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RESULTS ON WEAKLY UNISERIAL MODULES

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Abstract. In this paper, we study weakly uniserial modules, a concept recently introduced by Moradzadeh-Dehkordi et al., which extends the notion of uniserial modules. A module M is said to be weakly uniserial if for any submodules N and L of M, there exists a monomorphism $N \rightarrowtail L$ or $L \rightarrowtail N$. Our analysis explores the relationship between weakly uniserial modules and classical notions in ring and module theory, including preradicals, socle series, the singular submodule, injective hulls, and V-rings. In addition, we present further statements complementing the characterization provided by the aforementioned authors concerning rings over which every module is weakly uniserial. Finally, by using monomorphisms, we resolve the Schröder–Bernstein problem within the class of isoartinian modules.

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1. Introduction

In this article, all rings are assumed to be associative with identity. The category of left (right) unitary modules over the ring R is denoted by R-Mod (Mod-R).

Foundations of module theory have evolved over time through various extensions and generalizations that aim to capture specific properties of modules. Among these, in 1935, G. Köthe defined *uniserial rings* as those whose lattice of ideals is totally ordered by inclusion [10]. In a natural way, a *uniserial module* was defined as one whose lattice of submodules is also totally ordered by inclusion. Over time, alternative definitions of uniserial rings or modules emerged, including *pro-uniserial rings*, introduced by L. E. P. Hupert [8] in 1981. These rings are those where every indecomposable projective R-module is uniserial. In 2016, M. Behboodi et al. [3] introduced and studied almost uniserial modules, defining a module M as

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almost uniserial if any two nonisomorphic submodules of M are linearly ordered by inclusion.

More recently, in 2023, A. Moradzadeh-Dehkordi et al. [11] gave a new extension of the concept of uniserial modules, *weakly uniserial modules*. These modules satisfy the property that for any pair of submodules, one is embedded into the other. Among other results, they showed that every left R-module is weakly uniserial if and only if every left 2-generated R-module is weakly uniserial, which is, in turn, equivalent to $R \cong M_n(D)$, the ring of $n \times n$ matrices over the division ring D. Moreover, they proved that the weakly uniserial property is Morita invariant. In line with this, we present further results concerning weakly uniserial modules and rings. To this end, the paper is organized as follows: Section 2 presents a brief review of weakly uniserial modules. In Section 3, we obtain some results relating these modules to other well-known concepts in ring and module theory. For instance, in Theorem 3.1, we show that for a hereditary preradical r, a weakly uniserial R-module M is either r-torsion-free or $r(M)$ is an essential submodule of M. In Theorem 3.7, we show that an R-module M is weakly uniserial if and only if $M/r(M)$ is weakly uniserial for any hereditary preradical r. In Theorem 3.16, we prove that a ring R is semisimple if and only if it is artinian and the class of weakly uniserial R-modules is closed under injective hulls. In Theorem 3.25, we present additional equivalent properties to the condition that every R-module is weakly uniserial to those given in [11]. Finally, a solution to Schröder–Bernstein problem using monomorphisms in the context of isoartinian modules is presented.

2. Preliminaries

In this section, we review some examples and basic properties of weakly uniserial modules and rings, which can also be found in [11] and [12].

Let M be a left R-module. We say that M is *left weakly uniserial* if for any two submodules N and L of M, either $N \rightarrowtail L$ or $L \rightarrowtail N$, meaning there exists a monomorphism between them. A left (resp., right) weakly uniserial ring is a ring which is weakly uniserial as a left (resp., right) R-module. A ring R is called weakly uniserial if it is both a left and a right weakly uniserial ring.

Remark 2.1. The class of weakly uniserial left R-modules is closed under submodules.

Example 2.2. Clearly, every left uniserial module is left weakly uniserial. But the converse is false: for any prime number p, the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ is weakly uniserial but is not uniserial.

Example 2.3. Every isosimple left R-module (i.e., a module isomorphic to each of its nonzero submodules) is left weakly uniserial. But again, the $\mathbb{Z}$-module $\mathbb{Z}_p \oplus \mathbb{Z}_p$ shows that the converse is not true in general.

Example 2.4. Additionally, every left almost uniserial R-module (a module whose any two nonisomorphic submodules are linearly ordered by inclusion) is left weakly uniserial. However, the converse is not true. Every vector space of dimension at least 3 over a division ring D is a weakly uniserial D-module, but it is by no means an almost uniserial D-module.

Remark 2.5. In general, the class of weakly uniserial left R-modules is not closed under quotients. For instance, the ring $\mathbb{Z}$ is isosimple and therefore weakly uniserial, as mentioned above. However $\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$, is not weakly uniserial. The same example shows in turn that the class of weakly uniserial modules is not closed under extensions.

Example 2.6. A module M is called *homogeneous semisimple* if it is isomorphic to a direct sum $\bigoplus_I S$, where S is a single simple module. Every homogeneous semisimple module is weakly uniserial.

Remark 2.7. The notion of weakly uniserial module is preserved under isomorphism.

Theorem 2.8. *For a ring R, the following statements hold:*

(a) *If R is a left weakly uniserial ring, then R has no nontrivial central idempotent.*

(b) *If $R \cong \prod_{i\in I} R_i$, where $|I| \geq 2$ and each R_i is a ring, then R is neither a left nor a right weakly uniserial ring.*

Theorem 2.9. *For a ring R, the following statements hold:*

(a) *If every nonzero left ideal of R contains a left regular element, then R is a left weakly uniserial.*

(b) *Every domain is a left and right weakly uniserial ring.*

(c) *If R is either a principal left ideal domain or a left hereditary local ring, then every projective left R-module is weakly uniserial.*

A left R-module M is called *semiartinian* if every nonzero factor module of M has nonzero socle. A ring R is a *left semiartinian ring* if R itself is a semiartinian left R-module. A ring R is left semiartinian if and only if every nonzero left R-module contains an essential socle, which is equivalent to say that every nonzero left R-module has a simple submodule.

Theorem 2.10. *Let R be a ring.*

(a) *For any weakly uniserial left R-module M, the following statements hold:*
  (i) $\mathrm{Soc}(M) = 0$ *or* $\mathrm{Soc}(M) \leq_e M$.
  (ii) *If M is semisimple, then M is homogeneous semisimple.*
  (iii) *M does not include nonzero singular submodules and nonsingular submodules simultaneously.*
  (iv) *The singular submodule $Z(M)$ is essential in M or $Z(M) = 0$.*
  (v) *If R is a left nonsingular ring, then M is singular or nonsingular.*
  (vi) *If M satisfies (C_2[1]), then M does not contain a nontrivial fully invariant direct summand.*

(b) *For any left weakly uniserial ring R, the following statements hold:*
  (vii) *If M is a left R-module that $Z(M) = M$, then M contains a weakly uniserial submodule.*
  (viii) *R is prime if and only if it is semiprime.*

A ring R is called *left Kasch* if every simple left R-module can be embedded in $_RR$. A *right Kasch* ring is defined analogously. As is common, R is called a *Kasch* ring if it is both left and right Kasch. Theorem below states a connection between commutative weakly uniserial rings and fields, local rings, artinian and Kasch rings.

Theorem 2.11. *For a ring R, the following statements hold:*

(a) *If R is a commutative (semi) prime weakly uniserial ring, then R is a field or* $\mathrm{Soc}(R) = 0$.
(b) *Every commutative Kasch weakly uniserial ring is local.*
(c) *Every commutative artinian weakly uniserial ring is local.*

Finally, the following theorem presents a series of equivalent statements that connect simple rings with weakly uniserial modules.

Theorem 2.12. *For a ring R, the following statements are equivalent:*

(a) *$R \cong M_n(D)$ where D is a division ring.*
(b) *Every left R-module is weakly uniserial.*
(c) *Every finitely generated left R-module is weakly uniserial.*
(d) *Every* 2*-generated left R-module is weakly uniserial.*
(e) *Every injective left R-module is weakly uniserial.*
(f) *The right-left symmetric of* (b), (c), (d) *and* (e).

[1] An R-module M is said to satisfy (C_2) if every submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.

3. Results on weakly uniserial modules

We present the results of this work concerning weakly uniserial modules, which build upon those established by Shirzadi et al. in [11] and [12].

Recall that a *preradical* r of R-Mod assigns to each R-module M a submodule $r(M)$ such that every morphism $M \to N$ induces $r(M) \to r(N)$ by restriction. In other words, a preradical is a subfunctor of the identity functor in the category R-Mod. In addition, a left R-module M is an *r-torsion module* if $r(M) = M$ and an *r-torsion-free module* if $r(M) = 0$.

The preradical r is said to be *hereditary* if the equality $r(N) = r(M) \cap N$ holds for every submodule N of M. Equivalently, r is a hereditary preradical if it is a left exact functor, which, in turn, is equivalent to r being an idempotent preradical and the class of r-torsion modules is closed under submodules (see [4, Proposition I.2.1]).

The following result provides insight into how weakly uniserial modules interact with hereditary preradicals, particularly regarding their torsion properties.

Theorem 3.1. *Let r be a hereditary preradical of R-Mod, and let M be a weakly uniserial left R-module. Then M is r-torsion-free or $r(M)$ is an essential submodule of M.*

Proof. Suppose that $r(M) \neq 0$, i.e., M is not r-torsion-free. We aim to show that $r(M)$ is an essential submodule of M. Let N be any nonzero submodule of M. Then $N \rightarrowtail r(M)$ or $r(M) \rightarrowtail N$. Now, $r(N) = r(M) \cap N$ and $rr(M) = r(M)$. In the first case, we have $r(N) = N \neq 0$, and we are done. In the second case, applying r to the second monomorphism, we have $rr(M) = r(M) \rightarrowtail r(N)$. In particular, $r(N) \neq 0$. $\square$

It is a well-known fact that taking the socle, the singular submodule and the second singular submodule of a module are hereditary preradicals. The above theorem leads to the following corollary (cf. [12, Proposition 2.1]).

Corollary 3.2. *If M is a weakly uniserial left R-module and N is* $\mathrm{Soc}(M)$*, $Z(M)$ or $Z_2(M)$, then $N = 0$ or N is an essential submodule of M.*

A ring R is called *left semiartinian* if $\mathrm{Soc}(M) \neq 0$ for every nonzero left R-module M. Moreover, for a left R-module M, the *injective hull* $E(M)$ is, up to isomorphism, the minimal injective essential extension of M.

Corollary 3.3. *For a ring R, the following statements hold:*

(a) *The injective hull of any left simple R-module is weakly uniserial if and only if the injective hull of any left weakly uniserial R-module with a simple socle is weakly uniserial.*

(b) *If R is a left semiartinian ring, and the injective hull of every homogeneous semisimple left R-module is weakly uniserial, then the injective hull of every left weakly uniserial R-module is weakly uniserial.*

Proof. The result can be deduced from Corollary 3.2 and [2, Proposition 18.12 (2)]. □

An R-module M is *finitely cogenerated* if for every family of submodules of M with trivial intersection, there exists a finite subfamily with also trivial intersection.

Corollary 3.4. *A nonzero weakly uniserial left R-module M is finitely cogenerated if and only if* $\mathrm{Soc}(M)$ *is a nonzero and finitely cogenerated submodule.*

Proof. By [2, Theorem 10.4 (2)], if M is finitely cogenerated, then $\mathrm{Soc}(M)$ is a finitely cogenerated and essential submodule of M. In particular, $\mathrm{Soc}(M)$ is a nonzero finitely cogenerated submodule of M.
On the other hand, if $\mathrm{Soc}(M)$ is a nonzero finitely cogenerated submodule, then, since the socle is a hereditary preradical, the result follows from Theorem 3.1 and [2, Proposition 10.4 (2)]. □

A module M is *cocyclic* if it contains an essential simple submodule.

Corollary 3.5. *A weakly uniserial left R-module M is cocyclic if and only if* $\mathrm{Soc}(M)$ *is simple.*

Proof. If M is cocyclic, then it is an essential extension of a simple module. Then $\mathrm{Soc}(M)$ is simple. Conversely, if $\mathrm{Soc}(M)$ is simple, then by Corollary 3.2, it is an essential submodule of M. □

Lemma 3.6. *Let r be a preradical, and let M and N be two R-modules. Then the following statements hold:*

(a) *If $f : M \to N$ is an isomorphism, then $f(r(M)) = r(N)$.*

(b) *If $N \leq M$, then $r(N) \leq r(M)$.*

(c) *If r is hereditary, $N \leq M$ and $r(M) \leq N$, then $r(M) = r(N)$.*

Proof. (a) and (b) follow directly from the definition of a preradical.
(c) Let $N \leq M$ and $r(M) \leq N$. As r is hereditary, $r(M) = rr(M) = r(N) \cap r(M) = r(N)$. □

Theorem 3.7. *Let r be a hereditary preradical, and let M be a weakly uniserial left R-module. Then $M/r(M)$ is a left weakly uniserial R-module.*

Proof. Let M be a weakly uniserial left R-module, and consider $K/r(M)$ and $N/r(M)$ two submodules of $M/r(M)$. According to Lemma 3.6-(c), $r(M) = r(N) = r(K)$. We can suppose that there exists a monomorphism, say, $f : N \to K$. As $f(N) \leq M$ and r is a hereditary preradical, $r(f(N)) = r(M) \cap f(N)$. Furthermore, by Lemma 3.6-(a), $r(f(N)) = f(r(N)) = f(r(M))$. Hence

$$f(r(M)) = r(f(N)) = r(M) \cap f(N) \leq r(M).$$

Now, the last inequality induces a morphism $\overline{f} : N/r(M) \to K/r(M)$. It remains to show that this morphism is actually a monomorphism. To this end, suppose that $\overline{f}(\overline{x}) = 0$ for some $x \in N$, then $f(x) \in r(M) \cap f(N) = r(f(N)) = f(r(M))$. Therefore, $x \in r(M)$ and consequently $\overline{x} = 0$. □

If M is an almost uniserial left R-module, using similar arguments to those in the proof of the previous theorem, we can establish the following result:

Theorem 3.8. *Let r be a hereditary preradical and let M be an almost uniserial left R-module. Then $M/r(M)$ is also an almost uniserial left R-module.*

Given two preradicals r and s, the preradical $r : s$ is defined by the relation

$$(r : s)(M)/r(M) = s(M/r(M)).$$

By [4, Proposition I.4.8 (7)], $r : s$ is a hereditary preradical, provided that r and s are hereditary preradicals.

For a nonlimit ordinal α, set $r_\alpha = r : r_{\alpha-1}$; otherwise, take $r_\alpha = \bigcup_{\beta<\alpha} r_\beta$. This construction produces an increasing sequence of preradicals r_α.

Lemma 3.9. *If r is a hereditary preradical, then r_α is also a hereditary preradical for every ordinal α.*

Proof. The proof will be carried out by transfinite induction, with the base case being straightforward.
If α is a nonlimit ordinal and $r_{\alpha-1}$ is a hereditary preradical, then $r_\alpha = r : r_{\alpha-1}$ is a hereditary preradical.
If α is a limit ordinal and r_β is a hereditary preradical for every $\beta < \alpha$, consider M a left R-module and N a submodule of M. Then

$$r_\alpha(N) = \bigcup_{\beta<\alpha} r_\beta(N) = \bigcup_{\beta<\alpha} (r_\beta(M) \cap N) = (\bigcup_{\beta<\alpha} r_\beta(M)) \cap N = r_\alpha(M) \cap N,$$

showing that r_α is a hereditary preradical. □

Theorems 2.10, 3.7, and 3.8, together with the previous lemma, lead to the following theorem.

Theorem 3.10. *Let r be a hereditary preradical, M a left weakly uniserial R-module, and α an ordinal. Then the following statements hold:*

(a) *M is r_α-torsion-free or $r_\alpha(M)$ is an essential submodule of M.*

(b) *$M/r_\alpha(M)$ is a left weakly uniserial R-module.*

(c) *Moreover, if M is an almost uniserial left R-module, then $M/r_\alpha(M)$ is also an almost uniserial left R-module.*

If M is an almost uniserial left R-module, it follows from (c) of the preceding theorem that $M/\operatorname{Soc}_\alpha(M)$ is an almost uniserial left R-module for every ordinal α (cf. [5, Theorem 2.12]).

Theorem 3.11. *A left R-module M is weakly uniserial if and only if for every submodules $B, D \leq M$ and every R-R-bimodule N, one of* $\operatorname{Hom}_R(N, B)$ *and* $\operatorname{Hom}_R(N, D)$ *is embedded in the other.*

Proof. First, suppose that M is a left weakly uniserial R-module, and let B, D be submodules of M. We can suppose that there exists a monomorphism, say, $\alpha : B \to D$. Now, if N is any R-R-bimodule, then α induces an embedding $\operatorname{Hom}_R(N, \alpha) : \operatorname{Hom}_R(N, B) \rightarrowtail \operatorname{Hom}_R(N, D)$.
For the converse, the result is established after realizing that $\operatorname{Hom}_R(R, B) \cong B$ and $\operatorname{Hom}_R(R, D) \cong D$. □

Theorem 3.12. *If M is a weakly uniserial left R-module, then for every pair of submodules B, $D \leq M$ and every injective left R-module E, one of* $\operatorname{Hom}_R(B, E)$ *and* $\operatorname{Hom}_R(D, E)$ *is a quotient of the other.*

Proof. Let M be a weakly uniserial left R-module, and consider two submodules B and D of M. We can suppose, for example, that there exists a monomorphism $\alpha : B \rightarrowtail D$. Now, for every $\gamma \in \operatorname{Hom}_R(B, E)$, there exists $\overline{\gamma} \in \operatorname{Hom}_R(D, E)$ such that $\overline{\gamma}\alpha = \gamma$. This shows that $\operatorname{Hom}_R(\alpha, E)$ is surjective. □

Let M be a left R-module and N an R-R-bimodule; in the following results, the notation M^* and M^{**} refer to the right R-module $\operatorname{Hom}_R(M, N)$ and the left R-module $\operatorname{Hom}_R(\operatorname{Hom}_R(M, N), N)$, respectively.

Theorem 3.13. *Let M be a noetherian and hereditary left R-module. If for any submodules B and D of M, there exists an R-R-bimodule N, such that one of the* *left R-modules B^{**} and D^{**} is embedded in the other, then M is weakly uniserial. The converse holds if there exists a left injective R-R-bimodule N.*

Proof. Let M be a noetherian and hereditary left R-module, B and D submodules of M and N an R-R-bimodule for which there is, say, an embedding of left R-modules $f : B^{**} \to D^{**}$.
Now, the submodules B and D are reflexive, as they are finitely generated. If $g : B \to B^{**}$ and $h : D^{**} \to D$ are isomorphisms, then the result follows by considering hfg.
On the other hand, let M be a weakly uniserial left R-module, and let N be an R-R-bimodule that is left injective. Now, given two submodules B and D of M, we can suppose that there exists a monomorphism of left R-modules, say, $f : B \to D$. Accordingly, there exists an epimorphism of right R-modules $f^* : D^* \to B^*$. Finally, since the functor $\mathrm{Hom}_R(\_, N)$ is contravariant, it follows that $\mathrm{Hom}_R(f^*, N) : B^{**} \to D^{**}$ is a monomorphism of left R-modules. □

Corollary 3.14. *Let R be a noetherian and hereditary ring and M a finitely generated projective left R-module. If for any submodules B and D of M, there exists an R-R-bimodule N, such that one of the left R-modules B^{**} and D^{**} is embedded in the other, then M is a weakly uniserial left R-module. The converse holds if there exists a left injective R-R-bimodule N.*

Corollary 3.15. *Let R be a semisimple ring. A finitely generated left R-module M is weakly uniserial if and only if for any pair of submodules B and D of M, there exist an R-R-bimodule N and a morphism $f : B \to D$ or $f : D \to B$ such that the induced map $f^* : B^* \to D^*$ or $f^* : D^* \to B^*$ is an epimorphism of right R-modules.*

Proof. Since R is a semisimple ring, any finitely generated left R-module is noetherian and hereditary. The result can be now deduced from Theorem 3.13. □

In what follows, we explore the injective hulls of weakly uniserial modules and provide a characterization of semisimple rings. To this end, recall that a ring R is called a *left V-ring* if every simple left R-module is injective.

Theorem 3.16. *A ring R is semisimple if and only if it is artinian, and the class of weakly uniserial left R-modules is closed under injective hulls.*

Proof. If R is a semisimple ring, then R is artinian and every left R-module is injective and coincides with its injective hull. Hence, the class of weakly uniserial left R-modules is closed under injective hulls.

For the converse, let R be an artinian ring and suppose that the class of weakly uniserial left R-modules is closed under injective hulls. Consider S a left simple R-module and $E(S)$ its injective hull. First, note that $E(S \oplus S) = E(S) \oplus E(S)$. Since $S \oplus S$ is weakly uniserial, it follows that $E(S) \oplus E(S)$ must also be weakly uniserial. Now, consider $S \oplus S$ and $E(S)$ as submodules of $E(S) \oplus E(S)$, then either $S \oplus S \rightarrowtail E(S)$ or $E(S) \rightarrowtail S \oplus S$. Since $\mathrm{Soc}(E(S)) = S$, the submodule $S \oplus S$ cannot embed in $E(S)$ without contradicting the fact that the socle of $E(S)$ is precisely S. Thus, the first case cannot occur. On the other hand, if $E(S)$ is embedded in $S \oplus S$, then either $E(S) \cong S$ or $E(S) \cong S \oplus S$. We claim that the second possibility cannot occur; otherwise, S would not be essential in $E(S)$. Therefore, $E(S) \cong S$ for every left simple R-module S, meaning that R is a left V-ring. Since R is also left artinian, it follows from the Artin-Wedderburn Theorem that R is semisimple. □

Corollary 3.17. *If R is an artinian ring, and the class of left weakly uniserial R-modules is closed under injective hulls, then it is closed under quotients.*

Proof. By Theorem 3.16, R is a semisimple ring. Since every weakly uniserial left R-module is homogeneous semisimple, the result follows. □

The converse of the corollary above fails, as shown by the following example. Consider $R = \mathbb{Z}_4$, which is a Σ-cyclic ring. The cyclic $\mathbb{Z}_4$-modules are either 0, $\mathbb{Z}_4$ or (2), so any $\mathbb{Z}_4$-module has the form ${\mathbb{Z}_4}^{(I)} \bigoplus {\mathbb{Z}_2}^{(J)}$, with I and J some index sets. On the other hand, if M is a weakly uniserial $\mathbb{Z}_4$-module, then it has the form $M = 0, \mathbb{Z}_4$, or ${\mathbb{Z}_2}^{(J)}$, whose quotients are weakly uniserial. However, $\mathbb{Z}_4$ is not a semisimple ring, so that the class of weakly uniserial left R-modules is not closed under injective hulls.

Theorem 3.18. *R is a left V-ring if and only if $E(S)^n$ is weakly uniserial for each simple left R-module S and each $n \in \mathbb{Z}^+$.*

Proof. Let R be a left V-ring and let $n \in \mathbb{Z}^+$. If S is a simple left R-module, then S is injective, and hence $E(S) = S$. Therefore $E(S)^n = S^n$, which is a weakly uniserial left R-module.

Conversely, suppose that $E(S)^n$ is a weakly uniserial left R-module for each simple left R-module S and every $n \in \mathbb{Z}^+$. The submodule $E(S) \oplus E(S) \leq E(S)^n$ is then weakly uniserial. Following the proof of Theorem 3.16, we conclude that $E(S) \cong S$, i.e., S is injective. Thus, R is a left V-ring. □

Theorem 3.19. *R is a semisimple ring if and only if R is semilocal and the class of weakly uniserial left R-modules is closed under injective hulls.*

Proof. If R is a semisimple ring, then by Theorem 3.16, the class of weakly uniserial left R-modules is closed under injective hulls. Moreover, since R is semisimple, the quotient $R/J(R)$ is semisimple, and thus R is semilocal.
For the converse, suppose that R is semilocal and that the class of weakly uniserial left R-modules is closed under injective hulls. Again, following the proof of Theorem 3.16, we obtain that R is a left V-ring. Finally, since R is semilocal and a left V-ring, it follows that R is semisimple. □

Theorem 3.20. *Let R be a ring for which the class of weakly uniserial left R-modules is closed under injective hulls. For a nonzero weakly uniserial left R-module M, the following statements are equivalent:*

(a) *M is finitely cogenerated.*
(b) *M is homogeneous semisimple and finitely cogenerated.*
(c) *M is homogeneous semisimple with finitely many direct summands.*
(d) *M has finite length.*
(e) *M is artinian.*
(f) *M is noetherian and* $\mathrm{Soc}(M) \neq 0$.
(g) $\mathrm{Soc}(M)$ *is nonzero and finitely generated.*

Consequently, M is cocyclic if and only if it is simple.

Proof. $(a) \Rightarrow (b)$ Let M be a weakly uniserial and finitely cogenerated left R-module. By [9, Theorem 9.4.3], we have

$$E(M) = E(S_1) \oplus \cdots \oplus E(S_n),$$

for some simple left R-modules $S_1, \ldots, S_n$ and some $n \in \mathbb{N}$. By the proof of Theorem 3.16, R is a V-ring and consequently

$$E(M) = S_1 \oplus \cdots \oplus S_n.$$

Since the class of weakly uniserial modules is closed under injective hulls, $E(M)$ is weakly uniserial, and by Theorem 2.10 (a)(ii), $E(M)$ is homogeneous semisimple. As M is an essential submodule of the semisimple module $E(M)$, it follows that $M = E(M)$. Hence, M is homogeneous semisimple and finitely cogenerated.

(b) ⇒ (c) and (c) ⇒ (d) follow from [9, Theorem 8.1.6].

(d) ⇒ (e) This is immediate.

(e) ⇒ (f) Suppose that M is artinian. Then $\mathrm{Soc}(M)$ is nonzero, essential in M, and has finitely many direct summands. A similar argument to that used in (a) ⇒

(b) shows that M is a homogeneous semisimple module with finitely many direct summands, and the result follows from [9, Theorem 8.1.6].

(f) $\Rightarrow$ (g) This is clear.

(g) $\Rightarrow$ (a) Assume that $\mathrm{Soc}(M)$ is nonzero and finitely generated. By [9, Theorem 8.1.6], $\mathrm{Soc}(M)$ is finitely cogenerated. Since M is weakly uniserial, $\mathrm{Soc}(M)$ is essential in M. Therefore, by [9, Theorem 9.4.3], M is finitely cogenerated.

Finally, using the equivalence between (a) and (c), it follows that M is cocyclic if and only if it is simple. □

Theorem 3.21. *Let R be a strongly von Neumann regular ring. If R is left weakly uniserial, then R is a division ring.*

Proof. Let $I \leq R$ be a left ideal and $x \in I$. Since R is strongly von Neumann regular, the left ideal $Rx \leq I$ is a principal left ideal generated by a central idempotent element. Without loss of generality, assume that x is such an element. By Theorem 2.8, it follows that $I = 0$ or $I = R$, proving that R is a division ring. □

A class of R-modules is called a *torsion class* if it is closed under quotients, direct sums and extensions. A ring R is called *left local* if all simple left R-modules are isomorphic, or equivalently, if there exists a simple left R-module S such that $E(S)$ is a cogenerator for R-Mod. Additionally, R is called *left Max* if every left R-module has a simple quotient. In [1], a ring R is defined as *left BKN* if it satisfies $\mathrm{Hom}_R(M, N) \neq 0$ for every pair of nonzero left R-modules M and N. In [4], it is shown that a ring is left BKN if and only if it is left local, left semiartinian, and left Max.

Theorem 3.22. *For a ring R, the following conditions are equivalent:*

(a) *R is left local.*
(b) *Every left R-module of length 2 is almost uniserial.*
(c) *Every left R-module of length 2 is weakly uniserial.*

Proof. (a) $\Rightarrow$ (b) Let R be a left local ring and M a left R-module of length 2. If N_1 and N_2 are submodules of M such that $N_1 \nleq N_2$ and $N_2 \nleq N_1$, then

$$0 \leq N_1 \leq M \quad \text{and} \quad 0 \leq N_2 \leq M$$

are composition series of M. Hence N_1 and N_2 are simple R-modules. In particular, they are isomorphic as left R-modules.

(b) $\Rightarrow$ (c) It follows at once since every almost uniserial left R-module is weakly uniserial.

(c) $\Rightarrow$ (a) Let S_1 and S_2 be simple left R-modules. The left R-module $S_1 \oplus S_2$ has length 2, and therefore it is weakly uniserial. Hence either S_1 embeds into S_2 or S_2 embeds into S_1. In either case, $S_1 \cong S_2$. □

Corollary 3.23. *If R is a ring with length 2 as a left R-module, then the following statements are equivalent:*

(a) *Every left R-module with length 2 is almost uniserial.*
(b) *R is a left local ring.*
(c) *R is left weakly uniserial.*

Proof. The implications (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) follow from the previous theorem.
(c) $\Rightarrow$ (a) Let M be a left R-module of length 2. Consider two submodules $N_1, N_2 \leq M$ that are not comparable by inclusion, so that

$$0 \leq N_1 \leq M \quad \text{and} \quad 0 \leq N_2 \leq M$$

are isomorphic composition series of M. Thus, there exist maximal left ideals I_1 and I_2 such that $R/I_1 \cong N_1$ and $R/I_2 \cong N_2$. If $I_1 = I_2$, then $N_1 \cong N_2$. Otherwise, we have two isomorphic composition series

$$0 \leq I_1 \leq R \quad \text{and} \quad 0 \leq I_2 \leq R,$$

and once again $N_1 \cong N_2$. Therefore, M is almost uniserial. □

Corollary 3.24. *If R is a left principal ideal ring of length 2 as a left R-module, then the following statements are equivalent:*

(a) *R is left weakly uniserial.*
(b) *R is almost left uniserial.*
(c) *R is left uniserial or $R \cong M_2(D)$ where D is a division ring.*

Proof. For the implication (a) $\Rightarrow$ (b) use (c) $\Rightarrow$ (a) of the above corollary.
(b) $\Rightarrow$ (c) Follows from [3, Proposition 2.7].
(c) $\Rightarrow$ (a) If R is left uniserial, we are done. Otherwise, if $R \cong M_2(D)$ use Theorem 3.25 below. □

In the following result, we present a series of equivalences for a simple ring R (cf. Theorem 2.12).

Theorem 3.25. *The following properties of a ring R are equivalent:*

(a) *All left R-modules are weakly uniserial.*
(b) *All 2-generated left R-modules are weakly uniserial.*
(c) *R is a simple ring ($R \cong M_n(D)$ for some division ring D).*

(d) *R is a left BKN ring and a left V-ring.*

(e) *The class of weakly uniserial left R-modules is a torsion class, and there exists a left maximal ideal of R that belongs to this class.*

(f) *The class of weakly uniserial left R-modules is closed under submodules, quotients, direct sums, injective hulls, and there exists a weakly uniserial left ideal I of R such that R/I is weakly uniserial.*

(g) *The class of weakly uniserial left R-modules is a torsion class, and every maximal left ideal belongs to this class.*

(h) *The class of weakly uniserial left R-modules is closed under submodules, quotients, direct sums, injective hulls, and every left ideal I is weakly uniserial with R/I weakly uniserial.*

Proof. For (a) $\Leftrightarrow$ (b) and (b) $\Leftrightarrow$ (c) see [11, Theorem 3.5].
(a) $\Rightarrow$ (d) Suppose that every left R-module is weakly uniserial. Thus, the class of weakly uniserial left R-modules is closed under injective hulls. As in the proof of Theorem 3.16, we obtain that R is a V-ring, and hence a left Max ring. On the other hand, let S and S' be two simple left R-modules. The weakly uniserial left R-module $S \oplus S'$ is a homogeneous semisimple left R-module. Hence, $S \cong S'$ and R is a left local ring.
Now, let M be a nonzero left R-module and consider the left R-module $M \oplus S$, where S is a simple left R-module. Since $M \oplus S$ is weakly uniserial, there exists a monomorphism either from M into S or from S into M. In either case, it follows that $\mathrm{Soc}(M) \neq 0$. Therefore, R is a left semiartinian ring. Accordingly, we conclude that R is a left BKN ring.
(d) $\Rightarrow$ (c) Assume that R is a left BKN ring and a left V-ring. Let S be a simple left R-module and consider a nonzero monomorphism $\alpha : S \to R$. Since $\alpha(S)$ is an injective submodule of R, it follows that $\alpha(S)$ is a direct summand of R, and hence a projective R-module. Thus, the (up to isomorphism) unique simple left R-module is projective, which implies that R is a simple ring.
(c) $\Rightarrow$ (a) This is because every left R-module is homogeneous semisimple.
(e) $\Leftrightarrow$ (a) If I is a weakly uniserial maximal left ideal of R, then R/I is a simple left R-module, and hence weakly uniserial. Since the class of weakly uniserial left R-modules is closed under extensions, R is a left weakly uniserial ring and we conclude that every left R-module is weakly uniserial. The converse is clear. $\square$

Theorem 3.26. *Let M be a projective and isosimple left R-module and let N be a weakly uniserial left R-module. If ρ is the projection of $N \oplus M$ onto M, then for all* *submodules K, L of $N \oplus M$ such that $\rho(K)$, $\rho(L) \neq 0$, there exists a monomorphism $K \rightarrowtail L$ or $L \rightarrowtail K$.*

Proof. Let M be a projective and isosimple left R-module and let N be a weakly uniserial R-module. If ρ is the projection of $N \oplus M$ onto M and K is a submodule of $N \oplus M$ such that $\rho(K) \neq 0$, denote by ρ_K the restriction of ρ to K. Since $\operatorname{Im} \rho_K$ is a projective R-module, it follows that ρ_K is a splitting epimorphism. If $\tau_K : \operatorname{Im} \rho_K \to K$ is a right inverse of ρ_K, then $K = \operatorname{Ker} \rho_K \oplus \operatorname{Im} \tau_K$. Finally, if L is another submodule of $N \oplus M$ such that $\rho(L) \neq 0$, then either $\operatorname{Ker} \rho_K \rightarrowtail \operatorname{Ker} \rho_L$ or $\operatorname{Ker} \rho_L \rightarrowtail \operatorname{Ker} \rho_K$ and we also have $\operatorname{Im} \tau_K \cong \operatorname{Im} \tau_L$. Thus, in any case $K \rightarrowtail L$ or $L \rightarrowtail K$, as required. $\square$

In set theory, the Schröder–Bernstein Theorem states that if there exist injective maps $f : A \to B$ and $g : B \to A$ between sets A and B, then there exists a bijective map $h : A \to B$. Inspired by this theorem, it is of common interest to determine when its statement holds in mathematical contexts other than set theory. A remarkable result in this direction is due to W. T. Gowers [7], who constructed an example of two nonisomorphic Banach spaces, each of which is a complemented subspace of the other, thereby showing that the Schröder–Bernstein problem has a negative solution in the context of Banach spaces. We conclude this work by addressing this problem in the context of isoartinian modules.

A left R-module M is called *left isoartinian* if every descending chain of submodules $M \geq M_1 \geq M_2 \geq \cdots$ eventually becomes stationary up to isomorphism, i.e., there exists an index n such that $M_n \cong M_i$ for all $i \geq n$. For background information on isoartinian modules, the reader is directed to [6].

Theorem 3.27. *A left R-module M is isoartinian if and only if every descending chain of monomorphisms between submodules of M eventually becomes stationary up to isomorphism.*

Proof. Suppose that M is an isoartinian left R-module, and consider the descending chain of monomorphisms between submodules of M:

$$N_1 \overset{\alpha_1}{\leftarrowtail} N_2 \overset{\alpha_2}{\leftarrowtail} N_3 \cdots .$$

Define $N_1' = N_1$, and for $i \geq 2$, set $N_i' = \alpha_1 \alpha_2 \cdots \alpha_{i-1}(N_i)$. Since each α_i is a monomorphism, it follows that $N_i' \cong N_i$ for all i. This gives rise to a descending chain:

$$N_1' \geq N_2' \geq N_3' \geq \cdots .$$

Since M is isoartinian, this chain must eventually become stationary up to isomorphism at some index n, meaning that $N'_i \cong N'_{i+1}$ for all $i \geq n$. Since the N'_i's are isomorphic to the N_i's, the original chain of monomorphisms must also become stationary up to isomorphism. Conversely, suppose that M satisfies the given condition. Consider any descending chain of submodules:

$$N_1 \geq N_2 \geq N_3 \geq \cdots .$$

Each inclusion $N_{i+1} \leq N_i$ can be regarded as a monomorphism. By assumption, this sequence must become stationary up to isomorphism at some index n, proving that M is isoartinian. □

Theorem 3.28. *Let M and N be left R-modules such that there exist monomorphisms $M \rightarrowtail N$ and $N \rightarrowtail M$. If, in addition, either M or N is isoartinian, then $M \cong N$.*

Proof. Let M and N be left R-modules. Consider monomorphisms $\alpha : N \rightarrowtail M$ and $\beta : M \rightarrowtail N$. Assuming without loss of generality that M is isoartinian, we construct a descending chain of monomorphisms:

$$M \leftarrowtail N' \leftarrowtail N \leftarrowtail M' \leftarrowtail M \leftarrowtail N' \leftarrowtail \cdots ,$$

where $N' = \alpha(N)$ and $M' = \beta(M)$.

By Theorem 3.27, this sequence becomes stationary up to isomorphism, which means that $M \cong N$. □

According to [13], a left R-module M is *compressible* if for every nonzero submodule N of M, there exists a monomorphism from M to N.

Theorem 3.29. *If M is a compressible left R-module, then M is weakly uniserial. Moreover, if M is isoartinian, then M is isosimple.*

Proof. Let M be a compressible left R-module, and let N be a nonzero proper submodule of M. Since M is compressible, there exists a monomorphism $\alpha : M \rightarrowtail N$. If K is another nonzero proper submodule of M, then the restriction $\alpha|_K$ is a monomorphism from K to N, showing that M is weakly uniserial.
For the second assertion, consider the monomorphisms $\alpha : M \rightarrowtail N$ and the inclusion $N \rightarrowtail M$. By Theorem 3.28, it follows that M is isosimple. □

Corollary 3.30. *A commutative ring R is a PID if and only if it is isoartinian and compressible as a left R-module.*

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