International Electronic Journal of Algebra
Volume 8 (2010) 140-152

WEAK GORENSTEIN GLOBAL DIMENSION

Driss Bennis

Received: 12 February 2010; Revised: 16 June 2010
Communicated by Abdullah Harmancı

Abstract. In this paper, we investigate the weak Gorenstein global dimension. We are mainly interested in studying the problem when the left and right weak Gorenstein global dimensions coincide. We first show, for GF-closed rings, that the left and right weak Gorenstein global dimensions are equal when they are finite. Then, we prove that the same equality holds for any two-sided coherent ring. We conclude with some examples and a brief discussion of the scope and limits of our results.

Mathematics Subject Classification (2000): 16E05, 16E10, 16E30, 16E65
Keywords: Gorenstein flat dimension, weak Gorenstein global dimension, weak global dimension, GF-closed rings

1. Introduction

Throughout the paper, all rings are associative with identity, and all modules are unitary. Let R be a ring and let M be an R-module. The injective (resp., flat) dimension of M is denoted by $\mathrm{id}_R(M)$ (resp., $\mathrm{fd}_R(M)$). We use M^* to denote the character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M.

A left (resp., right) R-module M is called *Gorenstein flat*, if there exists an exact sequence of flat left (resp., right) R-modules

$$\mathrm{F} = \ \cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots ,$$

such that $M \cong \mathrm{Im}(F_0 \to F^0)$ and such that the sequence $I \otimes_R \mathrm{F}$ (resp., $\mathrm{F} \otimes_R I$) remains exact whenever I is an injective right (resp., left) R-module. The sequence F is called *a complete flat resolution.*

For a positive integer n, we say that M has *Gorenstein flat dimension* at most n, and we write $\mathrm{Gfd}_R(M) \leq n$, if there is an exact sequence of R-modules

$$0 \to G_n \to \cdots \to G_0 \to M \to 0,$$

where each G_i is Gorenstein flat (please see [8,11,13]).

The notion of Gorenstein flat modules was introduced and studied over Gorenstein rings, by Enochs, Jenda, and Torrecillas [12], as a generalization of the notion of flat modules in the sense that an R-module is flat if and only if it is Gorenstein flat with finite flat dimension. In [7], Chen and Ding generalized known characterizations of Gorenstein flat modules (then of the Gorenstein flat dimension) over Gorenstein rings to n-FC rings (coherent with finite self-FP-injective dimension). Then, in [13], Holm generalized the study of the Gorenstein flat dimension to coherent rings. In the same direction, the study of Gorenstein flat dimension is generalized, in [1], to a larger class of rings called GF-closed: a ring R is called left (resp., right) GF-closed, if the class of all Gorenstein flat left (resp., right) R-modules is closed under extensions; that is, for every short exact sequence of left (resp., right) R-modules $0 \to A \to B \to C \to 0$, the condition A and C are Gorenstein flat implies that B is Gorenstein flat. A ring is called GF-closed, if it is both left and right GF-closed. The class of GF-closed rings includes strictly the one of coherent rings and also the one of rings of finite weak global dimension [1, Example 3.6].

In this paper, we are concerned with the left and right weak Gorenstein global dimension of rings, $l.\mathrm{wGgldim}(R)$ and $r.\mathrm{wGgldim}(R)$, which are introduced in [3] as follows:

$$l.\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \textit{ is a left } R\textit{-module}\} \text{ and}$$
$$r.\mathrm{wGgldim}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \textit{ is a right } R\textit{-module}\}.$$

In the classical case we have, for any ring R, the following well-known equality [16, Theorem 9.15]:

$$\sup\{\mathrm{fd}_R(M) \mid M \textit{ is a right } R\textit{-module}\} = \sup\{\mathrm{fd}_R(M) \mid M \textit{ is a left } R\textit{-module}\}.$$

The common value of these equal terms is called weak global dimension of R and denoted by $\mathrm{wgldim}(R)$.

In [3, Corollary 1.2], we have: if $\mathrm{wgldim}(R) < \infty$, then

$$l.\mathrm{wGgldim}(R) = \mathrm{wgldim}(R) = r.\mathrm{wGgldim}(R).$$

This naturally leads to the following conjecture:

Conjecture 1.1. *For any ring R, $l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R)$.*

The main purpose of this paper is to prove that this conjecture holds for a large class of rings. First, we prove that the conjecture is true for GF-closed rings which have finite both left and right weak Gorenstein global dimensions (Theorem 2.1). Then, we prove the conjecture is true for two-sided coherent rings (Theorem 2.8). In Proposition 2.14, we prove that, for a ring R, $l.\mathrm{wGgldim}(R) = 0$ if and only if $r.\mathrm{wGgldim}(R) = 0$; and, in this case, R is an IF ring (i.e., R satisfies: every injective right R-module I is flat and every injective left R-module I is flat). We conclude with some examples and a brief discussion of the scope and limits of our results (Remark 2.17, Proposition 2.18, and Example 2.19).

2. Main results

We begin with the first main result which says that Conjecture 1.1 is true for GF-closed rings with finite both left and right weak Gorenstein global dimensions.

Theorem 2.1. *If R is a GF-closed ring with finite both left and right weak Gorenstein global dimensions, then* $l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R)$.

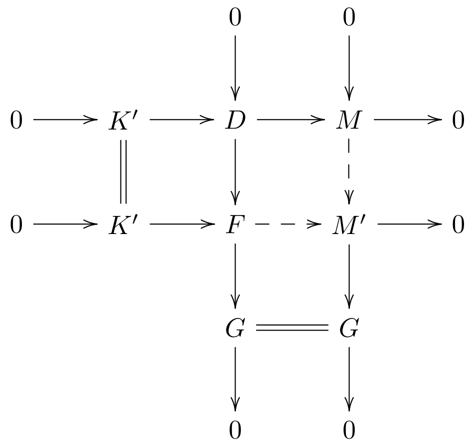
To prove this theorem, we need the following results. The following lemma generalizes [9, Lemma 2.19].

Lemma 2.2. *Assume that R is a left (resp., right) GF-closed ring. If M is a left (resp., right) R-module with $\mathrm{Gfd}_R(M) < \infty$, then there exists a short exact sequence of left (resp., right) R-modules $0 \to M \to M' \to G \to 0$, such that $\mathrm{fd}_R(M') = \mathrm{Gfd}_R(M)$ and G is Gorenstein flat.*

Proof. We only prove the case of left modules, and the case of right modules is proved similarly. Let $\mathrm{Gfd}_R(M) = n$ for some positive integer n. We prove the result by induction on n. The case $n = 0$ holds by the definition of the Gorenstein flat module. Then, suppose that $n > 0$ and pick a short exact sequence of left R-modules: $0 \to K \to G_0 \to M \to 0$, where G_0 is Gorenstein flat and $\mathrm{Gfd}_R(K) = n - 1$. By induction, there exists a short exact sequence of R-modules: $0 \to K \to K' \to H \to 0$, such that $\mathrm{fd}_R(K') = \mathrm{Gfd}_R(K) = n - 1$ and H is Gorenstein flat. Consider the pushout diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & K & \to & G_0 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & K' & \dashrightarrow & D & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & H & = & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By the middle vertical sequence and since R is left GF-closed, D is Gorenstein flat. Then, there exists a short exact sequence of left R-modules $0 \to D \to F \to G \to 0$, where F is flat and G is Gorenstein flat. Consider the pushout diagram:



By the middle horizontal sequence $\mathrm{fd}_R(M') = \mathrm{fd}_R(K') + 1 = n$ (since F is flat). Therefore, the right vertical sequence is the desired sequence. $\square$

Compare the following result to [14, Theorem 2.6 (ii)].

Corollary 2.3. *Assume that R is a left (resp., right) GF-closed ring. If M is an injective left (resp., right) R-module, then* $\mathrm{fd}_R(M) = \mathrm{Gfd}_R(M)$.

Proof. It is known that $\mathrm{Gfd}_R(M) \leq \mathrm{fd}_R(M)$ for every (left or right) R-module and over any associative ring R. Conversely, assume that $\mathrm{Gfd}_R(M)$ is finite, then by Lemma 2.2 there exists a short exact sequence of R-modules: $0 \to M \to M' \to G \to 0$ such that $\mathrm{fd}_R(M') = \mathrm{Gfd}_R(M)$. Since M is injective, this sequence splits and therefore $\mathrm{fd}_R(M) \leq \mathrm{fd}_R(M') = \mathrm{Gfd}_R(M)$. $\square$

Lemma 2.4. *If R is a left GF-closed ring with $l.\mathrm{wGgldim}(R) < \infty$, then, for a positive integer n, the following conditions are equivalent:*

(1) $l.\mathrm{wGgldim}(R) \leq n$;
(2) $\mathrm{Gfd}_R(M) \leq n$ *for every finitely presented left R-module M;*
(3) $\mathrm{Gfd}_R(R/I) \leq n$ *for every finitely generated left ideal I of R;*
(4) $\mathrm{fd}_R(E) \leq n$ *for every injective right R-module E;*
(5) $\mathrm{fd}_R(E') \leq n$ *for every right R-module E' with finite injective dimension;*

Consequently, the left weak Gorenstein global dimension of R is also determined by the formulas:

$$\begin{aligned} l.\mathrm{wGgldim}(R) &= \sup\{\mathrm{Gfd}_R(R/I)\,|\,I \textit{ is a finitely generated left ideal of } R\} \\ &= \sup\{\mathrm{Gfd}_R(M)\,|\,M \textit{ is a finitely presented left } R\textit{-module}\} \\ &= \sup\{\mathrm{fd}_R(E)\,|\,E \textit{ is an injective right } R\textit{-module}\} \\ &= \sup\{\mathrm{fd}_R(E')\,|\,E' \textit{ is a right } R\textit{-module with } \mathrm{id}_R(E) < \infty\}. \end{aligned}$$

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3$ are trivial. The implication $3 \Rightarrow 4$ follows from [1, Theorem 2.8 ($1 \Rightarrow 2$)]. The implication $4 \Rightarrow 5$ is proved by induction on $\mathrm{id}_R(E')$ using the flat counterpart of [5, Corollary 2, p. 135]. Finally, the implication $5 \Rightarrow 1$ is a simple consequence of [1, Theorem 2.8 ($3 \Rightarrow 1$)]. □

Similarly we obtain the right version of Lemma 2.4.

Lemma 2.5. *If R is a right GF-closed ring with $r.\mathrm{wGgldim}(R) < \infty$, then, for a positive integer n, the following conditions are equivalent:*

(1) $r.\mathrm{wGgldim}(R) \leq n$*;*
(2) $\mathrm{Gfd}_R(M) \leq n$ *for every finitely presented right R-module M;*
(3) $\mathrm{Gfd}_R(R/I) \leq n$ *for every finitely generated right ideal I of R;*
(4) $\mathrm{fd}_R(E) \leq n$ *for every injective left R-module E;*
(5) $\mathrm{fd}_R(E') \leq n$ *for every left R-module E' with finite injective dimension;*

Consequently, the right weak Gorenstein global dimension of R is also determined by the formulas:

$$\begin{aligned} r.\mathrm{wGgldim}(R) &= \sup\{\mathrm{Gfd}_R(R/I)\,|\,I \textit{ is a finitely generated right ideal of } R\} \\ &= \sup\{\mathrm{Gfd}_R(M)\,|\,M \textit{ is a finitely presented right } R\textit{-module}\} \\ &= \sup\{\mathrm{fd}_R(E)\,|\,E \textit{ is an injective left } R\textit{-module}\} \\ &= \sup\{\mathrm{fd}_R(E')\,|\,E' \textit{ is a left } R\textit{-module with } \mathrm{id}_R(E) < \infty\}. \end{aligned}$$

Proof of Theorem 2.1. Assume that $r.\mathrm{wGgldim}(R) = n$ is finite. From Corollary 2.3, $\mathrm{fd}_R(E) = \mathrm{Gfd}_R(E) \leq n$ for every injective right R-module E. Then, from Lemma 2.4, $l.\mathrm{wGgldim}(R) \leq n = r.\mathrm{wGgldim}(R)$.
The converse inequality is proved similarly. □

Under the condition of Theorem 2.1, the classical left and right finitistic flat dimension are equal and they are also equal to the left and right weak Gorenstein global dimensions. Recall that the left finitistic flat dimension, $l.\mathrm{FFD}(R)$, of a ring R is defined as follows:

$$l.\mathrm{FFD}(R) = \sup\{\mathrm{fd}_R(M) \mid M \textit{ is a left } R\textit{-module with } \mathrm{fd}_R(M) < \infty\}.$$

The right finitistic dimension $r.\mathrm{FFD}(R)$ is defined similarly.

Proposition 2.6. *If R is a GF-closed ring with finite both left and right weak Gorenstein global dimensions, then* $l.\mathrm{FFD}(R) = l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R) = r.\mathrm{FFD}(R)$.

Proof. This follows from Theorem 2.1 and the following result. □

Recall that the left finitistic Gorenstein flat dimension, $l.\mathrm{FGFD}(R)$, of a ring R is defined as follows:

$$l.\mathrm{FGFD}(R) = \sup\{\mathrm{Gfd}_R(M) \mid M \textit{ is a left } R\textit{-module with } \mathrm{Gfd}_R(M) < \infty\}.$$

The right finitistic dimension $r.\mathrm{FGFD}(R)$ is defined similarly.
The following result is a generalization of [13, Theorem 3.24].

Proposition 2.7. *For any ring R, we have* $l.\mathrm{FFD}(R) \leq l.\mathrm{FGFD}(R)$ *and* $r.\mathrm{FFD}(R) \leq r.\mathrm{FGFD}(R)$.
Furthermore, if R is left (resp., right) GF-closed, then $l.\mathrm{FFD}(R) = l.\mathrm{FGFD}(R)$ *(resp.,* $r.\mathrm{FFD}(R) = r.\mathrm{FGFD}(R)$*).*

Proof. The inequalities follow immediately by the fact that $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M)$ for every R-module M with finite flat dimension [2, Theorem 2.2].
Now, assume that R is left GF-closed (the right version is proved similarly). It remains to prove the converse inequality $l.\mathrm{FGFD}(R) \leq l.\mathrm{FFD}(R)$. For that, we can assume that $l.\mathrm{FFD}(R) = n$ is finite. Let M be a left R-module with finite Gorenstein flat dimension. By Lemma 2.2, there exists a short exact sequence of left R-modules $0 \to M \to M' \to G \to 0$ such that $\mathrm{Gfd}_R(M) = \mathrm{fd}_R(M') \leq n$. This implies the desired inequality. □

Now we give the second main result which says that Conjecture 1.1 is true for two-sided coherent rings. For that we use the following notions:
From [10], a ring R is called *right* (resp., left) *IF*, if every injective right (resp., left) R-module I is flat. A ring R is called *IF*, if it is both left and right IF. Then, let us call a ring R is *right* (resp., left) *n-IF*, for $n \geq 0$, if $\mathrm{fd}_R(E) \leq n$ for every injective right (resp., left) R-module E. And R is called *n-IF*, if it is both left and right n-IF.

Obviously, 0-IF rings are just the IF rings. And, from [11, Theorem 9.1.11], the n-IF Noetherian rings are the same as the well-known n-Gorenstein rings.

Theorem 2.8. *If R is a right and left coherent ring, then, for a positive integer n, the following conditions are equivalent:*

(1) $l.\mathrm{wGgldim}(R) \leq n$;
(2) *R is n-IF;*
(3) $r.\mathrm{wGgldim}(R) \leq n$.

Consequently, for any two-sided coherent ring R, $l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R)$.

The proof of this theorem uses the notion of a flat preenvelope of modules which is defined as follows:

Definition 2.9 ([18])**.** Let R be a ring and let F be a flat R-module. For an R-module M, a homomorphism (or F) $\varphi : M \to F$ is called *a flat preenvelope*, if for any homomorphism $\varphi' : M \to F'$ with F' is a flat module, there is a homomorphism $f : F \to F'$ such that $\varphi' = f\varphi$.

Note that if M embeds in a flat module, then its flat preenvelope (if it exists) is injective.

The coherent rings is also characterized by the notion of a flat preenvelope of modules as follows:

Lemma 2.10. [18, Theorem 2.5.1] *A ring R is coherent if and only if every R-module has a flat preenvelope.*

Also we use the notion of a flat cover of modules which is defined as follows:

Definition 2.11 ([18])**.** Let R be a ring and let F be a flat R-module. For an R-module M, a homomorphism (or F) $\varphi : F \to M$ is called *a flat precover*, if for any homomorphism $\varphi' : M \to F'$ with F' is a flat module, there is a homomorphism $f : F' \to F$ such that $\varphi' = \varphi f$.
A flat precover $\varphi : F \to M$ of M is called *flat cover*, if every endomorphism f of F with $\varphi = \varphi f$ must be an automorphism.

Recall that an R-module M is called *cotorsion*, if $\mathrm{Ext}^1_R(F, M) = 0$ for every flat R-module F.

Lemma 2.12. ([4], [11, Lemma 5.3.25]) *For any ring R, every R-module M has a flat cover $\varphi : F \to M$ such that $\mathrm{Ker}(\varphi)$ is cotorsion.*

Note that every flat cover is surjective.

From its proof, [13, Proposition 3.22] is stated, as we need here, as follows:

Lemma 2.13. *Let R be a right (resp., left) coherent ring. If T is a left (resp., right) R-module such that $\mathrm{Tor}_1^R(I,T)=0$ (resp. $\mathrm{Tor}_1^R(T,I)=0$) for every injective right (resp., left) R-module, then $\mathrm{Ext}_R^1(T,K)=0$ for every cotorsion left (resp., right) R-module K with finite flat dimension.*

Proof of Theorem 2.8. We prove the implication $1\Rightarrow 2$. The implication $3\Rightarrow 2$ has a similar proof. Since every coherent ring is GF-closed, Lemmas 2.4 implies that $\mathrm{fd}_R(E)\leq n$ for every injective right R-module E. Now, consider an injective left R-module I. From Corollary 2.3, $\mathrm{fd}_R(I)=\mathrm{Gfd}_R(I)\leq n$. Therefore, R is n-IF.

We prove the implication $2\Rightarrow 1$. The implication $3\Rightarrow 1$ has a similar proof. Let M be a left R-module, and consider an exact sequence of left R-modules:

$$(*)\quad 0\to G\to P_{n-1}\to\cdots\to P_0\to M\to 0,$$

where each P_i is projective. We have to prove that G is Gorenstein flat.
First note that, using the above sequence $(*)$, we have:

$$\mathrm{Tor}_k^R(E,G)\cong \mathrm{Tor}_{n+k}^R(E,M)\qquad \text{for every } k\geq 1 \text{ and every right R-module } E.$$

If E is an injective right R-module, then $\mathrm{fd}_R(E)\leq n$ (since R is n-IF), and so by the above isomorphism we get:

$$(**)\quad \mathrm{Tor}_k^R(E,G)=0\qquad \text{for every } k\geq 1 \text{ and every injective right R-module } E.$$

Then, by [13, Theorem 3.6 ($i\Leftrightarrow iii$)], it remains to construct a right flat resolution of G:

$$\mathbf{F}=0\to G\to F^0\to F^1\to\cdots,$$

such that the sequence $\mathrm{Hom}_R(\mathbf{F},F)$ is exact whenever F is a flat left R-module. Equivalently, for every positive integer i, $G^i\to F^i$ is a flat preenvelope of G^i, where $G^0=G$ and $G^i=\mathrm{Ker}(F^i\to F^{i+1})$ for $i\geq 1$.
Consider a short exact sequence of left R-modules $0\to G\to I\to L\to 0$, where I is injective. From Lemma 2.12, there exists a short exact sequence of left R-modules $0\to K\to F\to I\to 0$, where F is flat and K is cotorsion. Since R is n-IF,

$\mathrm{fd}_R(I) \leq n$ and so $\mathrm{fd}_R(K) < \infty$. Consider the pullback diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & D & \dashrightarrow & F & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & G & \to & I & \to & L & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

From Lemma 2.13 and $(**)$, we have $\mathrm{Ext}^1_R(G, K) = 0$. Then, the left vertical sequence splits, and so G embeds in the flat module F. Thus, G admits an injective flat preenvelope $G \to F^0$, which gives the desired first flat preenvelope.

Now, for $G^1 = \mathrm{Coker}(G \to F^0)$ we prove that $\mathrm{Ext}^1_R(G^1, K) = 0$ for every cotorsion left R-module K with finite flat dimension. This gives, using the same argument above, the desired second flat preenvelope $G^1 \to F^1$, and recursively we obtain the remaining flat preenvelopes.

Let K be a cotorsion left R-module with finite flat dimension. By Lemma 2.12, there exists a flat cover $F \to K$ of K such that we obtain a short exact sequence of left R-modules $0 \to K' \to F \to K \to 0$, where K' is cotorsion with finite flat dimension (since $\mathrm{fd}_R(K) < \infty$). By [18, Proposition 3.1.2], F is cotorsion. Then, we get the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(G^1, K') & \to & \mathrm{Hom}(F^0, K') & \to & \mathrm{Hom}(G, K') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(G^1, F) & \to & \mathrm{Hom}(F^0, F) & \to & \mathrm{Hom}(G, F) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}(G^1, K) & \to & \mathrm{Hom}(F^0, K) & \to & \mathrm{Hom}(G, K) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

with exact rows and columns. Indeed, the middle vertical sequence is exact since $F \to K$ is a flat cover of K; the right vertical sequence is exact since $\mathrm{Ext}^1_R(G, K') = 0$ since K' is cotorsion with finite flat dimension; and the middle horizontal sequence is exact since $G \to F^0$ is a flat preenvelope. Then, the sequence $0 \to \mathrm{Hom}(G^1, K) \to \mathrm{Hom}(F^0, K) \to \mathrm{Hom}(G, K) \to 0$ is exact. This implies, using $\mathrm{Ext}^1_R(F^0, K) = 0$, that $\mathrm{Ext}^1_R(G^1, K) = 0$, and this completes the proof. □

For the case where $l.\mathrm{wGgldim}(R) = 0$ or $r.\mathrm{wGgldim}(R) = 0$ we have the following generalization of [7, Theorem 6 (1 ⇔ 2)].

Proposition 2.14. *For a ring R, the following conditions are equivalent:*

(1) $l.\mathrm{wGgldim}(R) = 0$;
(2) *R is IF;*
(3) $r.\mathrm{wGgldim}(R) = 0$.

Proof. The implications 1 ⇒ 2 and 3 ⇒ 2 follow from Theorem 2.8.

We prove the implication 2 ⇒ 1. The implication 3 ⇒ 1 has a similar proof. We prove that every left R-module M is Gorenstein flat. For that, we have to construct a complete flat resolution $\mathbf{F}$ such that $M \cong \mathrm{Im}(F_0 \to F^0)$. Since R is IF, we can consider any flat resolution of M as the "left half" of $\mathbf{F}$. For that "right half" of $\mathbf{F}$, consider an injective resolution $\mathbf{I}$ of M. Since R is IF, the sequence $\mathbf{I}$ is a right flat resolution of M such that the sequence $I \otimes_R \mathbf{I}$ is exact whenever I is an injective (then flat) right R-module, as desired. □

Naturally, for a ring R which satisfies $l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R) = 0$ we denote $\mathrm{wGgldim}(R)$ for the common value of these equal terms and we call it *weak Gorenstein global dimension of R*. As a generalization of [7, Theorem 7], the following result gives a characterization of $\mathrm{wGgldim}(R)$ when R is a two sided coherent rings using the notion of FP-injective dimension of modules, which are defined as follows [17]: We say that an R-module M has *FP-injective (or pure) dimension at most n* (for some $n \geq 0$), denoted by $\mathrm{FP\text{-}id}_R(M) \leq n$, if and only if, $\mathrm{Ext}^{n+1}_R(P, M) = 0$ for all finitely presented R-modules P. The modules of FP-injective dimension 0 are called *FP-injective* (or absolutely pure [15]).

Recall also that a ring R is said to be *n-FC*, for some positive integer n, if it is left and right coherent and it has self-FP-injective dimension at most n on both the left and the right sides .

In the proof of Theorem 2.16, we use the following Cheatham and Stone's characterization of coherent rings.

Lemma 2.15. ([6, Theorem 2, (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4)]) *Let R be a ring. The following conditions are equivalent:*

(1) *R is left (resp., right) coherent;*

(2) *M is an FP-injective left R-module if and only if M^* is a flat right R-module;*

(3) *M is an flat left R-module if and only if $(M^*)^*$ is a flat left R-module.*

Theorem 2.16. *Let R be a left and right coherent ring and let n be a positive integer. The following conditions are equivalent:*

(1) $\text{wGgldim}(R) \leq n$*;*

(2) *FP-*$\text{id}_R(F) \leq n$ *for every flat left R-module F and every flat right R-module F;*

(3) *R is n-FC.*

Proof. The equivalence $1 \Leftrightarrow 3$ follows from Theorem 2.8 and [7, Theorem 7].

We prove the equivalence $1 \Leftrightarrow 2$.

$(1 \Rightarrow 2)$. Let F be a flat left R-module. Then, F^* is an injective right R-module (from [16, Theorem 3.52]). Since R is left coherent (then left GF-closed) with $\text{wGgldim}(R) \leq n$, we get from Lemma 2.4, $\text{fd}_R(F^*) \leq n$. Therefore, using [16, Lemma 3.51] and Lemma 2.15 (1) $\Leftrightarrow$ (2), we get $\text{FP-id}_R(F) = \text{fd}_R(F^*) \leq n$. Similarly we get $\text{FP-id}_R(F) \leq n$ for every flat right R-module F.

$(2 \Rightarrow 1)$. Let E be an injective left R-module, then it is FP-injective and so, by Lemma 2.15 (1) $\Leftrightarrow$ (2), E^* is a flat right R-module. Then, by (2), $\text{FP-id}_R(E^*) \leq n$. Now, pick an exact sequence of left R-modules:

$$0 \to K \to F_{m-1} \to \cdots \to F_1 \to F_0 \to E \to 0$$

where each F_i is flat. Then, we have the following exact sequence of right R-modules:

$$0 \to (E)^* \to (F_0)^* \to (F_1)^* \to \cdots \to (F_{m-1})^* \to (K)^* \to 0$$

where each $(F_i)^*$ is injective (from [16, Theorem 3.52]). And since $\text{FP-id}_R(E^*) \leq n$, $(K)^*$ is FP-injective. Then, from Lemma 2.15 (1) $\Leftrightarrow$ (2), $((K)^*)^*$ is flat, which is means, by Lemma 2.15 (1) $\Leftrightarrow$ (3), that K is flat. Therefore, $\text{fd}_R(E) \leq n$. Similarly we get $\text{fd}_R(F) \leq n$ for every injective right R-module F. Therefore, R is n-IF and so, by Theorem 2.8, $\text{wGgldim}(R) \leq n$. □

Remark 2.17. In [10, Example 2], Colby gave an example of a left and right coherent ring R which is right IF but not left IF. Then, $l.\text{wGgldim}(R) = r.\text{wGgldim}(R) =$

∞. Indeed, if $l.\mathrm{wGgldim}(R) < \infty$, then, by Lemma 2.4, $l.\mathrm{wGgldim}(R) = 0$, and from Theorem 2.8, $l.\mathrm{wGgldim}(R) = r.\mathrm{wGgldim}(R) = 0$. But, this contradicts the fact that R is not left IF (Proposition 2.14). Consequently:

1. To get the implication 4 (or 5) $\Rightarrow$ 1 in Lemma 2.4, the condition "$l.\mathrm{wGgldim}(R) < \infty$" can not be dropped.
2. In [13, Theorem 3.14], the condition "$\mathrm{Gfd}(M) < \infty$" can not be dropped to get the implication iii(or ii) $\Rightarrow$ i. Indeed, since $l.\mathrm{wGgldim}(R) = \infty$, there exists, using [13, Proposition 3.13], a left R-module with $\mathrm{Gfd}_R(M) = \infty$. However, $\mathrm{Tor}_i^R(I, M) = 0$ for every $i > 0$ and every injective (then flat) right R-module I.

Finally, to construct examples of GF-closed rings of finite weak Gorenstein global dimension which are neither coherent nor of finite weak global dimension, we need the following result:

Proposition 2.18. *For any family of rings $\{R_i\}_{i=1,\dots,m}$, we have:*

$$l.\mathrm{wGgldim}(\prod_{i=1}^{m} R_i) = \sup\{l.\mathrm{wGgldim}(R_i), 1 \leq i \leq m\} \quad \textit{and}$$

$$r.\mathrm{wGgldim}(\prod_{i=1}^{m} R_i) = \sup\{r.\mathrm{wGgldim}(R_i), 1 \leq i \leq m\}.$$

Proof. The result is a consequence of [1, Theorem 3.4]. □

Example 2.19. Consider an IF ring R_1 with infinite weak global dimension, and consider a non-coherent ring R_2 with finite weak global dimension. Then, the direct product $R_1 \times R_2$ is GF-closed (by [1, Proposition 3.5]) with finite weak Gorenstein global dimension (by Proposition 2.18), but it is neither coherent nor of finite weak global dimension.

References

[1] D. Bennis, *Rings over which the class of Gorenstein flat modules is closed under extensions*, Comm. Algebra, 37 (2009), 855–868.

[2] D. Bennis, *A Note on Gorenstein Flat Dimension.* Accepted for Publication in Algebra Coll. Available from arXiv:0811.2650v1.

[3] D. Bennis and N. Mahdou, *Global Gorenstein Dimensions*, Proc. Amer. Math. Soc., 138 (2010), 461–465.

[4] L. Bican, R. El Bashir and E. Enochs, *All modules have flat covers*, Bull. London Math. Soc., 33 (2001), 385–390.

[5] N. Bourbaki, Algèbre Homologique, Chapitre 10, Masson, Paris, 1980.

[6] T. J. Cheatham and D. R. Stone, *Flat and projective character modules*, Proc. Amer. Math. Soc., 81 (1981), 175–177.

[7] J. Chen and N. Ding, *Coherent rings with finite self-FP-injective dimension*, Comm. Algebra, 24 (1996), 2963–2980.

[8] L. W. Christensen, Gorenstein Dimensions, Lecture Notes in Math., Springer-Verlag, Berlin, 2000.

[9] L. W. Christensen, A. Frankild and H. Holm, *On Gorenstein projective, injective and flat dimensions - a functorial description with applications*, J. Algebra, 302 (2006), 231–279.

[10] R. R. Colby, *On Rings which have flat injective modules*, J. Algebra, 35 (1975), 239–252.

[11] E. E. Enochs and O. M. G. Jenda, Relative Homological Algebra, de Gruyter Expositions in Mathematics, Walter de Gruyter & Co., Berlin, 2000.

[12] E. E. Enochs and O. M. G. Jenda, Torrecillas, B. *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan, 10 (1993), 1–9.

[13] H. Holm, *Gorenstein homological dimensions*, J. Pure Appl. Algebra, 189 (2004), 167–193.

[14] H. Holm, *Rings with finite Gorenstein injective dimension*, Proc. Amer. Math. Soc., 132 (2004), 1279–1283.

[15] B. Madox, *Absolutely pure modules.* Proc. Amer. Math. Soc., 18 (1967), 155–158.

[16] J. J. Rotman, An Introduction to Homological Algebra, Academic Press, New York, 1979.

[17] B. Stenström, *Coherent rings and FP-injective modules*, J. London Math. Soc., 2 (1970), 323–329.

[18] J. Xu, Flat Covers of Modules, Lecture Notes in Math., Springer-Verlag, Berlin, 1996.

Driss Bennis
Department of Mathematics,
Faculty of Science and Technology of Fez, Box 2202,
University S. M. Ben Abdellah Fez, Morocco,
e-mail: driss_bennis@hotmail.com