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# A NOTE ON FULLY $(m,n)$-STABLE MODULES

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Abstract. Let $R$ be a commutative ring with non-zero identity element. For two fixed positive integers $m$ and $n$. For two fixed positive integers $m$ and $n$, a right $R$-module $M$ is called fully $(m,n)$-stable, if $\theta(N) \subseteq N$ for each $n$-generated submodule $N$ of $M^m$ and $R$-homomorphism $\theta : N \to M^m$. In this paper we give some characterization theorems and properties of fully $(m,n)$-stable modules which generalize the results of fully stable modules. Also we study and describe the maximal submodules of fully $(m,n)$-stable modules.

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## 1. Introduction

Throughout, $R$ is an commutative ring with non-zero identity and all modules are unitary. We use the notation $R^{m\times n}$ for the set of all $m \times n$ matrices over $R$. For $A \in R^{m\times n}$, $A^T$ will denote the transpose of A. In general, for an $R$-module $N$, we write $N^{m\times n}$ for the set of all formal $m \times n$ matrices whose entries are elements of $N$. Let $M$ be a right $R$-module and $N$ be a left $R$-module. For $x \in M^{l\times m}$, $s \in R^{m\times n}$ and $y \in N^{n\times k}$, under the usual multiplication of matrices, $xs$ (resp. $sy$) is a well defined element in $M^{l\times m}$ (resp. $N^{n\times k}$). If $X \in M^{l\times m}$, $S \in R^{m\times n}$ and $Y \in N^{n\times k}$, define

$$\begin{aligned}
\ell_{M^{l\times m}}(S) &= \{u \in M^{l\times m} : us = 0, \forall s \in S\} \\
r_{N^{n\times k}}(S) &= \{v \in N^{n\times k} : sv = 0, \forall s \in S\} \\
\ell_{R^{m\times n}}(Y) &= \{s \in R^{m\times n} : sy = 0, \forall y \in Y\} \\
r_{R^{m\times n}}(X) &= \{s \in R^{m\times n} : xs = 0, \forall x \in X\}
\end{aligned}$$

We will write $N^n = N^{1\times n}$, $N_n = N^{n\times 1}$. Fully stable module have been discussed in [1], an $R$-module $M$ is called *fully stable* if $\theta(N) \subseteq N$ for each submodule $N$ of $M$ and $R$-homomorphism $\theta$ from $N$ into $M$. It is an easy matter to see that $M$ is fully stable if and only if $\theta(xR) \subseteq xR$ for each $x$ in $M$ and $R$-homomorphism

$\theta : xR \to M$. In this paper, for two fixed positive integers $m$ and $n$, we introduce the concepts of fully $(m,n)$-stable modules and $(m,n)$-Baer criterion and we prove that an $R$-module $M$ is fully $(m,n)$-stable if and only if $(m,n)$-Baer criterion holds for $n$-generated submodules of $M^m$. Finally, the maximal submodules of fully $(m,n)$-stable will be discussed. Let $M$ be a fully $(m,n)$-stable $R$-module and $U$ be a uniform element of $R^{m\times n}$. It is shown that $M_U$ the unique maximal left submodule of $M^m$ which contains $\ell_{M^m}(U)$.

## 2. Results

**Definition 2.1.** An $R$-module $M$ is called *fully* $(m,n)$*-stable* if $\theta(N) \subseteq N$ for each $n$-generated submodule $N$ of $M^m$ and $R$-homomorphism $\theta : N \to M^m$. The ring $R$ is *fully* $(m,n)$*-stable* if $R$ is fully $(m,n)$-stable as $R$-module.

It is clear that $M$ is fully $(1,1)$-stable if and only if $M$ is fully stable.

It is an easy matter to see that an $R$-module $M$ is fully $(m,n)$-stable if and only if it is fully $(m,q)$-stable for all $1 \le q \le n$ if and only if it is fully $(p,n)$ for all $1 \le p \le m$ if and only if it is fully $(p,q)$-stable for all $1 \le p \le m$ and $1 \le q \le n$.

Rutter ([6, Example 1]) gave an example of fully $(1,1)$-stable ring which is not fully $(1,2)$-stable.

An $R$-module $M$ is fully $(m,n)$-stable if and only if for each $\theta : N(=\sum_{i=1}^n \alpha_i R) \to M^m$ (where $\alpha_i \in M^m$) and each $w \in N$, there exists $t = (t_1,\dots,t_n) \in R^n$ such that $\theta(w) = \sum_{i=1}^n \alpha_i t_i = (\alpha_1,\dots,\alpha_n)t^T$, if $r = (r_1,\dots,r_n) \in R^n$ , then $\theta((\alpha_1,\dots,\alpha_n)r^T) = (\alpha_1,\dots,\alpha_n)t^T$.

**Proposition 2.2.** *An $R$-module $M$ is fully $(m,n)$-stable, if and only if any two $m$-element subsets $\{\alpha_1,\dots,\alpha_m\}$ and $\{\beta_1,\dots,\beta_m\}$ of $M^n$ , if $\beta_j \notin \sum_{i=1}^n \alpha_i R$, for each $j = 1,\dots,m$ implies $r_{R_n}\{\alpha_1,\dots,\alpha_m\} \not\subseteq r_{R_n}\{\beta_1,\dots,\beta_m\}$.*

**Proof.** Assume that $M$ is fully $(m,n)$-stable $R$-module and there exist two $m$-element subsets $\{\alpha_1,\dots,\alpha_m\}$ and $\{\beta_1,\dots,\beta_m\}$ of $M^n$ such that $\beta_j \notin \sum_{i=1}^n \alpha_i R$, $\forall j = 1,\dots,m$ and $r_{R_n}\{\alpha_1,\dots,\alpha_m\} \subseteq r_{R_n}\{\beta_1,\dots,\beta_m\}$. Define $f : \sum_{i=1}^n \alpha_i R \to M^m$ by $f(\sum_{i=1}^n \alpha_i r_i) = \sum_{i=1}^n \beta_i r_i$. Let $\alpha_i = (a_{1i}, a_{2i},\dots,a_{ni})$. If $\sum_{i=1}^n \alpha_i r_i = 0$, then $\sum_{i=1}^n a_{ij} r_i = 0$, $j = 1,\dots,m$ implies that $\alpha_j\ r^T = 0$ where $r = (r_1,\dots,r_n)$ and hence $r^T \in r_{R_n}\{\alpha_1,\dots,\ \alpha_m\}$. By assumption $\beta_j\ r^T = 0, j = 1,\dots,m$ so $\sum_{i=1}^n \beta_i r_i = 0$. This shows that $f$ is well defined. It is an easy matter to see that $f$ is $R$-homomorphism. Fully $(m,n)$-stability of $M$ implies that there exists $t = (t_1,\dots,t_n) \in R^n$ such that

$$f(\sum_{i=1}^n \alpha_i r_i) = \sum_{k=1}^n (\sum_{i=1}^n \alpha_i r_i) t_k = \sum_{k=1}^n \sum_{i=1}^n \alpha_i (r_i t_k)$$

for each $\sum_{i=1}^{n}\alpha_i r_i \in \sum_{i=1}^{n}\alpha_i R$. Let $r_i = (0,\dots,0,1,0,\dots,0) \in R^n$ where 1 in the *ith* position and 0 otherwise. $\beta_i = f(\alpha_i) = \sum_{k=1}^{n}\alpha_i t_k \in \sum_{i=1}^{n}\alpha_i R$ which is contradiction. Conversely assume that there exists $n$-generated submodule of $M^m$ and $R$-homomorphism $\theta : \sum_{i=1}^{n}\alpha_i R \to M^m$ such that $\theta(\sum_{i=1}^{n}\alpha_i R) \not\subset \sum_{i=1}^{n}\alpha_i R$. Then there exists an element $\beta(=\sum_{i=1}^{n}\alpha_i r_i) \in \sum_{i=1}^{n}\alpha_i R$ such that $\theta(\beta) \notin \sum_{i=1}^{n}\alpha_i R$. Take $\beta_j = \beta, j = 1,\dots,m$, then we have m-element subset $\{\theta(\beta),\dots,\theta(\beta)\}$, such that $\theta(\beta) \notin \sum_{i=1}^{n}\alpha_i R, j = 1,\dots,m$. Let $\eta = (t_1,\dots,t_n)^T \in r_{R_n}\{\alpha_1,\dots,\alpha_m\}$, then $\alpha_j\eta = 0$, i.e $\sum_{i=1}^{n}a_{ij}t_i = 0$, $\forall j = 1,\dots,m$, $\alpha_j = (a_{1j},a_{2j},\dots,a_{nj})$ and $\{\theta(\beta),\dots\theta(\beta)\}\ \eta$

$$= \sum_{k=1}^{n}\theta(\beta)t_k = \sum_{k=1}^{n}\theta(\sum_{i=1}^{n}\alpha_i r_i)t_k = \sum_{k=1}^{n}\theta(\sum_{i=1}^{n}\alpha_i r_i t_k) = 0,$$

hence $r_{R_n}\{\alpha_1,\dots,\alpha_m\} \subseteq r_{R_n}\{\theta(\beta),\dots,\theta(\beta)\}$, thus

$$r_{R_n}\{\alpha_1,\dots,\alpha_m\} \subseteq r_{R_n}\{\theta(\beta_1),\dots,\theta(\beta_m)\}$$

which is a contradiction. Thus $M$ is fully $(m,n)$-stable $R$-module. □

**Corollary 2.3.** *Let $M$ be fully $(m,n)$-stable $R$-module, then for any two m-element subsets $\{\alpha_1,\dots,\alpha_m\}$ and $\{\beta_1,\dots,\beta_m\}$ of $M^n$, $r_{R_n}\{\alpha_1,\dots,\alpha_m\} = r_{R_n}\{\beta_1,\dots,\beta_m\}$ implies $\alpha_1 R + \cdots + \alpha_m R = \beta_1 R + \cdots + \beta_m R$.*

**Corollary 2.4.** [1] *Let $M$ be a fully stable $R$-module, then for each $x,y$ in $M$, $r_R(x) = r_R(y)$ implies $(x) = (y)$*

A submodule $N$ of an $R$-module $M$ satisfies *Baer criterion* if for every $R$-homomorphism $f : N \to M$ , there exists an element $r \in R$ such that $f(n) = rn$ for each $n \in N$ . An $R$-module $M$ is said to satisfy *Baer criterion* if each submodule of $M$ satisfies Baer criterion and it is proved that an $R$-module $M$ satisfies Baer criterion for cyclic submodules if and only if $M$ is fully stable [1].

**Definition 2.5.** For a fixed positive integers $n$ and $m$, we say that an $R$-module $M$ satisfies *$(m,n)$-Bear criterion* if for any $n$-generated submodule $N$ of $M^m$ and any $R$-homomorphism $\theta : N \to M^m$ there exists $t \in R$ such that $\theta(x) = xt$ for each $x$ in $N$.

It is clear that if $M$ satisfies $(m,n)$-Baer criterion, then $M$ satisfies $(p,q)$-Baer criterion, for all $1 \le p \le m$ and $1 \le q \le n$.

**Proposition 2.6.** *If $M$ satisfies (m,1)-Bear criterion and $r_R(N \cap K) = r_R(N) + r_R(K)$ for each two n-generated submodules of $M^m$, then $M$ satisfies $(m,n)$-Baer criterion.*

**Proof.** Let $L = x_1R+x_2R+\cdots+x_nR$ be an $n$-generated submodule of $M^m$ and $f : L \to M^m$ an $R$-homomorphism. We use induction on $n$. It is clear that $M$ satisfies $(m,n)$-Bear criterion, if $n = 1$. Suppose that $M$satisfies $(m,n)$-Bear criterion for all $k$-generated submodule of $M^m$, for $k \leq n-1$. Write $N = x_1R, K = x_2R+\cdots+x_nR$, then for each $w_1 \in N$ and $w_2 \in K, f\mid_N (w_1) = w_1y_1, f\mid_K (w_2) = w_2y_2$ for some $y_1, y_2 \in R$. It is clear $y_1 - y_2 \in r_R(N \cap K) = r_R(N) + r_R(K)$. Suppose that $y_1 - y_2 = z_1 + z_2$ with $z_1 \in r_R(N), z_2 \in r_R(K)$ and let $y = y_1 - z_1 = y_2 + z_2$. Then for any $w = w_1 + w_2 \in L$ with $w_1 \in N$ and $w_2 \in K, f(w) = f(w_1) + f(w_2) = w_1y_1 + w_2y_2 = w_1(y_1 - z_1) + w_2(y_2 + z_2) = w_1y + w_2y = (w_1 + w_2)y = wy$. □

**Proposition 2.7.** *Let $M$ be an $R$-module. Then $M$ satisfies $(m,n)$-Baer criterion, if and only if $l_{M^n}r_{R_n}(\alpha_1R+\cdots+\alpha_nR) = \alpha_1R+\cdots+\alpha_nR$ for any $n$-element subset $\{\alpha_1,\ldots,\alpha_n\}$ of $M^n$.*

**Proof.** First assume that $(m,n)$-Baer criterion holds for $n$-generated submodule of $M^m$ , let $\alpha_i = (a_{i1}, a_{i2},\ldots,a_{im})$, for each $i = 1,\ldots,n$ and $\beta = \{\beta_1,\ldots,\beta_n\} \in l_{M^n}r_{R_n}(\alpha_1R+\ldots+\alpha_nR)$, $\beta_i = (a_{1i}, a_{2i},\ldots,a_{ni})$. Define $\theta : \alpha_1R+\cdots+\alpha_nR \to M^m$ by $\theta(\sum_{i=1}^n \alpha_ir_i) = \sum_{i=1}^n \beta_ir_i$. If $\sum_{i=1}^n \alpha_ir_i = 0$, then $\sum_{i=1}^n a_{ij}r_i = 0, j = 1,\ldots,m$, this implies that $\alpha_ir^T = 0$ where $r = (r_1,\ldots,r_n)$ and hence $r^T \in r_{R_n}(\alpha_1R + \cdots + \alpha_nR)$. By assumption $\beta_ir^T = 0,\forall i = 1,\ldots,n$ so $\sum_{i=1}^n \beta_ir_i = 0$ . This show that $f$ is well defined. It is an easy matter to see that $\theta$ is an $R$-homomorphism. By assumption there exists $t \in R$ such that $\theta(\sum_{i=1}^n \alpha_ir_i) = (\sum_{i=1}^n \alpha_ir_i)t = \sum_{i=1}^n \alpha_i(r_it)$ for each $\sum_{i=1}^n \alpha_ir_i \in \sum_{i=1}^n \alpha_iR$. Let $r_i = (0,\ldots,0,1,0,\ldots,0) \in R^n$ where 1 in the *ith* position and 0 otherwise. $\beta_i = \theta(\alpha_i) = \sum_{i=1}^n \alpha_it \in \sum_{i=1}^n \alpha_iR$ which is contradiction. This implies that $\ell_{M^n}r_{R_n}(\alpha_1R + \cdots + \alpha_nR) \subseteq \alpha_1R + \cdots + \alpha_nR$,the other inclusion is trivial. Conversely, assume that $\ell_{M^n}r_{R_n}(\alpha_1R + \cdots+\alpha_nR) = \alpha_1R+\cdots+ \alpha_nR$,for each $\{\alpha_1,\ldots,\alpha_n\}$ in $M^n$. Then for each $R$-homomorphism $f : \alpha_1R+\cdots+ \alpha_nR \to M^m$ and $s = (s_1,\ldots,s_n) \in r_{R_n}(\alpha_1R + \cdots + \alpha_nR)$,$\sum_{k=1}^n(\sum_{i=1}^n \alpha_ir_i)s_k = 0$ for each $\sum_{i=1}^n \alpha_ir_i \in \sum_{i=1}^n \alpha_iR$, hence

$$\sum_{k=1}^n f(\sum_{i=1}^n \alpha_ir_i)s_k = \sum_{k=1}^n f(\sum_{i=1}^n \alpha_ir_is_k) = 0,$$

thus $f(\sum_{i=1}^n \alpha_ir_i) \in \ell_{M^n}r_{R_n}(\alpha_1R+\cdots+\alpha_nR) = \alpha_1R+\cdots+\alpha_nR$, then $f(\sum_{i=1}^n \alpha_ir_i) = \sum_{i=1}^n \alpha_it$, for some $t \in R$. Then $M$ satisfies $(m,n)$-Baer criterion. □

**Corollary 2.8.** *An $R$-module $M$ is fully $(m,n)$-stable if and only if $l_{M^n}r_{R_n}(\alpha_1R+ \cdots + \alpha_nR) = \alpha_1R + \ldots + \alpha_nR$ for any $n$-element subset $\{\alpha_1,\ldots,\alpha_n\}$ of $M^n$*

The following proposition gives other characterizations of fully $(m,n)$-stable modules.

**Proposition 2.9.** *The following statements are equivalent for an $R$-module $M$.*

(1) *$M$ is fully $(m, n)$-stable.*

(2) $l_{M^n} r_{R_n}(\alpha_1 R + \cdots + \alpha_n R) = \alpha_1 R + \cdots + \alpha_n R$ *for any $n$-element subset* $\{\alpha_1, \ldots, \alpha_n\}$ *of* $M^n$.

(2′) $l_{M^n} r_{R_n}(A) = R^m A$ *where* $A \in M^{m\times n}$.

(3) $r_{R_n}\{\alpha_1, \ldots, \alpha_m\} \subseteq r_{R_n}\{\beta_1, ..., \beta_m\}$ *for each $m$-element two subsets* $\{\alpha_1, \ldots, \alpha_m\}$ *and* $\{\beta_1, \ldots, \beta_m\}$ *of* $M^n$ *implies* $\alpha_1 R + \ldots + \alpha_m R \subseteq \beta_1 R + \cdots + \beta_m R$.

(3′) $r_{R_n}(A) \subseteq r_{R_n}(B)$ *where* $A, B \in M^{m\times n}$ *implies* $R^m B \subseteq R^m A$.

(4) *If* $z \in M^n$ *and* $A \in M^{m\times n}$ *satisfy* $r_{R_n}(A) \subseteq r_{R_n}(z)$*,then* $z \in R^m A$.

(5) *$(m, n)$-Baer criterion holds for $n$-generated submodules of $M^m$.*

(6) $l_{M^k}[BR_n \cap r_{R_k}(A)] = l_{M^k}(B) + R^m A$, *where* $B \in R^{k\times n}, A \in M^{m\times k}$.

(6′) $l_{M^n}[BR_n \cap r_{R_n}(A)] = l_{M^n}(B) + R^m A$, *where* $B \in R^{m\times n}, A \in M^{m\times n}$.

**Proof.** (1) ⇔ (2) is by Corollary(2.8). (2) ⇔ (5) is by Proposition(2.7). (2) ⇔ (2′), (3) ⇔ (3′) and (6) ⇒ (6′) ⇒ (2′) ⇒ (3′) are trivial.

(3′) ⇒ (4) Let $B = \begin{pmatrix} z \\ 0 \end{pmatrix} \in M^{m\times n}$. Then $r_{R_n}(A) \subseteq r_{R_n}(z) = r_{R_n}(B)$ and $R^m B = Rz$. By (3′), we have $Rz = R^m B \subseteq R^m A$. Therefore $z \in R^m A$.

(6) ⇒ (4) it is clear.

(4) ⇒ (6) Let $w \in l_{M^k}[BR_n \cap r_{R_k}(A)]$, then $r_{R_n}(AB) \subseteq r_{R_n}(wB)$. So we have by (3′), $wB = sAB$ for some $s \in R^m$. Thus $w - sA \in l_{M^k}(B)$, and hence $w \in l_{M^k}(B) + R^m A$. The other inclusion is clear. □

**Corollary 2.10.** [2, Theorem 1] *The following statements are equivalent for an $R$-module $M$.*

(1) *$M$ is fully-stable.*

(2) $\ell_M r_R(x) = xR$ *for each $x$ in $M$.*

(3) $r_R(x) \subseteq r_R(y)$ *implies that* $yR \subseteq xR$ *for each $x, y$ in $M$.*

(4) *Baer criterion holds for cyclic submodules of $M$.*

(5) $\ell_M[yR \cap r_R(x)] = \ell_M(y) + xR$ *for each $x$ in $M$ and $y$ in $R$.*

In the following theorem summarize the above results.

**Theorem 2.11.** *Given an $R$-module $M_R$. Then $M_R$ is fully $(m, n)$-stable, if and only if the right $R^{n\times n}$-module $M^{m\times n}$ is fully-stable.*

**Proof.** (⇒) Let $A, B \in M^{m\times n}$ with $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B)$ and write $B = \begin{pmatrix} B_1 \\ \vdots \\ B_m \end{pmatrix}$.

Then for each $i = 1, \ldots, m$, $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B_i)$. consequently, $r_{R_n}(A) \subseteq$

$r_{R_n}(B_i)$. Since M is fully $(m,n)$-stable, by Proposition(2.9)(3′), $B_i \in R^m A, (i = 1, \ldots, m)$, $B_i = C_i A$ for some $C_i \in R^m$. So $B = CA$ where $C = \begin{pmatrix} C_1 \\ \vdots \\ C_m \end{pmatrix} \in R^{m\times m}$. Therefore the right $R^{n\times n}$-module $M^{m\times n}$ is fully-stable by [2].

($\Leftarrow$) Suppose that $z \in M^n$ and $r_{R_n}(A) \subseteq r_{R_n}(z)$. Let $B = \begin{pmatrix} z \\ 0 \end{pmatrix} \in M^{m\times n}$. Then $r_{R^{n\times n}}(A) \subseteq r_{R^{n\times n}}(B)$. Since $M^{m\times n}_{R^{n\times n}}$ is fully stable, $B = CA$ for some $C \in R^{m\times m}$ by [3, Theorem 1]. It follows that $z \in R^m A$ by Proposition 2.9(4). Then $M$ is fully $(m,n)$-stable. □

Recall an $R$-module $M$ is *semi-fully stable* if for each cyclic submodule $N$ of $M$ and $R$-homomorphism $f : N \to M$, there exists $g \in End(M)$ such that $f(n) = g \cdot n$ for each $n \in N$ [3]. This is equivalent to saying that each $R$-homomorphism of a cyclic submodule of $M$ into $M$ is extendable to an $R$-endomorphism of $M$, that is, $M$ is principally quasi injective [5]. Known that every fully-stable is semi-fully stable [3]. So we have the following corollary.

**Corollary 2.12.** *Given an R-module $M_R$. If $M_R$ is fully $(m,n)$-stable, then the right $R^{n\times n}$ -module $M^{m\times n}$ is semi-fully stable.*

**Corollary 2.13.** *Given an R-module $M_R$. If $M_R$ is fully $(m,n)$-stable, then the right $R^{n\times n}$ -module $M^{m\times n}$ is principally quasi-injective.*

It is proved in [7] that, $M_R$ is $(m,n)$-quasi injective if and only if the right $R^{n\times n}$-module $M^{m\times n}$ is principally quasi-injective.

The following theorem follows from Theorem (2.11) and Theorem (1.9) in [7]

**Theorem 2.14.** *Given an R-module $M_R$. If $M_R$ is fully $(m,n)$-stable, then the right R-module M is $(m,n)$-quasi injective.*

For the proof of the following lemma see Proposition (2.2).

**Lemma 2.15.** *R is fully $(m,n)$-stable, if and only if for all $A \in R^{m\times n}$, $\ell_{R^n} r_{R_n}(A) = R^m A$.*

It is proved in [5] that $R$ is $(m,n)$-injective, if and only if $\ell_{R^n} r_{R_n}(A) = R^m A$, for all $A \in R^{m\times n}$. Thus $R$ is $(m,n)$-injective, if and only if $R$ is fully $(m,n)$-stable.

In the next part we consider the converse of Theorem (2.14).

Recall that an $R$-module $M$ is multiplication, if each submodule of $M$ of the form $IM$ for some ideal of $R$ [4]. This is equivalent to saying that, every cyclic submodule of $M$ of the form $MI$ for some I of R [4].

Now, we introduce the following concept.

**Definition 2.16.** An $R$-module $M$ is called *$(m,n)$-multiplication*, if each $n$- generated submodule of $M^m$ is of the form $M^mI$ for some ideal $I$ of $R^{m\times n}$

**Proposition 2.17.** *Let $M$ be an $(m,n)$-multiplication $R$-module. If $M$ is $(m,n)$-quasi injective, then $M$ is a fully $(m,n)$-stable module.*

**Proof.** Let $N$ be any $n$-generated submodule of $M^m$ and $f : N \to M^m$ any $R$-homomorphism. Since $M$ is $(m,n)$-multiplication, then $N = M^mI$ for some $I \in R^{m\times n}$ , By $(m,n)$-quasi injectivity of $M$, $M^m$ is $n$-quasi-injective [8], thus $f$ can be extended to an $R$-homomorphism $g : M^m \to M^m$. Now $f(N) = g(N) = g(M^mI) = g(M^m)I \subseteq M^mI = N$ . Thus $M$ is fully $(m,n)$-stable module. □

Recall that an $R$-module $M$ is uniform, if every non-zero submodules of $M$ has non-zero intersection with every non-zero submodule of $M$.

Next, we study the maximal submodule of fully $(m,n)$-stable modules. First we introduce the following concept.

**Definition 2.18.** An element $U \in R^{m\times n}$ is called *uniform*, if $U \neq 0$ and $UR_n$ is a uniform ideal of $R_m$ and write $M_U = \{x \in M^m : r_{R_m}(x) \cap UR_n \neq 0\}$

**Proposition 2.19.** *Let $M$ be a fully $(m,n)$-stable $R$-module and $U$ be a uniform element of $R^{m\times n}$. Then $M_U$ is the unique maximal left submodule of $M^m$ which contains $\ell_{M^m}(U)$.*

**Proof.** For each $x, y \in M_U$ . Since $UR_n$ is a uniform, then $r_{R_m}(x+y) \cap UR_n \neq 0$ and $r_{R_m}(tx) \cap UR_n \neq 0$ for each $t = (t_1,\dots,t_n) \in R_n$. Then $M_U$ is a left submodule of $R$-module $M^m$. Furthermore for each $w \in \ell_{M^m}(U)$ then $wU = 0$, hence $0 \neq U \in r_{R_m}(w) \cap UR_n$, so $w \in M_U$. For each $A \notin M_U$, then $r_{R_m}(A) \cap UR_n = 0$, so $\ell_{M^m}[r_{R_m}(A) \cap UR_n] = M^m$. Let $\bar{A} = \begin{pmatrix} A \\ 0 \end{pmatrix} \in M^{m\times m}$. Then $r_{R_m}(\bar{A}) = r_{R_m}(A)$ and $R^m(\bar{A}) = RA$. By Proposition(2.9) we have $\ell_{M^m}(U) + RA = M^m$. This shows that $M_U$ is maximal. Finally, if $\ell_{M^m}(U) \subseteq L$ for some maximal left submodule of $M^m$ and, if $v \in L/M_U$, then as before $\ell_{M^m}(U) + RA = M^m$, so $L = M^m$ which is contradiction. □

An $R$-module $M$ is called *dual-distinguished* (simply *d-distinguished*), if $r_R(N) \neq 0$ for every maximal submodule $N$ of $M$. This concept was introduced in [2].

**Definition 2.20.** An $R$-module $M$ is called *$m$-dual-distinguished* (simply *$m$-$d$-distinguished*), if $r_{R_m}(N) \neq 0$ for every maximal submodule $N$ of $M^m$.

**Theorem 2.21.** *Let $R$ be a ring such that every non-zero ideal in $R^{m\times n}$ contains a uniform ideal and $M$ be a fully $(m,n)$-stable $m$-$d$-distinguished $R$-module. Then every maximal left submodule $N$ of $M^m$ has the form $M_U$ for some uniform element $U$ in $R^{m\times n}$.*

**Proof.** Since $M$ is $m$-$d$-distinguished $R$-module, then $r_{R_m}(N) \neq 0$. The hypothesis implies that there is a uniform ideal $UR_n$ of $R^m$ such that $UR_n \subseteq r_{R_m}(N)$. For each $x \in M_U$, then $W = r_{R_m}(x) \cap UR_n \neq 0$ , then $\ell_{M^m}(W) = \ell_{M^m}[r_{R_m}(x) \cap UR_n]$. Let $\bar{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in M^{m\times m}$. Then $r_{R_m}(\bar{x}) = r_{R_m}(x)$ and $R^m\bar{x} = Rx$. By Proposition (2.9) we have $\ell_{M^m}(W) = \ell_{M^m}(U) + Rx$, so $x \in \ell_{M^m}(W)$. But $W \subseteq UR_n \subseteq r_{R_m}(N)$, then $\ell_{M^m}(W) \supseteq \ell_{M^m}[r_{R_m}(N)] \supseteq N$. Maximality of $N$ gives that $\ell_{M^m}(W) = N$, hence $x \in N$, thus $M_U \subseteq N$. Again maximality of $M_U$ implies that $N = M_U$. □

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