer  $(b, c)$ -inverses.

**Corollary 3.2.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . For  $z \in \mathcal{R}$ , the following conditions are equivalent:*

- (i)  $z \in a\{l, go, b, c\}$ ;
- (ii)  $aza = a$  and  $zaa^{||(b,c)}a = a^{||(b,c)}a$ ;
- (iii)  $aza = a$  and  $zab = b$ ;
- (iv)  $aza = a$  and  $b\mathcal{R} \subseteq za\mathcal{R}$ .

**Corollary 3.3.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . For  $z \in \mathcal{R}$ , the following conditions are equivalent:*

- (i)  $z \in a\{r, go, b, c\}$ ;
- (ii)  $aza = a$  and  $aa^{||(b,c)}az = aa^{||(b,c)}$ ;
- (iii)  $aza = a$  and  $caz = c$ ;
- (iv)  $aza = a$  and  $(az)^\circ \subseteq c^\circ$ .

We can characterize left and right G-outer  $(b, c)$ -invertibility by idempotents as in Theorem 2.3.

**Corollary 3.4.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)}$ . The following conditions are equivalent:*

- (i)  $a\{l, go, b, c\} \neq \emptyset$ ;
- (ii) there are idempotents  $p, q \in \mathcal{R}$  such that

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad b\mathcal{R} \subseteq q\mathcal{R}.$$

Additionally, for arbitrary  $a^- \in a\{1\}$ , it is valid  $qa^-p \in a\{l, go, b, c\}$  and

$$q \cdot a\{1\} \cdot p \subseteq a\{l, go, b, c\}.$$

**Corollary 3.5.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)}$ . The following conditions are equivalent:*

- (i)  $a\{r, go, b, c\} \neq \emptyset$ ;
- (ii) there are idempotents  $p, q \in \mathcal{R}$  such that

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad \mathcal{R}p \subseteq \mathcal{R}c.$$

Additionally, for arbitrary  $a^- \in a\{1\}$ , it is valid  $qa^-p \in a\{r, go, b, c\}$  and

$$q \cdot a\{1\} \cdot p \subseteq a\{r, go, b, c\}.$$

If  $z$  and  $z'$  are arbitrary left (or right) G-outer  $(b, c)$ -inverses of  $a$ , we can show that  $zaz'$  is also left (or right) G-outer  $(b, c)$ -inverse of  $a$ .

**Corollary 3.6.** *Let  $b, c \in \mathcal{R}$  and  $a \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . Then*

$$a \{l, go, b, c\} \cdot a \cdot a \{l, go, b, c\} \subseteq a \{l, go, b, c\}$$

and

$$a \{r, go, b, c\} \cdot a \cdot a \{r, go, b, c\} \subseteq a \{r, go, b, c\}.$$

#### 4. G-outer partial orders

Applying the G-outer  $(b, c)$ -inverse and its one-sided versions, we define new binary relations in a ring.

**Definition 4.1.** Let  $b, c \in \mathcal{R}$  and  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . Then

(i)  $a_1$  is below  $a_2$  under the left G-outer  $(b, c)$ -relation ( $a_1 \leq^{l, go, b, c} a_2$ ) if there exist  $z_1, z_2 \in a_1 \{l, go, b, c\}$  such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2;$$

(ii)  $a_1$  is below  $a_2$  under the right G-outer  $(b, c)$ -relation ( $a_1 \leq^{r, go, b, c} a_2$ ) if there exist  $z_1, z_2 \in a_1 \{r, go, b, c\}$  such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2;$$

(iii)  $a_1$  is below  $a_2$  under the G-outer  $(b, c)$ -relation ( $a_1 \leq^{go, b, c} a_2$ ) if there exist  $z_1, z_2 \in a_1 \{go, b, c\}$  such that

$$a_1 z_1 = a_2 z_1 \quad \text{and} \quad z_2 a_1 = z_2 a_2.$$

In the first theorem of this section, we characterize the left G-outer  $(b, c)$ -relation.

**Theorem 4.2.** *Let  $b, c \in \mathcal{R}$  and  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . The following conditions are equivalent:*

- (i)  $a_1 \leq^{l, go, b, c} a_2$ ;
- (ii) there is  $z_3 \in a_1 \{l, go, b, c\}$  such that

$$a_1 z_3 = a_2 z_3 \quad \text{and} \quad z_3 a_1 = z_3 a_2;$$

- (iii) there is  $z_3 \in a_1 \{l, go, b, c\}$  such that

$$a_1 z_3 a_2 = a_1 = a_2 z_3 a_1;$$

- (iv) there are idempotents  $p, q \in \mathcal{R}$  such that

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad b\mathcal{R} \subseteq q\mathcal{R} \quad \text{and} \quad pa_2 = a_1 = a_2q.$$

**Proof.** (i)  $\Rightarrow$  (ii): If  $a_1 \leq^{l,go,b,c} a_2$ , there exist  $z_1, z_2 \in a_1 \{l, go, b, c\}$  satisfying  $a_1 z_1 = a_2 z_1$  and  $z_2 a_1 = z_2 a_2$ . By Corollary 3.6,  $z_3 = z_1 a_1 z_2$  is a left G-outer  $(b, c)$ -inverse of  $a_1$ . Now,  $a_1 z_3 = (a_1 z_1) a_1 z_2 = a_2 (z_1 a_1 z_2) = a_2 z_3$  and also  $z_3 a_1 = (z_1 a_1) z_2 a_1 = (z_1 a_2 z_2) a_2 = z_3 a_2$ .

(ii)  $\Rightarrow$  (iii): Because  $a_1 z_3 = a_2 z_3$  and  $z_3 a_1 = z_3 a_2$ , where  $z_3 \in a_1 \{l, go, b, c\}$ , we deduce that  $a_1 z_3 a_2 = a_1 z_3 a_1 = a_1$  and similarly  $a_2 z_3 a_1 = a_1$ .

(iii)  $\Rightarrow$  (i): Let's assume that  $z_3$  is a left G-outer  $(b, c)$ -inverse of  $a_1$  such that  $a_1 z_3 a_2 = a_1 = a_2 z_3 a_1$ . Then, by Corollary 3.6,  $z' = z_3 a_1 z_3$  is also a left G-outer  $(b, c)$ -inverse of  $a_1$ . We show that

$$a_1 z' = (a_1 z_3 a_1) z_3 = a_1 z_3 = a_2 (z_3 a_1 z_3) = a_2 z'.$$

Analogously, we verify that  $z' a_1 = z' a_2$  and so it is  $a_1 \leq^{l,go,b,c} a_2$ .

(ii)  $\Rightarrow$  (iv): Suppose that  $z_3 \in a_1 \{l, go, b, c\}$  satisfies  $a_1 z_3 = a_2 z_3$  and  $z_3 a_1 = z_3 a_2$ . For  $p = a_1 z_3$  and  $q = z_3 a_1$ , according to Corollary 3.4, we have  $p\mathcal{R} = a_1\mathcal{R}$ ,  $\mathcal{R}a_1 = \mathcal{R}q$  and  $b\mathcal{R} \subset q\mathcal{R}$ . Also,  $pa_2 = (a_1 z_3)a_2 = a_1(z_3 a_2) = a_1 z_3 a_1 = a_1$  and  $a_1 q = a_1(z_3 a_1) = a_1$ .

(iv)  $\Rightarrow$  (ii): If  $a_1^-$  is an inner inverse of  $a_1$ , by Corollary 3.4,  $z_3 = qa_1^- p$  is a left G-outer  $(b, c)$ -inverse of  $a_1$ . We now check  $a_1 z_3 = (a_1 q)a_1^- p = a_1 a_1^- p = a_2(qa_1^- p) = a_2 z_3$  and similarly  $z_3 a_1 = z_3 a_2$ .  $\square$

If  $(\mathcal{R}, \cdot)$  is a ring, then  $(\mathcal{R}, \star)$  is another ring if we define  $a \star b = b \cdot a$ . By this fact and Theorem 4.2, we can prove Theorem 4.3 about characterizations of the relation  $\leq^{r,go,b,c}$ .

**Theorem 4.3.** *Let  $b, c \in \mathcal{R}$  and  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . The following conditions are equivalent:*

- (i)  $a_1 \leq^{r,go,b,c} a_2$ ;
- (ii) *there is  $z_3 \in a_1 \{r, go, b, c\}$  such that*

$$a_1 z_3 = a_2 z_3 \quad \text{and} \quad z_3 a_1 = z_3 a_2;$$

- (iii) *there is  $z_3 \in a_1 \{r, go, b, c\}$  such that*

$$a_1 z_3 a_2 = a_1 = a_2 z_3 a_1;$$

- (iv) *there are idempotents  $p, q \in \mathcal{R}$  such that*

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad \mathcal{R}p \subseteq \mathcal{R}c \quad \text{and} \quad pa_2 = a_1 = a_2q.$$

Consequently, by Theorem 4.2 and Theorem 4.3, we characterize the relation  $\leq^{go,b,c}$ .

**Corollary 4.4.** *Let  $b, c \in \mathcal{R}$  and  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ . The following conditions are equivalent:*

- (i)  $a_1 \leq^{go,b,c} a_2$ ;
- (ii) *there is  $z_3 \in a_1 \{go, b, c\}$  such that*

$$a_1 z_3 = a_2 z_3 \quad \text{and} \quad z_3 a_1 = z_3 a_2;$$

- (iii) *there is  $z_3 \in a_1 \{go, b, c\}$  such that*

$$a_1 z_3 a_2 = a_1 = a_2 z_3 a_1;$$

- (iv) *there are idempotents  $p, q \in \mathcal{R}$  such that*

$$p\mathcal{R} = a_1\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a_1, \quad a^{||(b,c)}p = qa^{||(b,c)} \quad \text{and} \quad pa_2 = a_1 = a_2q.$$

By Theorem 4.2, Theorem 4.3 and Corollary 4.4, if  $a_1 \leq^{l,go,b,c} a_2$  or  $a_1 \leq^{r,go,b,c} a_2$  or  $a_1 \leq^{go,b,c} a_2$ , then  $a_1 \leq^- a_2$ . The relations defined in this section imply the following inclusions.

**Theorem 4.5.** *Let  $b, c \in \mathcal{R}$  and  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ .*

- (i) *If  $a_1 \leq^{l,go,b,c} a_2$ , then  $a_2 \{l, go, b, c\} \subseteq a_1 \{l, go, b, c\}$ .*
- (ii) *If  $a_1 \leq^{r,go,b,c} a_2$ , then  $a_2 \{r, go, b, c\} \subseteq a_1 \{r, go, b, c\}$ .*
- (iii) *If  $a_1 \leq^{go,b,c} a_2$ , then  $a_2 \{go, b, c\} \subseteq a_1 \{go, b, c\}$ .*

**Proof.** If  $a_1 \leq^{l,go,b,c} a_2$ , then according to Theorem 4.2, there is  $z_3 \in a_1 \{l, go, b, c\}$  satisfying  $a_1 z_3 = a_2 z_3$  and  $z_3 a_1 = z_3 a_2$ . Let  $f$  be a left G-outer  $(b, c)$ -inverse of  $a_2$ . We have equalities  $a_2 f a_2 = a_2$  and  $f a_2 a_2^{||(b,c)} = a_2^{||(b,c)}$ . Then

$$a_1 f a_1 = a_1 (z_3 a_1) f (a_1 z_3) a_1 = a_1 z_3 (a_2 f a_2) z_3 a_1 = a_1 z_3 a_2 z_3 a_1 = a_1 z_3 a_1 = a_1.$$

Since  $a_1^{||(b,c)} \mathcal{R} = b\mathcal{R} = a_2^{||(b,c)} \mathcal{R}$ , for some  $u \in \mathcal{R}$ , we show that

$$a_1^{||(b,c)} = a_2^{||(b,c)} u = a_2^{||(b,c)} a_2 (a_2^{||(b,c)} u) = a_2^{||(b,c)} a_2 a_1^{||(b,c)}.$$

Also, we have that it is

$$\begin{aligned} f a_1 a_1^{||(b,c)} &= f (a_1 z_3) a_1 a_1^{||(b,c)} = f a_2 (z_3 a_1 a_1^{||(b,c)}) = f a_2 a_1^{||(b,c)} \\ &= (f a_2 a_2^{||(b,c)}) a_2 a_1^{||(b,c)} = a_2^{||(b,c)} a_2 a_1^{||(b,c)} = a_1^{||(b,c)}. \end{aligned}$$

In this way, we conclude that  $f$  is the left G-outer  $(b, c)$ -inverse of  $a_1$  and we proved the inclusion in (i). Analogously, we prove parts (ii) and (iii).  $\square$

Now we prove that binary relations  $\leq^{l,go,b,c}$ ,  $\leq^{r,go,b,c}$  and  $\leq^{go,b,c}$  are partial orders on the set  $\mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ .

**Theorem 4.6.** *Let  $b, c \in \mathcal{R}$ . The left  $G$ -outer  $(b, c)$ -relation, right  $G$ -outer  $(b, c)$ -relation and  $G$ -outer  $(b, c)$ -relation are partial orders on the set  $\mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ .*

**Proof.** Evidently, the relation  $\leq^{l,go,b,c}$  is reflexive. To show the antisymmetry, suppose that, for  $a_1, a_2 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$ , it is valid  $a_1 \leq^{l,go,b,c} a_2$  and  $a_2 \leq^{l,go,b,c} a_1$ . It follows that the relations  $a_1 \leq^- a_2$  and  $a_2 \leq^- a_1$  hold. Since we have shown that the relation  $\leq^-$  is antisymmetric, we conclude that  $a_1 = a_2$ . So, the relation  $\leq^{l,go,b,c}$  is antisymmetric.

Now, we verify the transitivity of the relation  $\leq^{l,go,b,c}$ . Let  $a_1, a_2, a_3 \in \mathcal{R}^{||(b,c)} \cap \mathcal{R}^-$  satisfy  $a_1 \leq^{l,go,b,c} a_2$  and  $a_2 \leq^{l,go,b,c} a_3$ . By Theorem 4.2, there are left  $G$ -outer  $(b, c)$ -inverses  $e$  and  $f$  of  $a_1$  and  $a_2$ , respectively, such that

$$a_1 e = a_2 e, \quad e a_1 = e a_2, \quad a_2 f = a_3 f \quad \text{and} \quad f a_2 = f a_3.$$

Then, by Theorem 4.5,  $f$  is a left  $G$ -outer  $(b, c)$ -inverse of  $a_1$ . Thus,

$$a_1 = a_1 e a_1 = a_1 e a_2 = a_1 e a_2 (f a_2) = a_1 e a_2 (f a_3) = a_1 f a_3$$

and

$$a_1 = a_1 e a_1 = a_2 e a_1 = (a_2 f) a_2 e a_1 = a_3 f (a_2 e a_1) = a_3 f a_1.$$

According to Theorem 4.2(iii),  $a_1 \leq^{l,go,b,c} a_3$ .

The rest follows in a similar manner. □

## 5. Left and right G-Drazin inverses

As special versions of left and right  $G$ -outer  $(b, c)$ -inverses, we define left and right  $G$ -Drazin inverses for elements of a ring.

**Definition 5.1.** Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . An element  $z \in \mathcal{R}$  is:

(i) a left  $G$ -Drazin inverse of  $a$  if

$$a z a = a \quad \text{and} \quad z a^{k+1} = a^k;$$

(ii) a right  $G$ -Drazin inverse of  $a$  if

$$a z a = a \quad \text{and} \quad a^{k+1} z = a^k.$$

If  $c \in \mathcal{R}$  is both the left  $G$ -Drazin inverse and right  $G$ -Drazin inverse of  $a \in \mathcal{R}$ , then it is  $G$ -Drazin inverse of  $a$ . We denote the sets of all left and right  $G$ -Drazin inverses of  $a$  with  $a \{l, GD\}$  and  $a \{r, GD\}$ , respectively.

The following results are consequences of results of Section 3 and we can omit their proofs.

**Corollary 5.2.** *Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . For  $z \in \mathcal{R}$ , the following conditions are equivalent:*

- (i)  $z \in a\{l, GD\}$ ;
- (ii)  $aza = a$  and  $zaa^D = a^D$ ;
- (iii)  $aza = a$  and  $zaa^D a = a^D a$ ;
- (iv)  $aza = a$  and  $a^k \mathcal{R} \subseteq za\mathcal{R}$ .

**Corollary 5.3.** *Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . For  $z \in \mathcal{R}$ , the following conditions are equivalent:*

- (i)  $z \in a\{r, GD\}$ ;
- (ii)  $aza = a$  and  $a^D az = aa^D$ ;
- (iii)  $aza = a$  and  $aa^D az = aa^D$ ;
- (iv)  $aza = a$  and  $(az)^\circ \subseteq (a^k)^\circ$ .

Let us recall that, relative to an idempotent  $p \in \mathcal{R}$ ,  $a \in \mathcal{R}$  has a matrix form:

$$a = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_p,$$

where  $a_{11} = pap$ ,  $a_{12} = pa(1-p)$ ,  $a_{21} = (1-p)ap$ ,  $a_{22} = (1-p)a(1-p)$ .

It is known that  $a \in \mathcal{R}^d$  can be represented by

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p, \quad (1)$$

where  $a_1$  is invertible in  $p\mathcal{R}p$  and  $a_2^k = 0$ .

We show that a left G-Drazin invertible element has the following matrix form.

**Theorem 5.4.** *Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$ ,  $\text{ind}(a) = k$  and  $p = aa^d$ . If  $a$  is represented by (1), then the following conditions are equivalent:*

- (i)  $z \in a\{l, GD\}$ ;
- (ii)  $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p$ ,  $z_3 a_2 = 0$  and  $z_2 \in a_2\{1\}$ .

**Proof.** (i)  $\Rightarrow$  (ii): Let's assume  $z = \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p$ . We first use the equation  $za^{k+1} = a^k$  to get

$$\begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1^{k+1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} z_1 a_1^{k+1} & 0 \\ z_4 a_1^{k+1} & 0 \end{bmatrix}_p = \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p.$$

This implies  $z_1 a_1^{k+1} = a_1^k$  and  $z_4 a_1^{k+1} = 0$ . For the inverse  $a_{1,p\mathcal{R}p}^{-1}$  of  $a_1$  in  $p\mathcal{R}p$ , by  $z_1, z_4 \in \mathcal{R}p$ , we conclude  $z_1 = z_1 p = z_1 a_1 a_{1,p\mathcal{R}p}^{-1} = z_1 a_1^{k+1} (a_{1,p\mathcal{R}p}^{-1})^{k+1} =$

$a_1^k(a_{1,p\mathcal{R}p}^{-1})^{k+1} = a_{1,p\mathcal{R}p}^{-1}$  and  $z_4 = z_4p = z_4a_1^{k+1}(a_1^{-1})^{k+1} = 0$ . From  $aza = a$  and

$$aza = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p a = \begin{bmatrix} p & a_1z_3 \\ 0 & a_2z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & a_1z_3a_2 \\ 0 & a_2z_2a_2 \end{bmatrix}_p,$$

we obtain  $z_3a_2 = pz_3a_2 = a_{1,p\mathcal{R}p}^{-1}(a_1z_3a_2) = 0$  and  $z_2 \in a_2\{1\}$ .

(ii)  $\Rightarrow$  (i): By elementary computations, we show that it is  $z \in a\{l, GD\}$ .  $\square$

Analogously as Theorem 5.4, we get the following theorem.

**Theorem 5.5.** *Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$ ,  $\text{ind}(a) = k$  and  $p = a^da$ . If  $a$  is represented by (1), then the following conditions are equivalent:*

- (i)  $z \in a\{r, GD\}$ ;
- (ii)  $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ z_4 & z_2 \end{bmatrix}_p$ ,  $a_2z_4 = 0$  and  $z_2 \in a_2\{1\}$ .

Also, by results of Section 3, we have the following characterizations of left and right G-Drazin invertibility.

**Corollary 5.6.** *Let  $a \in \mathcal{R}^d$  and  $\text{ind}(a) = k$ .*

(a) *The following conditions are equivalent:*

- (i)  $a\{l, GD\} \neq \emptyset$ ;
- (ii) *there are idempotents  $p, q \in \mathcal{R}$  such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad a^k\mathcal{R} \subseteq q\mathcal{R}.$$

*Additionally, for arbitrary  $a^- \in a\{1\}$ , it is valid  $qa^-p \in a\{l, GD\}$  and  $q \cdot a\{1\} \cdot p \subseteq a\{l, GD\}$ .*

(b) *The following conditions are equivalent:*

- (i)  $a\{r, GD\} \neq \emptyset$ ;
- (ii) *there are idempotents  $p, q \in \mathcal{R}$  such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}a = \mathcal{R}q \quad \text{and} \quad \mathcal{R}p \subseteq \mathcal{R}a^k.$$

*Additionally, for arbitrary  $a^- \in a\{1\}$ , it is valid  $qa^-p \in a\{r, GD\}$  and  $q \cdot a\{1\} \cdot p \subseteq a\{r, GD\}$ .*

**Corollary 5.7.** *Let  $a \in \mathcal{R}^d \cap \mathcal{R}^-$ . Then  $a\{l, GD\} \cdot a \cdot a\{l, GD\} \subseteq a\{l, GD\}$  and  $a\{r, GD\} \cdot a \cdot a\{r, GD\} \subseteq a\{r, GD\}$ .*

## 6. Left and right G-Drazin partial orders

Using left and right G-Drazin inverses, we introduce left and right G-Drazin binary relations for elements of a ring.

**Definition 6.1.** Let  $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$ . Then

(i)  $a$  is below  $b$  under the left G-Drazin relation ( $a \leq^{l, GD} b$ ) if there exist  $z_1, z_2 \in a\{l, GD\}$  such that

$$az_1 = bz_1 \quad \text{and} \quad z_2a = z_2b;$$

(ii)  $a$  is below  $b$  under the right G-Drazin relation ( $a \leq^{r, GD} b$ ) if there exist  $z_1, z_2 \in a\{r, GD\}$  such that

$$az_1 = bz_1 \quad \text{and} \quad z_2a = z_2b.$$

**Theorem 6.2.** Let  $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . If  $a$  is represented by (1), then the following conditions are equivalent:

- (i)  $a \leq^{l, GD} b$ ;
- (ii) there is  $z \in a\{l, GD\}$  such that  $az = bz$  and  $za = zb$ ;
- (iii) there is  $z \in a\{l, GD\}$  such that  $azb = a = bza$ ;
- (iv) there are idempotents  $p, q \in \mathcal{R}$  such that

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a, \quad \mathcal{R}a^k \subseteq \mathcal{R}q \quad \text{and} \quad pb = a = bq.$$

(v) there is  $z \in \mathcal{R}$  such that for  $p = aa^d$ ,

$$z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} a_1 & -a_1z_3b_2 \\ 0 & b_2 \end{bmatrix}_p,$$

where  $z_3a_2 = 0$ ,  $z_2 \in a_2\{1\}$ ,  $a_2z_2 = b_2z_2$  and  $z_2a_2 = z_2b_2$ .

**Proof.** (i)  $\Leftrightarrow$  (ii)-(iv): It follows by Theorem 4.2.

(ii)  $\Rightarrow$  (v): Assume that there is  $z \in a\{l, GD\}$  such that  $az = bz$  and  $za = zb$ .

Using Theorem 5.4, we know that  $a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$  and  $z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p$ , where

$a_1$  is invertible in  $p\mathcal{R}p$ ,  $a_2^k = 0$ ,  $z_3a_2 = 0$ , and  $z_2 \in a_2\{1\}$ .

Let  $b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p$ . From  $az = bz$ ,

$$\begin{bmatrix} p & a_1z_3 \\ 0 & a_2z_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1a_{1,p\mathcal{R}p}^{-1} & b_1z_3 + b_3z_2 \\ b_4a_{1,p\mathcal{R}p}^{-1} & b_4z_3 + b_2z_2 \end{bmatrix}_p.$$

We conclude  $b_1 = b_1p = (b_1a_{1,p\mathcal{R}p}^{-1})a_1 = pa_1 = a_1$ ,  $b_4 = b_4p = (b_4a_{1,p\mathcal{R}p}^{-1})a_1 = 0$  and  $a_2z_2 = b_2z_2$ . The equality  $za = zb$  gives

$$\begin{bmatrix} p & z_3a_2 \\ 0 & z_2a_2 \end{bmatrix}_p = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2 \end{bmatrix}_p \begin{bmatrix} a_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p = \begin{bmatrix} p & a_{1,p\mathcal{R}p}^{-1}b_3 + z_3b_2 \\ 0 & z_2b_2 \end{bmatrix}_p.$$

We have now  $z_2a_2 = z_2b_2$  and, by  $z_3a_2 = 0$ ,  $a_{1,p\mathcal{R}p}^{-1}b_3 + z_3b_2 = 0$ , which implies  $b_3 = pb_3 = a_1(a_{1,p\mathcal{R}p}^{-1}b_3) = -a_1z_3b_2$ . So,  $b = \begin{bmatrix} a_1 & -a_1z_3b_2 \\ 0 & b_2 \end{bmatrix}_p$  and the implication is true.

(v)  $\Rightarrow$  (ii): Clearly, by direct computations.  $\square$

Analogously, the following theorem can be shown.

**Theorem 6.3.** *Let  $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . If  $a$  is represented by (1), then the following conditions are equivalent:*

- (i)  $a \leq^{r, GD} b$ ;
- (ii) *there is  $z \in a\{r, GD\}$  such that  $az = bz$  and  $za = zb$ ;*
- (iii) *there is  $z \in a\{r, GD\}$  such that  $azb = a = bza$ ;*
- (iv) *there are idempotents  $p, q \in \mathcal{R}$  such that*

$$p\mathcal{R} = a\mathcal{R}, \quad \mathcal{R}q = \mathcal{R}a, \quad p\mathcal{R} \subseteq a^k\mathcal{R} \quad \text{and} \quad pb = a = bq.$$

- (v) *there is  $z \in \mathcal{R}$  such that for  $p = a^da$ ,*

$$z = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ z_4 & z_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} a_1 & 0 \\ -b_2z_4a_1 & b_2 \end{bmatrix}_p,$$

where  $a_2z_4 = 0$ ,  $z_2 \in a_2\{1\}$ ,  $a_2z_2 = b_2z_2$  and  $z_2a_2 = z_2b_2$ .

We know that if  $a \leq^{l, GD} b$ , it is valid  $a \leq^- b$ . Under the given conditions, we prove the converse.

**Theorem 6.4.** *Let  $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . The following conditions are equivalent:*

- (i)  $a \leq^{l, GD} b$ ;
- (ii)  $a \leq^- b$  and  $ba^d = aa^d$ ;
- (iii)  $a \leq^- b$  and  $ba^k = a^{k+1}$ ;
- (iv)  $a \leq^- b$  and  $ba^k\mathcal{R} \subseteq a^k\mathcal{R}$ .

**Proof.** (i)  $\Rightarrow$  (ii): Let  $a \leq^{l, GD} b$ . By definition, it is valid  $a \leq^- b$ . Using Theorem 6.2, there is  $z \in \mathcal{R}$  such that  $a$ ,  $b$  and  $z$  are represented as in (v). Since  $b = \begin{bmatrix} a_1 & -a_1z_3b_2 \\ 0 & b_2 \end{bmatrix}_p$  and  $a^d = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p$ , we get the following:

$$ba^d = \begin{bmatrix} a_1 & -a_1 z_3 b_2 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = aa^d.$$

(ii)  $\Rightarrow$  (iii): The assumption  $ba^d = aa^d$  implies

$$ba^k = ba^d a^{k+1} = aa^d a^{k+1} = a^{k+1}.$$

(iii)  $\Rightarrow$  (iv): This is obvious.

(iv)  $\Rightarrow$  (i): For  $p = aa^d$ , we can write

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \quad \text{and} \quad b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where  $a_1$  is invertible in  $p\mathcal{R}p$  and  $a_2^k = 0$ . Since  $ba^k\mathcal{R} \subseteq a^k\mathcal{R}$ , we have

$$ba^k = a^k r = aa^d(a^k r) = aa^d ba^k,$$

for some  $r \in \mathcal{R}$ . Then

$$\begin{bmatrix} b_1 a_1^k & 0 \\ b_4 a_1^k & 0 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p \begin{bmatrix} a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} b_1 a_1^k & 0 \\ 0 & 0 \end{bmatrix}_p.$$

We deduce that  $b_4 a_1^k = 0$ , which implies

$$b_4 = b_4 p = b_4 a_1^k (a_{1,p\mathcal{R}p}^{-1})^k = 0.$$

From the assumption  $a \leq^- b$ , there is  $z \in a\{1\}$  satisfying  $az = bz$  and  $za = zb$ .

Let  $z = \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p$ . Then  $aza = a$  yields

$$\begin{bmatrix} a_1 z_1 a_1 & a_1 z_3 a_2 \\ a_2 z_4 a_1 & a_2 z_2 a_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p$$

which implies  $z_1 = a_{1,p\mathcal{R}p}^{-1}$ ,  $z_3 a_2 = 0$ ,  $a_2 z_4 = 0$  and  $z_2 \in a_2\{1\}$ . Using equality  $az = bz$ , we have:

$$\begin{bmatrix} p & a_1 z_3 \\ 0 & a_2 z_2 \end{bmatrix}_p = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 & b_3 \\ 0 & b_2 \end{bmatrix}_p \begin{bmatrix} z_1 & z_3 \\ z_4 & z_2 \end{bmatrix}_p = \begin{bmatrix} b_1 z_1 + b_3 z_4 & b_1 z_3 + b_3 z_2 \\ b_2 z_4 & b_2 z_2 \end{bmatrix}_p.$$

We concluded that  $b_2 z_4 = 0$ ,  $a_2 z_2 = b_2 z_2$ ,

$$p = b_1 z_1 + b_3 z_4 \quad \text{and} \quad a_1 z_3 = b_1 z_3 + b_3 z_2. \quad (2)$$

Also, by  $za = zb$ , we get

$$\begin{bmatrix} p & z_3 a_2 \\ z_4 a_1 & z_2 a_2 \end{bmatrix}_p = \begin{bmatrix} z_1 b_1 & z_1 b_3 + z_3 b_2 \\ z_4 b_1 & z_4 b_3 + z_2 b_2 \end{bmatrix}_p,$$

so we conclude that  $a_1 = b_1$ . Using (2),  $p = b_1z_1 + b_3z_4 = p + b_3z_4$ , which implies  $b_3z_4 = 0$ . Below,  $a_1z_3 = b_1z_3 + b_3z_2 = a_1z_3 + b_3z_2$ , which implies  $b_3z_2 = 0$ .

Also valid  $z_3a_2 = a_{1,p\mathcal{R}p}^{-1}b_3 + z_3b_2$  and  $z_2a_2 = z_4b_3 + z_2b_2$ . Thus,  $0 = a_{1,p\mathcal{R}p}^{-1}b_3 + z_3b_2$  implies  $b_3 = -a_1z_3b_2$ .

Let's define now  $z' = \begin{bmatrix} a_{1,p\mathcal{R}p}^{-1} & z_3 \\ 0 & z_2a_2z_2 \end{bmatrix}_p$ . Since  $z_2a_2z_2 \in a_2\{1\}$ , by Theorem 5.4,  $z' \in a\{l, GD\}$ .

Now we check the accuracy of the equalities  $az' = bz'$  and  $z'a = z'b$ . Equalities  $b_3z_2 = 0$  and  $a_2z_2 = b_2z_2$  imply

$$az' = \begin{bmatrix} p & a_1z_3 \\ 0 & a_2z_2a_2z_2 \end{bmatrix}_p = \begin{bmatrix} p & a_1z_3 + b_3z_2a_2z_2 \\ 0 & b_2z_2a_2z_2 \end{bmatrix}_p = bz'.$$

Analogously, by  $z_3a_2 = 0 = a_{1,p\mathcal{R}p}^{-1}b_3 + z_3b_2$  and  $z_2a_2z_2b_2 = z_2a_2(z_2a_2 - z_4b_3) = z_2a_2$ , we show  $z'a = z'b$ . Finally, it was shown to be valid  $a \leq^{l, GD} b$ .  $\square$

We similarly consider the following theorem related to the relation  $\leq^{r, GD}$ .

**Theorem 6.5.** *Let  $a, b \in \mathcal{R}^d \cap \mathcal{R}^-$  and  $\text{ind}(a) = k$ . The following conditions are equivalent:*

- (i)  $a \leq^{r, GD} b$ ;
- (ii)  $a \leq^- b$  and  $a^da = a^da$ ;
- (iii)  $a \leq^- b$  and  $a^kb = a^{k+1}$ ;
- (iv)  $a \leq^- b$  and  $\mathcal{R}a^kb \subseteq \mathcal{R}a^k$ .

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