International Electronic Journal of Algebra
Volume 40 (2026) 71-90
DOI: 10.24330/ieja.1891248

DUAL SPACE OF EXPONENTIAL VECTOR SPACE

Jayeeta Saha

Received: 12 September 2025; Revised: 12 November 2025; Accepted: 21 November 2025
Communicated by Handan Köse

Abstract. This paper explores the idea of the dual space of an exponential vector space (evs), focusing on its fundamental properties and structural features. Using the simplest topological evs $[0, \infty)$, over the field $\mathbb{K}$ of all real or complex numbers, we develop the general theory of order-functionals which are an analog of a functional-like structure in the evs-setting. The collection of all order-functionals on an evs is called the dual of the evs, whenever this collection is again an evs. We find a necessary and sufficient condition for this collection to be an evs. We compute the dual of some evs and finally present an application of order-functionals.

Mathematics Subject Classification (2020): 46A99, 06F99
Keywords: Topological exponential vector space, balanced exponential vector space, order-functional, dual, topological dual

1. Introduction

The concept of the dual space plays a crucial role in functional analysis and linear algebra; it provides a powerful framework for understanding the relationship between a linear space and the set of all linear functionals defined on it. In fact, properties of a linear space are studied with the help of linear functionals defined on it. In this paper, we shall explore the collection of all order-functionals on an exponential vector space, called its *dual*.

The concept of an exponential vector space generalizes the concept of a vector space by extending its axioms to accommodate exponential-type structures and operations. By examining the dual space of such a structure, one gains a clearer understanding of the order-functionals and how the exponential framework influences their form and properties. The aim of this paper is to explore the dual space by establishing fundamental properties, structural results, and possible isomorphisms. First, we recall the formal definition of an exponential vector space, which will serve as the foundation for our discussion.

In an earlier work [2], S. Ganguly, S. Mitra, and S. Jana introduced the term *quasi vector space* to describe what is now referred to as an *exponential vector* *space*. Later, Priti Sharma and Sandip Jana adopted the term *exponential vector space* in place of quasi vector space in a subsequent paper [8]. The definition of an exponential vector space is as follows:

Definition 1.1. [8]Let X be a nonempty set and '$\leq$' be a partial order in it. Let '$+$' be a binary operation on X [called *addition*] and '$\cdot$' : $K \times X \longrightarrow X$ be another composition [called *scalar multiplication*, K being a field]. If '$\leq$', '$+$' and '$\cdot$' satisfy the following axioms, we call $(X, \leq\ , +\ , \cdot)$ an *exponential vector space* [in short, *evs*] over the field K.

$A_1 : (X, +)$ is a commutative semigroup with identity θ

$A_2 : x \leq y\ (x, y \in X) \Rightarrow x + z \leq y + z$ and $\alpha \cdot x \leq \alpha \cdot y,\ \forall z \in X, \forall \alpha \in K$

A_3 : (i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$

(ii) $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x$

(iii) $(\alpha + \beta) \cdot x \leq \alpha \cdot x + \beta \cdot x$

(iv) $1 \cdot x = x$, where '1' is the multiplicative identity in K,

$\forall x, y \in X,\ \forall \alpha, \beta \in K$

$A_4 : \alpha \cdot x = \theta$ iff $\alpha = 0$ or $x = \theta$

$A_5 : x + (-1) \cdot x = \theta$ iff $x \in X_0 := \big\{z \in X :\ y \not\leq z, \forall y \in X \smallsetminus \{z\}\big\}$

A_6 : For each $x \in X$, $\exists\, y \in X_0$ such that $y \leq x$.

In the above definition, X_0 is precisely the set of all minimal elements (renamed as *primitive elements* in an evs) of the evs X with respect to the partial order of X and forms a vector space over the same field as that of X ([2]). We present below the definition of topological exponential vector space.

Definition 1.2. [8]An evs X over the field $\mathbb{K}$ of real or complex numbers is said to be a *topological exponential vector space* if X has a topological structure with respect to which the addition and scalar multiplication are continuous and the partial order '$\leq$' is closed. Here $\mathbb{K}$ is equipped with the usual topology. [The partial order '$\leq$' is said to be *closed* if its graph $G_{\leq}(X) := \{(x, y) \in X \times X : x \leq y\}$ is closed in $X \times X$ with respect to the product topology [5].]

In the present paper, we discuss the concept of order-functional on an evs, which we have introduced in the paper [3]. It is an analogue of functional-like structure in the evs-setting. To introduce such a concept we have used the space $[0, \infty)$ which is the simplest possible topological evs over the real or complex scalar field $\mathbb{K}$. In the 3rd section, we have developed the general theory of order-functional. The collection of all order-functionals on an evs is termed as *dual* of the evs, whenever the collection itself becomes an evs. The computation of dual of different evs is carried out in the 4th section. In the 5th section, we have discussed an application of order-functional in the settings of evs.

2. Prerequisites

Definition 2.1. [3]A subset Y of an evs X is said to be a *sub exponential vector space* or *subevs* of X if Y itself is an evs with all the compositions and partial order of X being restricted to Y.

Note 2.2. For any subset Y of an evs X, if $Y_0 := \big\{z \in Y : y \nleq z, \forall\, y \in Y \smallsetminus \{z\}\big\}$, then clearly $X_0 \cap Y \subseteq Y_0$. If Y is a subevs of X as well, then $Y_0 \subseteq X_0 \cap Y$. In fact, $y \in Y_0 \Rightarrow y + (-1) \cdot y = \theta$ and $y \in Y \Rightarrow y \in X_0$ and $y \in Y \Rightarrow y \in X_0 \cap Y$. Thus a subset Y of an evs X over the field K is a subevs of X iff it satisfies the following conditions:

(i) $\alpha x + \beta y \in Y$, $\forall\, x, y \in Y, \forall\, \alpha, \beta \in K$,
(ii) $Y_0 \subseteq X_0 \cap Y$,
(iii) For each $y \in Y$, $\exists\, y_0 \in Y_0$ such that $y_0 \leq y$.

Thus for a subevs Y of an evs X, we have $Y_0 = X_0 \cap Y$. Also we can say that if Y is a subevs of an evs X and Z is a subevs of Y, then Z must be a subevs of X. In fact, $Z_0 = Z \cap Y_0 = Z \cap Y \cap X_0 = Z \cap X_0$. The rest is obvious.

Notation: Let X be an evs. For any $A \subseteq X$ we denote

$$\downarrow A := \{x \in X : x \leq a \text{ for some } a \in A\}$$

and

$$\uparrow A := \{x \in X : x \geq a \text{ for some } a \in A\}.$$

Definition 2.3. [1]A map $\phi : X \longrightarrow Y$ (X, Y being two exponential vector spaces over the same field K) is said to be an *order-morphism* if

(i) $\phi(x + y) = \phi(x) + \phi(y),\ \forall x, y \in X$
(ii) $\phi(\alpha x) = \alpha\phi(x),\ \forall x \in X, \forall \alpha \in K$
(iii) $x \leq y\ \big(x, y \in X\big) \Rightarrow \phi(x) \leq \phi(y)$
(iv) $p \leq q\ \big(p, q \in \phi(X)\big) \Rightarrow \phi^{-1}(p) \subseteq \downarrow \phi^{-1}(q)$ and $\phi^{-1}(q) \subseteq \uparrow \phi^{-1}(p)$.

If moreover ϕ is bijective, then it is called an *order-isomorphism.* If X, Y be two topological evs, then $\phi : X \longrightarrow Y$ is said to be a *topological order-isomorphism* if it is a homeomorphism and order-isomorphism.

Definition 2.4. [8]A property of an evs is called an *evs property* if it remains invariant under order-isomorphism.

Definition 2.5. [6]Let X be an evs over the field $\mathbb{K}$ of real or complex numbers. An element $x \in X$ is said to be a *balanced element* if $\alpha x \leq x$, $\forall\, \alpha \in \mathbb{K}$ with $|\alpha| \leq 1$. The set $X_b := \{x \in X : x \text{ is a balanced element}\}$ is called the *balanced core* of X.

Definition 2.6. [6]An evs X is said to be a *balanced evs* if $X_b = X$.

We now give some examples of exponential vector space which we shall use later.

Example 2.7. [8]On the set $[0, \infty)^n$, $n \in \mathbb{N}$ define operations and partial order as follows: for $(x_1, \dots, x_n), (y_1, \dots, y_n) \in [0, \infty)^n$ and $\alpha \in \mathbb{K}$,

(i) $(x_1, \dots, x_n) + (y_1, \dots, y_n) := (x_1 + y_1, \dots, x_n + y_n)$
(ii) $\alpha(x_1, \dots, x_n) := (|\alpha|x_1, \dots, |\alpha|x_n)$
(iii) $(x_1, \dots, x_n) \leq (y_1, \dots, y_n)$ iff $x_1 \leq y_1, \dots, x_n \leq y_n$.

Then $[0, \infty)^n$ becomes a *balanced topological evs* with respect to the above operations, partial order and the subspace topology inherited from $\mathbb{R}^n$ with usual topology.

Example 2.8. [3]For any Hausdorff topological vector space $\mathcal{X}$ (over the field $\mathbb{K}$), the family $\mathscr{C}(\mathcal{X})$ of all nonempty compact subsets of $\mathcal{X}$ forms a topological evs with respect to usual addition, scalar multiplication of sets, set-inclusion as partial order and Vietoris topology ([4]) on it. If in particular, we consider the collection of all nonempty finite subsets of $\mathcal{X}$, denoted as $\mathscr{C}_F(\mathcal{X})$, then $\mathscr{C}_F(\mathcal{X})$ becomes a dense topological subevs of $\mathscr{C}(\mathcal{X})$. Also instead of $\mathscr{C}(\mathcal{X})$ if we consider $\mathscr{P}(\mathcal{X})$ [the collection of all nonempty subsets of $\mathcal{X}$] and define the operations and partial order same as in $\mathscr{C}(\mathcal{X})$, then $\mathscr{P}(\mathcal{X})$ becomes an evs over the field $\mathbb{K}$ with respect to these operations and partial order [7].

If $\mathcal{X}$ is a normed linear space, then the family $\mathscr{B}_0(\mathcal{X})$ of all closed balls with center at the origin in $\mathcal{X}$ forms an evs (in [3] we have used the symbol $\mathscr{C}_0(\mathcal{X})$ to designate the same family); if $\mathcal{X}$ is a finite dimensional Banach space, then $\mathscr{B}_0(\mathcal{X})$ becomes a topological subevs of $\mathscr{C}(\mathcal{X})$ and if $\mathcal{X}$ is an infinite dimensional normed linear space, then $\mathscr{B}_0(\mathcal{X})$ will be a subevs of $\mathscr{P}(\mathcal{X})$. Also for any normed linear space $\mathcal{X}$, $\mathscr{B}_0(\mathcal{X})$ is order-isomorphic to the evs $[0, \infty)$; if moreover, $\mathcal{X}$ is a finite dimensional Banach space, then $\mathscr{B}_0(\mathcal{X})$ is topologically order-isomorphic to the topological evs $[0,\infty)$.

Example 2.9. [6]Let us consider $\mathscr{D}^2([0,\infty)) := [0,\infty)\times[0,\infty)$. We define $+,\cdot,\leq$ on $\mathscr{D}^2([0,\infty))$ as follows:
For $(x_1,y_1),(x_2,y_2)\in\mathscr{D}^2([0,\infty))$ and $\alpha\in\mathbb{C}$, we define

(i) $(x_1,y_1)+(x_2,y_2) := (x_1+x_2, y_1+y_2)$

(ii) $\alpha\cdot(x_1,y_1) := (|\alpha|x_1, |\alpha|y_1)$

(iii) $(x_1,y_1)\leq(x_2,y_2)\iff$ either $x_1<x_2$ or if $x_1=x_2$ then $y_1\leq y_2$ [*dictionary order*].

Then $\left(\mathscr{D}^2([0,\infty)),+,\cdot,\leq\right)$ forms a *non-topological balanced evs* i.e., an evs which can never be made a topological evs with respect to any topology on it.

For a *well-ordered* set I and an evs X, if we consider $\mathscr{D}(X:I) := X^I$, then also, like above example, it forms a non-topological evs with dictionary order. If $I=\{1,2,\ldots,n\}$, we shall usually denote the evs $\mathscr{D}(X:I)$ as $\mathscr{D}^n(X)$. Moreover $\mathscr{D}(X:I)$ will be balanced iff X is so. We can also generalize this by taking different (balanced) evs i.e.,

$$\mathscr{D}(X_\alpha:\alpha\in I) := \prod_{\alpha\in I} X_\alpha,$$

which also becomes a (balanced) evs with dictionary order.

We now introduce an example of a *non-balanced topological evs* which will be useful in the sequel.

Example 2.10. Let X be an evs over the field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) and E be a vector space over the same field $\mathbb{K}$. We now define operations on $X\times E$ as follows: For $(x_1,e_1),(x_2,e_2),(x,e)\in X\times E$ and $\alpha\in\mathbb{K}$,

(i) $(x_1,e_1)+(x_2,e_2) := (x_1+x_2, e_1+e_2)$,

(ii) $\alpha(x,e) := (\alpha x, \alpha e)$.

The partial order '$\leq$' is defined as: $(x_1,e_1)\leq(x_2,e_2)$ iff $x_1\leq x_2$ and $e_1=e_2$. Then $X\times E$ becomes an evs over the field $\mathbb{K}$.

Justification:

$\mathbf{A_1}$: Clearly $(X\times E,+)$ is a commutative semigroup with identity (θ_X,θ_E) $\big[\theta_X,\theta_E$ being the identities in X,E, respectively$\big]$.

$\mathbf{A_2}$: $(x_1,e_1)\leq(x_2,e_2)\Rightarrow x_1\leq x_2$ and $e_1=e_2\Rightarrow$ for any $(x,e)\in X\times E$, $x_1+x\leq x_2+x$ and $e_1+e=e_2+e\Rightarrow(x_1,e_1)+(x,e)\leq(x_2,e_2)+(x,e)$. Also for any $\alpha\in\mathbb{K}$, $\alpha x_1\leq\alpha x_2$ and $\alpha e_1=\alpha e_2$. Therefore $\alpha(x_1,e_1)\leq\alpha(x_2,e_2)$.

$\mathbf{A_3}\,\mathbf{(i)}$: $\alpha\{(x_1,e_1)+(x_2,e_2)\} = \alpha(x_1+x_2,e_1+e_2) = (\alpha x_1+\alpha x_2, \alpha e_1+\alpha e_2)$ $\big[$by

axiom A_3 of evs X and since E is a vector space$\big]$ $= (\alpha x_1, \alpha e_1) + (\alpha x_2, \alpha e_2) = \alpha(x_1, e_1) + \alpha(x_2, e_2)$.
(ii) : $\alpha(\beta(x,e)) = \alpha(\beta x, \beta e) = (\alpha\beta x, \alpha\beta e) = \alpha\beta(x,e)$.
(iii) : $(\alpha+\beta)(x,e) = \big((\alpha+\beta)x, (\alpha+\beta)e\big) \leq (\alpha x + \beta x, \alpha e + \beta e)$ $\big[$since X is an evs, $(\alpha+\beta)x \leq \alpha x + \beta x\big]$ $= \alpha(x,e) + \beta(x,e)$.
(iv) : $1 \cdot (x,e) = (x,e)$.
$\mathbf{A_4}$: $\alpha(x,e) = (\theta_X, \theta_E) \Rightarrow (\alpha x, \alpha e) = (\theta_X, \theta_E) \Rightarrow \alpha x = \theta_X$ and $\alpha e = \theta_E$. Now if $\alpha \neq 0$, then $x = \theta_X$ and $e = \theta_E$ [since X and E are evs and vector space, respectively]. If $\alpha = 0$, then also $\alpha(x,e) = (\theta_X, \theta_E)$. The converse is obviously true.
$\mathbf{A_5}$: As for any $x_0 \in X_0$ and $e_0 \in E$, $\nexists\ (x,e) \in X \times E$ other than (x_0, e_0) such that $(x,e) \leq (x_0, e_0)$, so we can say that $[X \times E]_0 = \{(x,e) : x \in X_0, e \in E\} = X_0 \times E$. Now $(x,e) \in X \times E$ with $(x,e) + (-1)(x,e) = (\theta_X, \theta_E) \Rightarrow x + (-1)x = \theta_X$ and $e - e = \theta_E \Rightarrow x \in X_0$. So $(x,e) \in [X \times E]_0$. Also conversely, $(x,e) \in [X \times E]_0 = X_0 \times E \Rightarrow (x,e) + (-1)(x,e) = (x - x, e - e) = (\theta_X, \theta_E)$.
$\mathbf{A_6}$: For any $(x,e) \in X \times E$, $\exists\ (p_x, e) \in [X \times E]_0$ such that $(p_x, e) \leq (x,e)$.
Thus it follows that $\big(X \times E, +, \cdot, \leq\big)$ is an exponential vector space over the field $\mathbb{K}$. $\square$

If we take E as a Hausdorff topological vector space and X as a topological evs, then we can give the product topology on $X \times E$. We now show that it becomes a topological evs under this topology. In this product topology a net $\{(x_n, e_n)\}_n$ converges to (x,e) iff $x_n \to x$ in X and $e_n \to e$ in E.

Theorem 2.11. (1) *The addition '+': $(X \times E) \times (X \times E) \longrightarrow X \times E$ is continuous.*
(2) *The scalar multiplication '$\cdot$': $\mathbb{K} \times (X \times E) \longrightarrow X \times E$ is continuous.*
(3) *The partial order '$\leq$' is closed.*

Thus $X \times E$ becomes a topological evs. Moreover $X \times E$ is a non-balanced evs for any non-trivial vector space E.

Proof. (1) If we take two nets $\{(x_n, e_n)\}_n$ and $\{(y_n, f_n)\}_n$ such that $(x_n, e_n) \to (x,e)$ and $(y_n, f_n) \to (y, f)$, then $x_n \to x$, $y_n \to y$ in X and $e_n \to e$, $f_n \to f$ in E. As X and E are topological evs and topological vector space, respectively, so $x_n + y_n \to x + y$ in X and $e_n + f_n \to e + f$ in $E \Rightarrow (x_n + y_n, e_n + f_n) \to (x + y, e + f)$ in $X \times E$. This shows that the composition '+' is continuous.

(2) Let us take a net $\{(x_n, e_n)\}_n$ which converges to (x,e) in $X \times E$ and a net $\alpha_n \to \alpha$ in $\mathbb{K}$. Then $x_n \to x$ in X and $e_n \to e$ in E. So $\alpha_n x_n \to \alpha x$ in X and $\alpha_n e_n \to \alpha e$ in E [As E and X are topological vector space and topological evs

respectively] $\Rightarrow (\alpha_n x_n, \alpha_n e_n) \to (\alpha x, \alpha e)$ in $X \times E$. This shows that the composition '$\cdot$' is continuous.

(3) Let us take two nets $\{(x_n, e_n)\}_n$ and $\{(y_n, f_n)\}_n$ such that $(x_n, e_n) \leq (y_n, f_n)$, $\forall\, n$ and $(x_n, e_n) \to (x, e)$, $(y_n, f_n) \to (y, f) \Rightarrow x_n \leq y_n$ and $e_n = f_n$. Taking limit, we have $x \leq y$ and $e = f$ [since the order '$\leq$' in X is closed] $\Rightarrow (x, e) \leq (y, f)$. Thus the partial order is closed and hence the result.

If E is non-trivial, then $X \times E$ must be a *non-balanced topological evs*, even if X is balanced. In fact, for any $(x, e) \in X \times E$ and any $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, we cannot say $\alpha(x, e) = (\alpha x, \alpha e) \leq (x, e)$, even if $\alpha x \leq x$ for any balanced element x in X. $\square$

Example 2.12. [3]Let $\mathcal{X}$ be a vector space over the field $\mathbb{K}$ of real or complex numbers. Let $\mathscr{L}(\mathcal{X})$ be the set of all linear subspaces of $\mathcal{X}$. We now define $+, \cdot, \leq$ on $\mathscr{L}(\mathcal{X})$ as follows: For $\mathcal{X}_1, \mathcal{X}_2 \in \mathscr{L}(\mathcal{X})$ and $\alpha \in \mathbb{K}$, define

(i) $\mathcal{X}_1 + \mathcal{X}_2 := \operatorname{span}(\mathcal{X}_1 \cup \mathcal{X}_2)$,
(ii) $\alpha \cdot \mathcal{X}_1 := \mathcal{X}_1$, if $\alpha \neq 0$ and $\alpha \cdot \mathcal{X}_1 := \{\theta\}$, if $\alpha = 0$ (θ being the additive identity of $\mathcal{X}$),
(iii) $\mathcal{X}_1 \leq \mathcal{X}_2$ iff $\mathcal{X}_1 \subseteq \mathcal{X}_2$.

Then $\big(\mathscr{L}(\mathcal{X}), +, \cdot, \leq\big)$ is a non-topological balanced evs over $\mathbb{K}$.

3. Order-functional on evs

In this section, we discuss about order-functional on an evs. It is an analogue of functional-like structure in the evs-setting. To introduce such a concept we have used the evs $[0, \infty)$ which is the simplest topological evs over the field $\mathbb{K}$ of all real or complex numbers.

Definition 3.1. [3]Let X be an evs over the field $\mathbb{K}$ of real or complex numbers. Any order-morphism $f : X \longrightarrow [0, \infty)$ is said to be an *order-functional* on X.

The following necessary and sufficient condition for a function $f : X \longrightarrow [0, \infty)$ to be an order-functional will be useful in the sequel.

Theorem 3.2. [3] *A function $f : X \longrightarrow [0, \infty)$ [X is an evs over $\mathbb{K}$] is an order-functional on the evs X iff it satisfies the following*

(i) $f(x + y) = f(x) + f(y)$, $\forall x, y \in X$
(ii) $f(\alpha x) = |\alpha| f(x)$, $\forall \alpha \in \mathbb{K}$, $\forall x \in X$
(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$
(iv) $0 < c_1 < c_2 \Rightarrow f^{-1}(c_2) \subseteq\uparrow f^{-1}(c_1)$.

Lemma 3.3. [6] *Let X be a balanced evs. A function $f : X \longrightarrow [0, \infty)$ is an order-functional iff it satisfies the following:*

(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$
(ii) $f(\alpha x) = |\alpha| f(x)$, $\forall\, x \in X$, $\forall\, \alpha \in \mathbb{K}$
(iii) $x \leq y \Rightarrow f(x) \leq f(y)$, $\forall\, x, y \in X$.

Let us now consider the set $X^{\#}$ of all order-functionals on an evs X over $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$). Clearly the zero function 0 given by $0(x) := 0$, $\forall x \in X$ is a member of $X^{\#}$. We define addition, scalar multiplication and partial order on $X^{\#}$ in a very natural way:
for $f, g \in X^{\#}$ and $\alpha \in \mathbb{K}$,

(i) $(f + g)(x) := f(x) + g(x)$, $\forall\, x \in X$
(ii) $(\alpha f)(x) := |\alpha| f(x)$, $\forall\, x \in X$
(iii) $f \leq g \Leftrightarrow f(x) \leq g(x)$, $\forall\, x \in X$.

With respect to this partial order, the zero order-functional '0' is the only minimal element. So the set of all primitive elements is given by $[X^{\#}]_0 = \{0\}$.

The natural question that now arises is: "whether $X^{\#}$ with the aforesaid operations and partial order is an evs or not". The answer to this is not straightforward; in fact, $X^{\#}$ is not always closed under addition. So in general we cannot think about the evs structure of $X^{\#}$. We have found some conditions under which $X^{\#}$ becomes an evs over $\mathbb{K}$. Whenever $X^{\#}$ becomes an evs we call it the *dual* of X. If X is a topological evs and X^* denotes the collection of all continuous order-functionals on X, then we call X^* the *topological dual* of X, whenever X^* becomes an evs.

Result 3.4. *If $\phi : X \longmapsto Y$ (X, Y being two evs) is an order-isomorphism, then every order-functional f_X on X induces an order-functional f_Y on Y and conversely.*

Proof. Let $f_X \in X^{\#}$. Define $f_Y := f_X \circ \phi^{-1}$. Since ϕ is an order-isomorphism, ϕ^{-1} is also so from Y onto X. Thus f_Y is well-defined. We show that $f_Y \in Y^{\#}$. For this let $y_1, y_2 \in Y$ and $\alpha, \beta \in \mathbb{K}$. Then $f_Y(\alpha y_1 + \beta y_2) = (f_X \circ \phi^{-1})(\alpha y_1 + \beta y_2) = f_X\big(\alpha\phi^{-1}(y_1) + \beta\phi^{-1}(y_2)\big) = |\alpha|(f_X \circ \phi^{-1})(y_1) + |\beta|(f_X \circ \phi^{-1})(y_2) = |\alpha| f_Y(y_1) + |\beta| f_Y(y_2)$. Also $y_1 \leq y_2 \Rightarrow \phi^{-1}(y_1) \leq \phi^{-1}(y_2) \Rightarrow f_X(\phi^{-1}(y_1)) \leq f_X(\phi^{-1}(y_2)) \Rightarrow (f_X \circ \phi^{-1})(y_1) \leq (f_X \circ \phi^{-1})(y_2) \Rightarrow f_Y(y_1) \leq f_Y(y_2)$. Let $0 \neq c_1 < c_2$. Then $f_X^{-1}(c_2) \subseteq \uparrow f_X^{-1}(c_1)$ [By Theorem 3.2, since $f_X \in X^{\#}$] $\Rightarrow \phi(f_X^{-1}(c_2)) \subseteq \phi(\uparrow f_X^{-1}(c_1)) = \uparrow \phi(f_X^{-1}(c_1)) \Rightarrow (f_X \circ \phi^{-1})^{-1}(c_2) \subseteq \uparrow (f_X \circ \phi^{-1})^{-1}(c_1) \Rightarrow f_Y^{-1}(c_2) \subseteq \uparrow f_Y^{-1}(c_1)$. Therefore, by Theorem 3.2, $f_Y \in Y^{\#}$.

Conversely, if f_Y is an order-functional on Y, then $f_X := f_Y \circ \phi$ is the required order-functional on X. □

Corollary 3.5. *If ϕ is a topological order-isomorphism, whenever X, Y are topological evs, then f_Y is continuous iff f_X is continuous. Therefore $f_X \in X^* \iff f_Y \in Y^*$.*

Result 3.6. *For any order-morphism $\phi : X \longmapsto Y$ (X, Y being two evs), $\phi^{-1}(Y_0)$ is a subevs of X.*

Proof. Let us take $x_1, x_2 \in \phi^{-1}(Y_0)$. Then $\phi(x_1), \phi(x_2) \in Y_0$. As Y_0 is a vector space so for any $\alpha, \beta \in \mathbb{K}$, $\alpha\phi(x_1)+\beta\phi(x_2) \in Y_0 \Rightarrow \phi(\alpha x_1+\beta x_2) \in Y_0 \Rightarrow \alpha x_1+\beta x_2 \in \phi^{-1}(Y_0)$. We know that $\phi(X_0) \subseteq Y_0 \Rightarrow [\phi^{-1}(Y_0)]_0 = X_0 = \phi^{-1}(Y_0) \cap X_0$. For any $x \in \phi^{-1}(Y_0)$, $\exists\ x_0 \in X_0 = [\phi^{-1}(Y_0)]_0$ such that $x_0 \leq x$. □

Thus for any order-functional f on an evs X, $f^{-1}(0)$ is a subevs of X.

Result 3.7. *If X, Y are two order-isomorphic evs over $\mathbb{K}$ and f_X, f_Y are the order-functionals on X, Y respectively as in Result 3.4, then ${f_X}^{-1}(0)$ and ${f_Y}^{-1}(0)$ are order-isomorphic.*

Proof. By Result 3.6, both ${f_X}^{-1}(0)$ and ${f_Y}^{-1}(0)$ are subevs of X and Y, respectively. To show that ${f_X}^{-1}(0)$ and ${f_Y}^{-1}(0)$ are order-isomorphic let us consider the map $\Phi := \phi\big|_{{f_X}^{-1}(0)}$, where $\phi : X \to Y$ is the order-isomorphism, as in Result 3.4. Our claim is that Φ is an order-isomorphism between ${f_X}^{-1}(0)$ and ${f_Y}^{-1}(0)$. To justify this it is sufficient to prove that $\Phi\big({f_X}^{-1}(0)\big) = {f_Y}^{-1}(0)$. For any element $x \in {f_X}^{-1}(0)$, $f_X(x) = 0 \Rightarrow f_X\big(\phi^{-1}\phi(x)\big) = 0 \Rightarrow f_Y\big(\phi(x)\big) = 0 \Rightarrow \Phi(x) = \phi(x) \in {f_Y}^{-1}(0)$. Reversing the steps we get the required result. □

Let us now introduce the following notion which will be useful to prove that $X^{\#}$ is an evs again for an evs X.

Definition 3.8. An evs X is said to be *functionally additive* if $f + g \in X^{\#}$ for any two $f, g \in X^{\#}$.

Theorem 3.9. *Functionally additivity of an evs is an evs property.*

Proof. Let X be a functionally additive evs and Y be order-isomorphic with X. We show that Y is also functionally additive. For this let $f_Y, g_Y \in Y^{\#}$. Then by Result 3.4, $f_X := f_Y \circ \phi$ and $g_X := g_Y \circ \phi$ are order-functionals on X, where $\phi : X \to Y$ is an order-isomorphism. Since X is a functionally additive evs, we have $f_X + g_X \in X^{\#}$. Then $f_Y + g_Y = f_X \circ \phi^{-1} + g_X \circ \phi^{-1} = \big(f_X + g_X\big) \circ \phi^{-1} \in Y^{\#}$

[by Result 3.4]. This justifies that Y is functionally additive and hence functionally additivity is an evs property. $\square$

Theorem 3.10. *An evs X is functionally additive iff for any two $f, g \in X^{\#}$ and $0 < c_1 < c_2$, we have $(f+g)^{-1}(c_2) \subseteq\uparrow (f+g)^{-1}(c_1)$.*

Proof. For any $x, y \in X$, $(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$. Moreover for any $\alpha \in \mathbb{K}$, $(f+g)(\alpha x) = f(\alpha x) + g(\alpha x) = |\alpha| f(x) + |\alpha| g(x) = |\alpha|(f+g)(x)$. Also for $x \leq y$ $(x, y \in X)$, we have $(f+g)(x) = f(x)+g(x) \leq f(y)+g(y) = (f+g)(y)$. Thus in view of Theorem 3.2, we can say that $f+g$ will be an order-functional on X iff $(f+g)^{-1}(c_2) \subseteq\uparrow (f+g)^{-1}(c_1)$ whenever $0 < c_1 < c_2$. $\square$

Result 3.11. *Every balanced evs is functionally additive.*

Proof. Let X be a balanced evs and $f, g \in X^{\#}$. Let $0 < c_1 < c_2$. Let $x \in (f+g)^{-1}(c_2) \Rightarrow f(x) + g(x) = c_2 \Rightarrow f(\frac{c_1}{c_2}x) + g(\frac{c_1}{c_2}x) = c_1 \Rightarrow \frac{c_1}{c_2}x \in (f+g)^{-1}(c_1)$. Now $0 < \frac{c_1}{c_2} < 1 \Rightarrow \frac{c_1}{c_2}x \leq x$ [$\because X$ is balanced] $\Rightarrow x \in\uparrow (f+g)^{-1}(c_1) \Rightarrow (f+g)^{-1}(c_2) \subseteq\uparrow (f+g)^{-1}(c_1)$. Then by Theorem 3.10, X is functionally additive. $\square$

Result 3.12. *Let X be an evs and $f \in X^{\#}$. Then for any $\alpha \in \mathbb{K}$, we have $\alpha f \in X^{\#}$.*

Proof. If $\alpha = 0$, there is nothing to prove. So let $\alpha \neq 0$. For any $x, y \in X$, we have $(\alpha f)(x+y) = |\alpha| f(x+y) = |\alpha| f(x) + |\alpha| f(y) = (\alpha f)(x) + (\alpha f)(y)$. Also for any $\beta \in \mathbb{K}$, we have $(\alpha f)(\beta x) = |\alpha| f(\beta x) = |\alpha||\beta| f(x) = |\beta|(\alpha f)(x)$. Now for $x \leq y$ $(x, y \in X)$, we have $(\alpha f)(x) = |\alpha| f(x) \leq |\alpha| f(y) = (\alpha f)(y)$. Now let $0 < c_1 < c_2$ and $x \in (\alpha f)^{-1}(c_2)$. Then $|\alpha| f(x) = c_2$ which implies $x \in f^{-1}\big(\frac{c_2}{|\alpha|}\big) \subseteq\uparrow f^{-1}\big(\frac{c_1}{|\alpha|}\big)$ [by Theorem 3.2, since $f \in X^{\#}$]. So there is $y \in f^{-1}\big(\frac{c_1}{|\alpha|}\big)$ such that $y \leq x$. Now $(\alpha f)(y) = |\alpha| f(y) = c_1$. So $y \in (\alpha f)^{-1}(c_1)$. Consequently, $x \in\uparrow (\alpha f)^{-1}(c_1)$. Thus $(\alpha f)^{-1}(c_2) \subseteq\uparrow (\alpha f)^{-1}(c_1)$. Then by Theorem 3.2, $\alpha f \in X^{\#}$. $\square$

Theorem 3.13. *For an evs X over $\mathbb{K}$, $X^{\#}$ becomes an evs over $\mathbb{K}$ iff X is functionally additive. Moreover $X^{\#}$ is balanced.*

Proof. If $X^{\#}$ is an evs, then clearly X is functionally additive.

Conversely, let X be a functionally additive evs. Then for any $f, g \in X^{\#}$, their sum $f + g \in X^{\#}$. Also by Result 3.12, for any $\alpha \in \mathbb{K}$ and $f \in X^{\#}$, $\alpha f \in X^{\#}$.
$\mathbf{A_1}$: Since the zero function is an order-functional, it follows that $(X^{\#}, +)$ is a commutative semigroup with identity 0 (the zero order-functional).

A₂ : $f_1 \leq f_2 \Rightarrow f_1(x) \leq f_2(x), \forall x \in X$. So $f_1(x) + f_3(x) \leq f_2(x) + f_3(x)$, $\forall x \in X$ and for any $f_3 \in X^{\#} \Rightarrow f_1 + f_3 \leq f_2 + f_3$, $\forall f_1, f_2, f_3 \in X^{\#}$ with $f_1 \leq f_2$.

Again $(\alpha f_1)(x) = |\alpha| f_1(x) \leq |\alpha| f_2(x) = (\alpha f_2)(x)$, $\forall x \in X \Rightarrow \alpha f_1 \leq \alpha f_2$, $\forall \alpha \in \mathbb{K}$ and $\forall f_1, f_2 \in X^{\#}$ with $f_1 \leq f_2$.

$\mathbf{A_3}$ (i): $[\alpha(f_1 + f_2)](x) = |\alpha|(f_1(x) + f_2(x)) = |\alpha| f_1(x) + |\alpha| f_2(x) = (\alpha f_1 + \alpha f_2)(x)$, $\forall x \in X \Rightarrow \alpha(f_1 + f_2) = \alpha f_1 + \alpha f_2$, $\forall \alpha \in \mathbb{K}$ and $\forall f_1, f_2 \in X^{\#}$.

(ii) $[\alpha(\beta f)](x) = |\alpha|(\beta f)(x) = |\alpha||\beta| f(x) = [(\alpha\beta) f](x)$, $\forall x \in X \Rightarrow \alpha(\beta f) = (\alpha\beta) f$, $\forall \alpha, \beta \in \mathbb{K}$ and $\forall f \in X^{\#}$.

(iii) $[(\alpha + \beta) f](x) = |\alpha + \beta| f(x) \leq |\alpha| f(x) + |\beta| f(x) = (\alpha f + \beta f)(x)$, $\forall x \in X$ $\Rightarrow (\alpha + \beta) f \leq \alpha f + \beta f$, $\forall \alpha, \beta \in \mathbb{K}$ and $\forall f \in X^{\#}$.

(iv) Obviously $1 \cdot f = f$, $\forall f \in X^{\#}$, where 1 is the multiplicative identity in $\mathbb{K}$.

$\mathbf{A_4}$: $\alpha f = 0 \Rightarrow |\alpha| f(x) = 0$, $\forall x \in X$. If $\alpha \neq 0$, then $f(x) = 0$, $\forall x \in X \Rightarrow f \equiv 0$. So $\alpha f = 0 \Rightarrow$ either $\alpha = 0$ or $f = 0$. The converse is obvious.

$\mathbf{A_5}$: $[f + (-1) f] = 0 \Leftrightarrow [f + (-1) f](x) = 0$, $\forall x \in X \Leftrightarrow 2f(x) = 0, \forall x \in X \Leftrightarrow f \equiv 0$. Since $f \geq 0$, $\forall f \in X^{\#}$, it follows that $[X^{\#}]_0 = \{0\}$. Thus $f + (-1) f = 0 \Leftrightarrow f \in [X^{\#}]_0$.

$\mathbf{A_6}$: For each $f \in X^{\#}$, we have $0 \leq f$ where $0 \in [X^{\#}]_0$.

Thus $(X^{\#}, +, \cdot, \leq)$ forms an evs over $\mathbb{K}$.

Moreover for any $f \in X^{\#}$ and $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$, we have $(\alpha f)(x) = |\alpha| f(x) \leq f(x)$ for all $x \in X$. Thus $X^{\#}$ is always balanced, whenever it is an evs. □

Corollary 3.14. *For any balanced evs X, $X^{\#}$ is an evs.*

Proof. Since every balanced evs is functionally additive [by Result 3.11], the Corollary follows from Theorem 3.13. □

For a functionally additive evs X, we now introduce a topology on $X^{\#}$ which will make it a topological evs. We give the *topology of point-wise convergence* on $X^{\#}$. In this topology, a net $(f_\lambda)_{\lambda \in D}$ in $X^{\#}$ (D being a directed set) converges to $f \in X^{\#}$ iff $f_\lambda(x) \to f(x)$ for all $x \in X$. Then the following Theorem is evident.

Theorem 3.15. *For a functionally additive evs X, $X^{\#}$ is a topological evs with respect to the topology of point-wise convergence.*

Corollary 3.16. *For a functionally additive topological evs X, X^* is a topological evs with respect to the topology of point-wise convergence.*

Proof. This corollary follows from the fact that for any two $f, g \in X^*$, $f + g$ is a continuous order-functional on X, since X is functionally additive. □

Theorem 3.17. *If X and Y are two order-isomorphic evs and X is functionally additive, then $X^{\#}$ and $Y^{\#}$ are (topologically) order-isomorphic, the topology being the topology of point-wise convergence.*

Proof. Since X is functionally additive and Y is order-isomorphic to X, it follows that Y is also functionally additive [by Theorem 3.9]. So by Theorem 3.13, both $X^{\#}$ and $Y^{\#}$ are evs.

Let $\phi : X \to Y$ be an order-isomorphism. We now define a map $\Phi : X^{\#} \longrightarrow Y^{\#}$ by $\Phi(f) := f \circ \phi^{-1}$, $\forall f \in X^{\#}$. By Result 3.4, Φ is well-defined. As ϕ is bijective, Φ is injective. Again for $g \in Y^{\#}$, $g \circ \phi = g \circ (\phi^{-1})^{-1} \in X^{\#}$ [as $\phi^{-1} : Y \longmapsto X$ is an order-isomorphism] and $\Phi(g \circ \phi) = g$. So Φ is bijective.

Now for any $f_1, f_2 \in X^{\#}$ and any $\alpha, \beta \in \mathbb{K}$, we have $\Phi(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2) \circ \phi^{-1} = \alpha(f_1 \circ \phi^{-1}) + \beta(f_2 \circ \phi^{-1}) = \alpha\Phi(f_1) + \beta\Phi(f_2)$. Also $f_1 \leq f_2 \Leftrightarrow f_1 \circ \phi^{-1} \leq f_2 \circ \phi^{-1} \Leftrightarrow \Phi(f_1) \leq \Phi(f_2)$. So Φ is an order-isomorphism and hence $X^{\#}$ and $Y^{\#}$ are order-isomorphic.

Now if a net $f_\lambda \to f$ in $X^{\#}$ with respect to the topology of point-wise convergence, then $f_\lambda \circ \phi^{-1} \to f \circ \phi^{-1}$ in $Y^{\#}$ with the same topology and hence Φ is continuous. Similarly, $\Phi^{-1}(g) = g \circ \phi$ implies that Φ^{-1} is continuous. Thus Φ is a topological order-isomorphism. $\square$

In the above Theorem if X, Y are two topological evs and they are topologically order-isomorphic, ϕ is the topological order-isomorphism, then $f \circ \phi^{-1} \in Y^*$ for any $f \in X^*$. So we can easily have the following Corollary.

Corollary 3.18. *If X, Y are two topologically order-isomorphic evs, then X^* and Y^* are topologically order-isomorphic.*

Theorem 3.19. *Every order-functional on a topological evs is an open map.*

Proof. We can write any open set V of a topological evs X as $V = \bigcup_{y \in V} S(1, r_y)y$, where $S(1,t) := \{\alpha \in \mathbb{K} : |\alpha - 1| < t\}$, for any real $t > 0$. For any order-functional f on X, $f(V) = f\left(\bigcup_{y \in V} S(1, r_y)y\right) = \bigcup_{y \in V} f\big(S(1, r_y)y\big) = \bigcup_{y \in V} \big[(1 - r_y, 1 + r_y) \cap [0, \infty)\big] f(y)$. As each $\big[(1 - r_y, 1 + r_y) \cap [0, \infty)\big] f(y)$ is an open set in $[0, \infty)$, it follows that $f(V)$ is open. $\square$

4. Computation of dual of some evs

In this section, we shall compute the dual of some concrete evs, both topological and non-topological. We start with the computation of the non-topological evs $\mathscr{D}^n([0,\infty))$ explained in Example 2.9

Theorem 4.1. *Every order-functional on* $\mathscr{D}^n([0,\infty))$, $n-1 \in \mathbb{N}$ *is of the form*
$$(x_1, x_2, \ldots, x_n) \longmapsto \lambda x_1, \text{ for some } \lambda \in [0,\infty).$$

Proof. We first prove that for any $\lambda \in [0,\infty)$, $f_\lambda(x) := \lambda x_1, \forall x = (x_1, x_2, \ldots, x_n) \in \mathscr{D}^n([0,\infty))$ is an order-functional on $\mathscr{D}^n([0,\infty))$. As $\mathscr{D}^n([0,\infty))$ is a balanced evs, it is sufficient to prove that f_λ preserves addition, scalar multiplication and partial order of $\mathscr{D}^n([0,\infty))$, in view of Lemma 3.3. For two points $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$ and scalars α, β we have, $f_\lambda(\alpha x + \beta y) = \lambda(|\alpha|x_1 + |\beta|y_1) = |\alpha|\lambda x_1 + |\beta|\lambda y_1 = |\alpha| f_\lambda(x) + |\beta| f_\lambda(y)$. If $x \le y$, then either $x_1 < y_1$ or $x_1 = y_1$. In any case, $\lambda x_1 \le \lambda y_1 \Rightarrow f_\lambda(x) \le f_\lambda(y)$. Therefore f_λ becomes an order-functional on $\mathscr{D}^n([0,\infty))$.

We now show that any order-functional on $\mathscr{D}^n([0,\infty))$ is of the above form for some $\lambda \in [0,\infty)$. Let f be an order-functional. Then for any $x = (x_1, x_2, \ldots, x_n) \in \mathscr{D}^n([0,\infty))$, we can write $f(x) = x_1 f\big((1,0,\ldots,0)\big) + x_2 f\big((0,1,\ldots,0)\big) + \cdots + x_n f\big((0,0,\ldots,1)\big) = \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n$, where $\lambda_i = f\big((0,0,\ldots,1,\ldots,0)\big)$, 1 is in the i-th place. So $\lambda_i \ge 0$, $\forall\, i = 1,2,\ldots,n$. Now for any $m \in \mathbb{N}$, $(0,m,0,\ldots,0) \le (1,0,0,\ldots,0) \Rightarrow f\big((0,m,0,\ldots,0)\big) \le f\big((1,0,0,\ldots,0)\big) \Rightarrow m\lambda_2 \le \lambda_1$, for any $m \in \mathbb{N}$. Similarly, we can say that $m\lambda_i \le \lambda_1$, $\forall\, i = 3,4,\ldots,n$ and for any $m \in \mathbb{N}$. If $\lambda_1 = 0$, then all $\lambda_i = 0$ and in this case f becomes the zero order-functional. If $\lambda_1 > 0$, then also $\lambda_i = 0$, $\forall\, i = 2,3,\ldots,n$. In fact, for any $i \in \{2,3,\ldots,n\}$, $\lambda_i \le \frac{1}{m}\lambda_1$, $\forall\, m \in \mathbb{N}$. Taking limit $m \to \infty$, we get $\lambda_i = 0$ for $i = 2,3,\ldots,n$. So we can say that $f(x) = \lambda_1 x_1, \forall\, x = (x_1,\ldots,x_n) \in \mathscr{D}^n([0,\infty))$ where $\lambda_1 \ge 0$. □

From the above theorem, we can say that $\big[\mathscr{D}^n([0,\infty))\big]^{\#} = \big\{f_\lambda : \lambda \ge 0\big\}$. It becomes an evs over $\mathbb{K}$, by Corollary 3.14, since $\mathscr{D}^n([0,\infty))$ is balanced. Now if we consider the map $f_\lambda \longmapsto \lambda$, then the map becomes an order-isomorphism between $\big[\mathscr{D}^n([0,\infty))\big]^{\#}$ and $[0,\infty)$. Therefore we have the following Theorem. Since $\mathscr{D}^n([0,\infty))$ is a non-topological evs, $\big[\mathscr{D}^n([0,\infty))\big]^{*}$ does not exist.

Theorem 4.2. $\big[\mathscr{D}^n([0,\infty))\big]^{\#}$ *is order-isomorphic with* $[0,\infty)$ *for any* n *with* $n-1 \in \mathbb{N}$.

We will now compute the dual of the topological evs $[0,\infty)^n$, for any $n \in \mathbb{N}$, explained in Example 2.7.

Theorem 4.3. $\big[[0,\infty)^n\big]^{\#}$ *is order-isomorphic to* $[0,\infty)^n$*, for any* $n \in \mathbb{N}$.

Proof. As $[0,\infty)^n$ is balanced, $\big[[0,\infty)^n\big]^{\#}$ is a balanced evs, in view of Corollary 3.14 and Theorem 3.13. We first show that for any $\lambda = (\lambda_i) \in [0,\infty)^n$, $\exists$ an order-functional f_λ on $[0,\infty)^n$ defined by $f_\lambda(x_1, x_2, \ldots, x_n) := \sum_{i=1}^{n} \lambda_i x_i$. In fact, for any $x = (x_i), y = (y_i) \in [0,\infty)^n$ and any scalars $\alpha, \beta \in \mathbb{K}$, $f_\lambda(\alpha x + \beta y) = f_\lambda\big((|\alpha| x_i + |\beta| y_i)\big) = \sum_{i=1}^{n} \lambda_i(|\alpha| x_i + |\beta| y_i) = |\alpha| \sum_{i=1}^{n} \lambda_i x_i + |\beta| \sum_{i=1}^{n} \lambda_i y_i = |\alpha| f_\lambda(x) + |\beta| f_\lambda(y)$. Also $x \le y \Rightarrow x_i \le y_i, \forall\, i \Rightarrow \lambda_i x_i \le \lambda_i y_i, \forall\, i \Rightarrow \sum_{i=1}^{n} \lambda_i x_i \le \sum_{i=1}^{n} \lambda_i y_i \Rightarrow f_\lambda(x) \le f_\lambda(y)$. Now $[0,\infty)^n$ being balanced, f_λ is an order-functional on it, by Lemma 3.3.

For any order-functional f on $[0,\infty)^n$, for any $x = (x_i) \in [0,\infty)^n$, we can write $f(x) = \sum_{i=1}^{n} \lambda_i^f x_i$, where $\lambda_i^f = f(e_i)$ and $e_i = (\delta_j^i)_{j \in \{1,2,\ldots,n\}}$, where

$$\delta_j^i = \begin{cases} 1, \text{when } i = j, \\ 0, \text{when } i \neq j. \end{cases}$$

Thus if we choose $\lambda^f = (\lambda_i^f)_i \in [0,\infty)^n$, then $f = f_{\lambda^f}$. Also for distinct order-functionals f, g on $[0,\infty)^n$, $\exists\, j \in \{1, \ldots, n\}$ such that $f(e_j) \neq g(e_j) \Rightarrow \lambda_j^f \neq \lambda_j^g \Rightarrow \lambda^f \neq \lambda^g$. Thus we can say that the map $f \longmapsto \lambda^f$ where $\lambda^f := (\lambda_i^f)_i$, is a bijective map from $\big[[0,\infty)^n\big]^{\#}$ onto $[0,\infty)^n$.

We know that addition of two order-functionals is defined as $(f+g)(x) = f(x) + g(x), \forall\, x$. So $\sum_{i=1}^{n} \lambda_i^{f+g} x_i = \sum_{i=1}^{n} \lambda_i^f x_i + \sum_{i=1}^{n} \lambda_i^g x_i$, where $x = (x_i) \in [0,\infty)^n$. Taking $x = e_i$, we get $\lambda_i^{f+g} = \lambda_i^f + \lambda_i^g$. Varying i we get $\lambda^{f+g} = \lambda^f + \lambda^g$. Similarly, we can show that $\lambda_i^{\alpha f} = |\alpha| \lambda_i^f$ and hence $\lambda^{\alpha f} = |\alpha| \lambda^f$. If $f \le g$, then $f(e_i) \le g(e_i), \forall\, i$ $\Rightarrow \lambda_i^f \le \lambda_i^g, \forall\, i \Rightarrow \lambda^f \le \lambda^g$. Reversing the steps, we get $f \le g$ whenever $\lambda^f \le \lambda^g$, since each $x = (x_i) \in [0,\infty)^n$ can be written as $x = \sum_{i=1}^{n} x_i e_i$. All these show that the map $f \longmapsto \lambda^f$ is an order-isomorphism between $\big[[0,\infty)^n\big]^{\#}$ and $[0,\infty)^n$. $\square$

From the proof of the above theorem, it also follows that every order-functional on $[0,\infty)^n$ is continuous. Moreover, the map $f \mapsto \lambda^f$ is a homeomorphism. Thus we can easily have the following Corollary.

Corollary 4.4. $\big[[0,\infty)^n\big]^{*} = \big[[0,\infty)^n\big]^{\#}$ *which is topologically order-isomorphic with* $[0,\infty)^n$.

We now compute the dual of the non-balanced topological evs $X \times E$, explained in Example 2.10 and Theorem 2.11.

Theorem 4.5. *If X is a functionally additive evs, then $(X \times E)^{\#}$ is order-isomorphic with $X^{\#}$ for any vector space E.*

Proof. We first establish a one-to-one correspondence between $X^{\#}$ and $(X \times E)^{\#}$. For this let $f \in X^{\#}$ and define a function

$$\left.\begin{array}{rl} F_f : & X \times E \longrightarrow [0, \infty) \\ & (x, e) \longmapsto f(x) \end{array}\right\}.$$

We first show that $F_f \in (X \times E)^{\#}$. For any $(x, e), (y, d) \in X \times E$ and any two scalars α, β, we have $F_f\big(\alpha(x, e) + \beta(y, d)\big) = F_f\big((\alpha x + \beta y, \alpha e + \beta d)\big) = f(\alpha x + \beta y) = |\alpha| f(x) + |\beta| f(y) = |\alpha| F_f(x, e) + |\beta| F_f(y, d)$. If $(x, e) \leq (y, d)$, then $x \leq y$ and $e = d$ $\Rightarrow f(x) \leq f(y) \Rightarrow F_f(x, e) \leq F_f(y, d)$. Let $p \leq q$ in $[0, \infty)$ and $(x, e) \in F_f^{-1}(p)$ $\Rightarrow f(x) = p$. Now f is an order-functional on $X \Rightarrow x \in f^{-1}(p) \subseteq\downarrow f^{-1}(q) \Rightarrow (x, e) \in\downarrow f^{-1}(q) \times \{e\} \subseteq\downarrow F_f^{-1}(q) \Rightarrow F_f^{-1}(p) \subseteq\downarrow F_f^{-1}(q)$. Similarly, $F_f^{-1}(q) \subseteq\uparrow F_f^{-1}(p)$. Therefore $F_f \in (X \times E)^{\#}$.

Now let $f, f' \in X^{\#}$ such that $f \neq f'$. Then there is an $x \in X$ such that $f(x) \neq f'(x)$. So $F_f(x, e) \neq F_{f'}(x, e)$ for any $e \in E$. Thus $F_f \neq F_{f'}$ and hence the map $f \mapsto F_f$ from $X^{\#}$ to $(X \times E)^{\#}$ is injective.

Again for $F \in (X \times E)^{\#}$, define

$$\left.\begin{array}{rl} f_{F} : & X \longrightarrow [0, \infty) \\ & x \longmapsto F(x, \theta_E) \end{array}\right\}.$$

Then it becomes an order-functional on X. Also for any $(x, e) \in X \times E$, $F_{f_F}(x, e) = f_F(x) = F(x, \theta_E) = F(x, \theta_E) + F(\theta_X, e) = F(x, e)$ $\big[$since $(\theta_X, e) \in X_0 \times E$ and $F(X_0 \times E) = \{0\}\big] \Rightarrow F_{f_F} = F$. Thus the map $f \mapsto F_f$ from $X^{\#}$ to $(X \times E)^{\#}$ is surjective. This establishes a one-to-one correspondence between $X^{\#}$ and $(X \times E)^{\#}$.

We now claim that $X \times E$ is functionally additive iff X is functionally additive. In fact, if X is functionally additive, then for any $f, f' \in X^{\#}$, we have $f + f' \in X^{\#}$ and hence $F_{f+f'} \in (X \times E)^{\#}$. Now for any $(x, e) \in X \times E$, $F_{f+f'}(x, e) = (f + f')(x) = f(x) + f'(x) = F_f(x, e) + F_{f'}(x, e) = (F_f + F_{f'})(x, e)$. So $F_{f+f'} = F_f + F_{f'}$. Since the map $f \mapsto F_f$ is a bijection between $X^{\#}$ and $(X \times E)^{\#}$, our claim is justified.

Now in view of Theorem 3.13 and above discussion, $X^{\#}$ and hence $(X \times E)^{\#}$ both are evs over $\mathbb{K}$, if X is functionally additive.

We finally conclude that the map $f \mapsto F_f$ between $X^{\#}$ and $(X \times E)^{\#}$ is an order-isomorphism. To justify this conclusion, we only need to note the following: for any $\alpha \in \mathbb{K}$ and $f \in X^{\#}$, we have $F_{\alpha f}(x, e) = (\alpha f)(x) = |\alpha| f(x) = |\alpha| F_f(x, e)$

for all $(x, e) \in X \times E$ which implies $F_{\alpha f} = |\alpha| F_f$. Also for $f, g \in X^{\#}$ with $f \le g$, we have $f(x) \le g(x)$ for any $x \in X$ which implies $F_f(x, e) \le F_g(x, e)$ for any $e \in E$. Thus $F_f \le F_g$. Reversing the steps, we get $f \le g$ whenever $F_f \le F_g$. $\square$

There is a topological version of the above theorem as well.

Theorem 4.6. *If X is a functionally additive topological evs and E is a Hausdorff topological vector space, then $(X \times E)^*$ is topologically order-isomorphic with X^*.*

Proof. In the proof of the above Theorem 4.5, whenever $f \in X^*$, the order-functional $F_f : (x, e) \longmapsto f(x)$ on $X \times E$ also becomes continuous, since any net $\{(x_n, e_n)\}_n$ converges to (x, e) in $X \times E$ implies $f(x_n) \to f(x)$ [by continuity of f on X] and hence $F_f(x_n, e_n) = f(x_n) \to f(x) = F_f(x, e)$. Also if $F \in (X \times E)^*$, then the order-functional $f_{_F}$, defined in the proof of the above Theorem 4.5, is continuous on X.

In view of Theorem 4.5 it only remains to prove that $f \mapsto F_f$ is a homeomorphism. This is true since, a net $f_n \to f$ in X^* iff $f_n(x) \to f(x)$ for all $x \in X$ iff $F_{f_n}(x, e) \to F_f(x, e)$ for all $(x, e) \in X \times E$ iff $F_{f_n} \to F_f$ in $(X \times E)^*$. $\square$

Theorem 4.7. *For a Hausdorff topological vector space E, $\big[[0, \infty) \times E\big]^{\#} = \big[[0, \infty) \times E\big]^*$ which is topologically order-isomorphic with $[0, \infty)$.*

Proof. We first show that every order-functional on $[0, \infty) \times E$ is of the form

$$f_a : (\lambda, e) \longmapsto a\lambda, \text{ for a unique } a \ge 0.$$

In fact, if f is an order-functional on $[0, \infty) \times E$, then $f\big((\lambda, e)\big) = f\big(\lambda(1, 0) + (0, e)\big)$ $= \lambda f\big((1, 0)\big) = a\lambda$, $\forall\, (\lambda, e) \in [0, \infty) \times E$, where $a := f\big((1, 0)\big) \in [0, \infty)$ $\big[$since $(0, e)$ being a primitive element, $f\big((0, e)\big) = 0\big]$.

It can be easily shown that for any two order-functionals f, g on $[0, \infty) \times E$, $f\big((1, 0)\big) = g\big((1, 0)\big) \Rightarrow f = g$. So for any order-functional f on $[0, \infty) \times E$, there exists a unique $a := f\big((1, 0)\big)$ such that $f = f_a$. Conversely, for any $a \ge 0$, the map $f_a : (\lambda, e) \longmapsto a\lambda$ is clearly an order-functional on $[0, \infty) \times E$. Thus we can write $\big[[0, \infty) \times E\big]^{\#} = \big\{f_a : a \ge 0\big\}$. Also every order-functional on $[0, \infty) \times E$ is continuous and hence $\big[[0, \infty) \times E\big]^{\#} = \big[[0, \infty) \times E\big]^*$.

We now claim that $\big[[0, \infty) \times E\big]^{\#}$ is an evs. In fact, $f_a + f_b = f_{a+b}$ for any $a, b \ge 0$. This implies that $[0, \infty) \times E$ is functionally additive and hence by Theorem 3.13, $\big[[0, \infty) \times E\big]^{\#}$ is an evs.

To show that $\big[[0, \infty) \times E\big]^{\#}$ is topologically order-isomorphic with $[0, \infty)$, let us consider the map $\Phi : a \longmapsto f_a$, $a \ge 0$. Clearly, Φ is a bijection between $[0, \infty)$ and $\big[[0, \infty) \times E\big]^{\#}$ and preserves addition and scalar multiplication. Also $a \le b \Leftrightarrow$

$a\lambda \leq b\lambda$, $\forall\, \lambda \geq 0 \Leftrightarrow f_a \leq f_b$. This shows that Φ is an order-isomorphism between $[0, \infty)$ and $\big[[0, \infty) \times E\big]^{\#}$. Also Φ is a homeomorphism. $\square$

Remark 4.8. From the proof of the above Theorem 4.7 and Theorem 2.11, we can say that $[0, \infty) \times E$ is a non-balanced functionally additive evs for any non-trivial vector space E.

In view of Example 2.9, Theorem 2.11, Theorem 4.5 and Theorem 4.2 we can have the following Corollary.

Corollary 4.9. *For any non-trivial vector space E, $\mathscr{D}^n([0, \infty)) \times E$, where $n-1 \in \mathbb{N}$ is a non-balanced non-topological evs. Moreover, $\big[\mathscr{D}^n([0, \infty)) \times E\big]^{\#}$ is order-isomorphic with $[0, \infty)$ for any n with $n - 1 \in \mathbb{N}$.*

We now demonstrate the dual of the non-topological balanced evs $\mathscr{L}(\mathcal{X})$, explained in Example 2.12.

Theorem 4.10. *There does not exist any non-zero order-functional on $\mathscr{L}(\mathcal{X})$ i.e., $\big[\mathscr{L}(\mathcal{X})\big]^{\#} = \{0\}$.*

Proof. Let $f \in [\mathscr{L}(\mathcal{X})]^{\#}$. We know that for any $\mathcal{Y} \in \mathscr{L}(\mathcal{X})$ and any $\alpha \in \mathbb{K}$, $f(\alpha\mathcal{Y}) = |\alpha| f(\mathcal{Y})$. But $\alpha\mathcal{Y} = \mathcal{Y}$, for all non-zero α. So $f(\mathcal{Y}) = |\alpha| f(\mathcal{Y})$, for any non-zero scalar α. Therefore $f(\mathcal{Y}) = 0$, $\forall\, \mathcal{Y} \in \mathscr{L}(\mathcal{X}) \Rightarrow f \equiv 0$. $\square$

Since $\mathscr{L}(\mathcal{X})$ cannot be topological, its topological dual $\big[\mathscr{L}(\mathcal{X})\big]^*$ does not exist.

We now discuss the dual of the exponential vector spaces in hyperspace category, explained in Example 2.8.

Theorem 4.11. *For every order-functional f on $\mathscr{P}(\mathcal{X})$, $f \equiv 0$ on $\mathscr{C}_F(\mathcal{X})$.*

Proof. We shall prove this by induction on the number of points in each finite subset of $\mathcal{X}$ constituiting $\mathscr{C}_F(\mathcal{X})$. f being an order-functional, for each primitive element $\{x\}$ of $\mathscr{P}(\mathcal{X})$, $f\big(\{x\}\big) = 0$. Let us assume that $f(B) = 0$, $\forall B \in \mathscr{C}_F(\mathcal{X})$ with $1 \leq \text{card}(B) < n$, where $n - 1 \in \mathbb{N}$. Let $A \in \mathscr{C}_F(\mathcal{X})$ such that $\text{card}(A) = n$. We claim that $f(A) = 0$. If not, then $f(A) = c_1$ with $c_1 \neq 0$. Let $0 < c_2 < c_1$. Then by property of order-functional, there exists $B \in \mathscr{P}(\mathcal{X})$ with $B \subsetneqq A$ such that $f(B) = c_2$. Again $\text{card}(B) < \text{card}(A) = n$ and hence by hypothesis $f(B) = 0$, a contradiction. Therefore $f(A) = 0$. Then by the second principle of mathematical induction, we can say that $f(A) = 0$, $\forall\, A \in \mathscr{C}_F(\mathcal{X})$. $\square$

From Theorem 4.11, we can easily have the following Corollary.

Corollary 4.12. *For a Hausdorff topological vector space $\mathcal{X}$,*

$$\big[\mathscr{C}_F(\mathcal{X})\big]^{\#} = \big[\mathscr{C}_F(\mathcal{X})\big]^{*} = \{0\}.$$

Theorem 4.13. *There does not exist any continuous order-functional other than zero on the topological evs $\mathscr{C}(\mathcal{X})$, i.e., $\big[\mathscr{C}(\mathcal{X})\big]^{*} = \{0\}$.*

Proof. Let f be a continuous order-functional on $\mathscr{C}(\mathcal{X})$. For any $A \in \mathscr{C}(\mathcal{X})$, there exists a net $\{A_n\}_{n\in D}$ [D is a directed set] in $\mathscr{C}_F(\mathcal{X})$ which converges to A [since $\mathscr{C}_F(\mathcal{X})$ is dense in $\mathscr{C}(\mathcal{X})$]. So f being continuous, we have $f(A_n) \to f(A)$. But $f(A_n) = 0$, $\forall n \in D$, by above Theorem 4.11. Therefore $f(A) = 0$. So f is the zero order-functional on $\mathscr{C}(\mathcal{X})$. □

Remark 4.14. We know that for a finite dimensional Banach space $\mathcal{X}$, $\mathscr{B}_0(\mathcal{X})$ is order-isomorphic with $[0,\infty)$. Therefore, by Theorem 3.17, the dual of $\mathscr{B}_0(\mathcal{X})$ is topologically order-isomorphic with $[0,\infty)$, since $[0,\infty)^{\#}$ is topologically order-isomorphic with $[0,\infty)$, by Corollary 4.4. But there does not exist any non-zero continuous order-functional on $\mathscr{C}(\mathcal{X})$, by Theorem 4.13. Therefore in conclusion, we can say that extension of non-zero continuous order-functional from a subevs to a larger evs is not possible in general.

5. Application of order-functional

In this section, we shall discuss some application of order-functional on an evs.

Result 5.1. (i) *For any non-zero order-functional f on $\mathscr{D}^2([0,\infty))$, $f^{-1}(0)$ is order-isomorphic with $[0,\infty)$.*

(ii) *For any non-zero order-functional f on $\mathscr{D}^n([0,\infty))$ $(n-2 \in \mathbb{N})$, $f^{-1}(0)$ is order-isomorphic with $\mathscr{D}^{n-1}([0,\infty))$.*

Proof. (i) By Theorem 4.1, any non-zero order-functional f on $\mathscr{D}^2([0,\infty))$ is given by $f(x,y) = \lambda x$, $\forall\,(x,y) \in \mathscr{D}^2([0,\infty))$, for some $\lambda > 0$. So $f^{-1}(0) = \{0\} \times [0,\infty)$ which can be identified with the evs $[0,\infty) = \mathscr{D}^1([0,\infty))$ through the order-isomorphism $(0,y) \longmapsto y$.

(ii) By similar argument as above, here $f^{-1}(0) = \big\{(x_1,\dots,x_n) \in \mathscr{D}^n([0,\infty)) : x_1 = 0\big\}$ which is order-isomorphic with $\mathscr{D}^{n-1}([0,\infty))$, $(0,x_2,\dots,x_n) \longmapsto (x_2,\dots,x_n)$ being the order-isomorphism. □

Theorem 5.2. *$\mathscr{D}^n([0,\infty))$ is not order-isomorphic with $\mathscr{D}^m([0,\infty))$, where $1 \le m < n$.*

Proof. We prove this theorem by induction on m and n. Let us first note that $\mathscr{D}^n([0,\infty))$ is not order-isomorphic with $\mathscr{D}^1([0,\infty)) \equiv [0,\infty)$, for any $n > 1$. It is because, $[0,\infty)$ is a topological evs, whereas $\mathscr{D}^n([0,\infty))$ cannot be made topological for any $n>1$. Let us now assume that $\mathscr{D}^n([0,\infty))$ is not order-isomorphic with $\mathscr{D}^m([0,\infty))$, for $1\le m<n$. Now if $\mathscr{D}^{n+1}([0,\infty))$ is order-isomorphic with $\mathscr{D}^m([0,\infty))$, for some m satisfying $2\le m<n+1$ and $\phi:\mathscr{D}^{n+1}([0,\infty))\longrightarrow\mathscr{D}^m([0,\infty))$ is an order-isomorphism, then for any non-zero order-functional f on $\mathscr{D}^{n+1}([0,\infty))$ and for the induced order-functional $g:=f\circ\phi^{-1}$ on $\mathscr{D}^m([0,\infty))$ we have $f^{-1}(0)$ is order-isomorphic with $g^{-1}(0)$ [by Result 3.7]. Again by Result 5.1, $f^{-1}(0)$ is order-isomorphic with $\mathscr{D}^n([0,\infty))$ and $g^{-1}(0)$ is order-isomorphic with $\mathscr{D}^{m-1}([0,\infty))$. Therefore $\mathscr{D}^n([0,\infty))$ and $\mathscr{D}^{m-1}([0,\infty))$ are order-isomorphic, where $1\le m-1<n$ —— which contradicts our assumption. Thus it follows that $\mathscr{D}^{n+1}([0,\infty))$ cannot be order-isomorphic with $\mathscr{D}^m([0,\infty))$, for any m satisfying $2\le m<n+1$. So by the 2nd principle of induction, we can say that $\mathscr{D}^n([0,\infty))$ is not order-isomorphic with $\mathscr{D}^m([0,\infty))$, for any m,n with $1\le m<n$. $\square$

From above the theorem, we can say that $\mathscr{D}^m([0,\infty))$ and $\mathscr{D}^n([0,\infty))$ cannot be order-isomorphic whenever $m\ne n$. In this context, it should be noted that the dual of both these two spaces is order-isomorphic with $[0,\infty)$ by Theorem 4.2.

Conclusion

We conclude this article by outlining a plan for further research in this field.
(1) It is interesting to find a non-functionally additive evs.
(2) It is also an interesting problem to ensure the existence of a non-zero order-functional on an arbitrary evs.
(3) Under what conditions a non-zero order-functional can be extended from a subevs to a larger subevs of an evs?

Acknowledgement. I would like to express my sincere gratitude to my mentor Professor Sandip Jana, Department of Pure Mathematics, University of Calcutta, for his valuable guidance, encouragement, and insightful suggestions throughout this work. I am also thankful to the referee for his/her valuable feedback and comments.

Disclosure statement. The author reports that there are no competing interests to declare.

References

[1] S. Ganguly and S. Mitra, *More on topological quasi-vector space*, Rev. Acad. Canaria Cienc., 22(1-2) (2010), 45-58.

[2] S. Ganguly, S. Mitra and S. Jana, *An associated structure of a topological vector space*, Bull. Calcutta Math. Soc., 96(6) (2004), 489-498.

[3] S. Jana and J. Saha, *A study of topological quasi-vector spaces*, Rev. Acad. Canaria Cienc., 24(1) (2012), 7-23.

[4] E. Michael, *Topologies on spaces of subsets*, Trans. Amer. Math. Soc., 71 (1951), 152-182.

[5] L. Nachbin, Topology and Order, Van Nostrand Mathematical Studies, 4, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1965.

[6] J. Saha and S. Jana, *A Study of balanced quasi-vector space*, Rev. Acad. Canaria Cienc., 27 (2015), 8-28.

[7] J. Saha and S. Jana, *Algebraic characterisation of hyperspace corresponding to topological vector space*, J. Hyperstruct., 11(1) (2022), 48-64.

[8] P. Sharma and S. Jana, *An algebraic ordered extension of vector space*, Trans. A. Razmadze Math. Inst., 172 (2018), 545-558.

Jayeeta Saha
Department of Mathematics
Vivekananda College
Thakurpukur, Kolkata-700063
West Bengal, India
e-mail: jayeetasaha09@gmail.com